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Theory of Stochastic Pantograph Differential Equations with ϑ-Caputo Fractional Derivative

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Abstract: In this paper, we mainly study the existence of analytical solution of stochastic pantograph differential equations. The standard Picard's iteration method is used to obtain the theory.

Keywords: Pantograph differential equation, ψ-type fractional derivative, existence, Picard's iteration method.

1 Introduction

Fractional differential equations (FDEs) have been attracted much interesting in recent studies. This is happened due to improvement of the theory of fractional calculus and due to the broad spread to their applications in the engineering and natural, see [1,2,3]. In the literature, many researchers applied various complicated fractional operators such as the Riemann—Liouville, Caputo, Hadamard, Caputo—Hadamard, Fabrizio-Caputo and Hilfer fractional operators, etc. (see for example, [4,5,6,7,8,9,10,11,12,13]).

The pantograph equation is one of the most famous classes of differential equations and this type of equation is taken for as proportional delay functional differential equations and have many applications in pure and applied mathematics as it appear in a various contexts such as control systems, probability, electrodynamics, quantum mechanics, etc. Furthermore, a delay FDEs have established more actual interpretation of natural phenomena than those without delay. Thus, the studies of those equations has win much interesting, see [14,15,16,17,18]. Stochastic delay differential equations has an important applications in the physics, economics, finance and biology fields [19,20]. Ockendon and Tayler [21], studied a particular case which is stochastic pantograph differential equations and described how the electric current is collected by the pantograph of an electric locomotive, see [22,23]. Recently, existence, uniqueness and stability properties are the most important characteristics of stochastic systems and pantograph equations, for this regard we refer the readers to these works [20,23,24,25,26,27,28,29,30,31,32,33].

Very recently, Almeida [34] introduced a new fractional operator called by ϑ-fractional operator with respect to another function, which generalized the classical fractional operators and also discussed some properties like semigroup law, Taylor's Theorem and so on. Thereafter, Vivek et al. [23] initially studied a Cauchy problem for pantograph equations includes Hilfer fractional derivative.

Inspired by the papers [23,34,35], we consider the stochastic pantograph differential equations (SPDEs) via ϑ-Caputo fractional derivative of the version

$$ {}^{c}D^{\alpha;\vartheta}x(t) = Ax(t) + f(t,x(t),x(\lambda t)) + \sigma(t,x(t),x(\lambda t))\dot{W}(t), \quad t \in J := [0,T], \tag{1} $$

$$ x^{(k)}(0) = x_0^{(k)}, \quad k = 0,1,2,...,n-1, \tag{2} $$

where $0 < \lambda < 1$, $n-1 < \alpha \leq n$ and f,σ are given functions and A is the generator of strongly continuous semigroup $\{\tau(t) : t \geq 0\}$ on a Hilbert space $\mathscr{H}$.

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Observing that (1)-(2) is equivalent to the Volterra integral equation as follows:

$$x(t)=\begin{cases}\sum_{k=0}^{\lceil\alpha\rceil-1}\frac{x^{(k)}(0)}{k!}(\vartheta(t))^k+\frac{1}{\Gamma(\alpha)}\int_0^t\vartheta'(s)(\vartheta(t)-\vartheta(s))^{\alpha-1}Ax(s)ds\\ +\frac{1}{\Gamma(\alpha)}\int_0^t\vartheta'(s)(\vartheta(t)-\vartheta(s))^{\alpha-1}f(s,x(s),x(\lambda s))ds\\ +\frac{1}{\Gamma(\alpha)}\int_0^t\vartheta'(s)(\vartheta(t)-\vartheta(s))^{\alpha-1}\sigma(s,x(s),x(\lambda s))dW(s),\end{cases} \tag{3}$$

where $n-1<\alpha\leq n$ and $t\geq 0$.

Our current paper is coordinated as follows: An introduction is provided In Section 1. Some important prerequisite results are presented in Section 2. Furthermore, the main result is introduced in Section 3. Finally, Section 4 is devoted to propose a brief conclusion.

2 Prerequisite

Throughout this paper, the space $(\Omega,\Im,\mathbb{P})$ denotes a completely probability space, $\mathscr{H}$ denotes a separable Hilbert space with inner product $(\cdot,\cdot)$ norm $\|\cdot\|$. Thus, $\mathscr{L}_2(\Omega,\mathscr{H})$ be a Hilbert space of $\mathscr{H}$-valued random variables with the inner product $\mathbb{E}(\cdot,\cdot)$ and the norm $\left(\mathbb{E}\|\cdot\|^2\right)^{\frac{1}{2}}$ in which $\mathbb{E}$ denotes the expectation.

Furthermore, we consider the ϑ-type Caputo fractional derivative of order α for a vector-valued function $x(t)$, and the initial value problem (IVP) of an abstract SPDEs (1)-(2), where $f(t,x(t),x(\lambda t))$, $\sigma(t,x(t),x(\lambda t)):J\times\mathscr{R}^d\times\mathscr{R}^d\to\mathscr{R}^d$ with dimension $d\geq 1$. A state dependent random noise described by the term $\dot{W}(t)=\frac{dW}{dt}$ and a standard scalar brownian motion or Wiener process defined by $\{W(t)\}_{t\geq 0}$ in the filtered probability space $(\Omega,\Im,\Im_t,\mathbb{P})$ with a normal filteration $\{\Im_t\}_{t\geq 0}$, which is a continuous and increasing family of σ-algebra of $\Im$, contains the $\mathbb{P}$-null sets, and $W(t)$ is $\Im_t$-measurable for all $t\geq 0$.

Now, we will giving some important definitions related to our work. Further details can be found in [34].

Definition 1.[36] *The following Itô isometry property holds for* $u\in\mathscr{L}_2(\Omega,\mathscr{H})$*:*

$$\mathbb{E}\left\|\int_0^t u(s)dW(s)\right\|^2=\int_0^t E\|u(s)\|^2 ds, \tag{4}$$

such that $\{W(t)\}_{t\geq 0}$ *is the Wiener (Brownian motion) process.*

Definition 2. *The* ϑ*-type Riemann-Liouville fractional integral of order* $\alpha>0$ *for a function* f *defined by*

$$I^{\alpha,\vartheta}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\vartheta'(s)\left(\vartheta(t)-\vartheta(s)\right)^{\alpha-1}f(s)ds,\quad a.e\quad t\in J,$$

where the symbol $\Gamma(\cdot)$ *stands for the Euler's gamma function.*

Definition 3. *The* ϑ*-type Caputo fractional derivative of order* α *for a function* f *defined by*

$${}^cD^{\alpha;\vartheta}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\vartheta'(s)\left(\vartheta(t)-\vartheta(s)\right)^{n-\alpha-1}f^{(n)}(s)ds,$$

where $t>0$, $n-1<\alpha\leq n$.

Remark. The connection relationship between ϑ-type Riemann-Liouville factional derivative and the ϑ-type Caputo factional derivative given by

$${}^cD^{\alpha;\vartheta}f(t)={}^RD^{\alpha;\vartheta}f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}(\vartheta(t))^k.$$

Lemma 1. *The solution of the equation* (3) *is equivalent to the solution of the IVP* (1)-(2) *for* $\alpha\in(0,1]$*, and vice versa.*

Particularly, if $0<\alpha\leq 1$, the Volterra integral equation (3) reduce to

$$x(t)=\begin{cases}x(0)+\frac{1}{\Gamma(\alpha)}\int_0^t\vartheta'(s)(\vartheta(t)-\vartheta(s))^{\alpha-1}Ax(s)ds\\ +\frac{1}{\Gamma(\alpha)}\int_0^t\vartheta'(s)(\vartheta(t)-\vartheta(s))^{\alpha-1}f(s,x(s),x(\lambda s))ds\\ +\frac{1}{\Gamma(\alpha)}\int_0^t\vartheta'(s)(\vartheta(t)-\vartheta(s))^{\alpha-1}\sigma(s,x(s),x(\lambda s))dW(s).\end{cases} \tag{5}$$

Lemma 2. *The IVP (1)-(2) is equivalent to the integral equation (5), for $\alpha \in (0,1]$ with $k=0$ and vice versa.*

Proof. For proof, see e.g. [1].

Lemma 3. *([36]) If the function $\mathbb{L}(t,x(\cdot),x(\cdot))$ is continuous non-decreasing in x for each fixed $t \in J$ and is locally integrable in t for each fixed $x \in [0,\infty)$, for all $\delta >$, $x_0 \geq 0$, then the integral equation*

$$x(t) = x_0 + \delta \int_0^t \mathbb{L}(s,x(s),x(\lambda s))ds,$$

has a global solution on J.

Lemma 4. *([36]) The function $K(t,x(\cdot),x(\cdot))$ is continuous non-decreasing in x for each fixed $t \in J$ and is locally integrable in t for each fixed $x \in [0,\infty)$, for $K(t,0,0)=0$ and $\gamma > 0$, if a non-negative continuous function $\phi(t)$ satisfies*

$$\phi(t) \leq \gamma \int_0^t K(s,x(s),x(\lambda s))ds, \quad t \in \mathscr{R},$$
$$\phi(0) = 0,$$

then $\phi(t) = 0$ for all $t \in J$.

3 Main results

First of all, regarding to study the existence and uniqueness of the solution for the IVP (1)-(2) for $\alpha \in (0,1]$, we list the following hypotheses:

(H1) Assume that $\tau(\cdot)$ be a C_0-semigroup generated by the unbounded operator A, let $M = \max_{t\in J} \|\tau(t)\|_{\mathscr{H}}$.

(H2) The functions f and σ are measurable and continuous in $\mathscr{H}$ for all fixed $t \in J$ and there is a bounded function $\mathbb{L} : J \times [0,\infty) \times [0,\infty) \to [0,\infty)$, $(t,x,y) \to \mathbb{L}(t,x,y)$ such that

$$\mathbb{E}\left(\|f(t,u,v)\|^2\right) \leq \mathbb{L}\left(t,\mathbb{E}\left(\|u\|^2\right),\mathbb{E}\left(\|v\|^2\right)\right), \tag{6}$$

and

$$\mathbb{E}\left(\|\sigma(t,u,v)\|^2\right) \leq \mathbb{L}\left(t,\mathbb{E}\left(\|u\|^2\right),\mathbb{E}\left(\|v\|^2\right)\right), \tag{7}$$

for all $t \in \mathscr{R}$ and $u,v \in \mathscr{L}_2(\Omega,\mathscr{H})$.

(H3) There exists a bounded function $K : J \times [0,\infty) \times [0,\infty) \to [0,\infty)$ such that

$$\mathbb{E}\left(\|f(t,u,v) - f(t,\overline{u},\overline{v})\|^2\right) \leq K\left(t,\mathbb{E}\left(\|u-\overline{u}\|^2\right),\mathbb{E}\left(\|v-\overline{v}\|^2\right)\right),$$

and

$$\mathbb{E}\left(\|\sigma(t,u,v) - \sigma(t,\overline{u},\overline{v})\|^2\right) \leq K\left(t,\mathbb{E}\left(\|u-\overline{u}\|^2\right),\mathbb{E}\left(\|v-\overline{v}\|^2\right)\right),$$

for all $t \in \mathscr{R}$ and $u,\overline{u},v,\overline{v} \in \mathscr{L}_2(\Omega,\mathscr{H})$.

Now, will use Picard's iteration method in order to study the existence and uniqueness of the solution of equation (5). The sequence of stochastic process $\{x_n\}_{n\geq 0}$ is constructed as follows:

$$x_0(t) = x_0$$
$$x_{n+1}(t) = x_0 + G_1(x_n)(t) + G_2(x_n)(t), \quad n \geq 1,$$

in which

$$G_1(x_n)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\alpha-1} f(s,x_n(s),x_n(\lambda s))ds,$$
$$G_2(x_n)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\alpha-1} \sigma(s,x_n(s),x_n(\lambda s))dW(s).$$

Lemma 5.*([36]) The sequence $\{x_n\}_{n\geq 0}$ be a bounded stochastic sequence processes in $\mathscr{L}_2(\Omega,\mathscr{H})$.*

*Proof.*Due to the following inequality

$$(a_1+a_2+a_3)^n \leq 3^{n-1}(a_1^n+a_2^n+a_3^n), \quad n\geq 1.$$

We get

$$\mathbb{E}\|x_{n+1}(t)\|^2 \leq 3\mathbb{E}\|x_0\|^2+3\mathbb{E}\|G_1(x_n)(t)\|^2+3\mathbb{E}\|G_2(x_n)(t)\|^2. \tag{8}$$

By using the *Hölder's* inequality, (H2) and $\alpha>\frac{1}{2}$, we have

$$\begin{aligned}\mathbb{E}\|G_1(x_n)(t)\|^2 &\leq \frac{1}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_0^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\alpha-1}f(s,x_n(s),x_n(\lambda s))ds\right\|^2\\ &\leq \frac{1}{\Gamma^2(\alpha)}\frac{(\vartheta(t))^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\|f(s,x_n(s),x_n(\lambda s))\|^2 ds\\ &\leq k_1\int_0^t \mathbb{L}\left(s,\|x_n(s)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})},\|x_n(s)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})}\right)ds,\end{aligned}$$

where $k_1=\frac{1}{\Gamma^2(\alpha)}\frac{(\vartheta(T))^{2\alpha-1}}{2\alpha-1}$.

In view of the *Itô*isometry property (4), the *Hölder's* inequality, (H2) and $\alpha>\frac{1}{2}$, we get

$$\begin{aligned}\mathbb{E}\|G_2(x_n)(t)\|^2 &\leq \frac{1}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_0^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\alpha-1}\sigma(s,x_n(s),x_n(\lambda s))ds\right\|^2\\ &\leq \frac{1}{\Gamma^2(\alpha)}\frac{(\vartheta(t))^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\|\sigma(s,x_n(s),x_n(\lambda s))\|^2 ds\\ &\leq k_1\int_0^t \mathbb{L}\left(s,\|x_n(s)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})},\|x_n(s)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})}\right)ds.\end{aligned}$$

Hence, using the above relation into the inequality (8), we have

$$\|x_{n+1}(t)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})} \leq c_1+c_2\int_0^t \mathbb{L}\left(s,\|x_n(s)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})},\|x_n(s)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})}\right)ds, \tag{9}$$

in which $c_1=3\mathbb{E}\|x_0\|^2$ and $c_2=6k_1$.

Therefore, we consider the following integral equation:

$$u(t)=c_1+c_2\int_0^t \mathbb{L}(s,u(s),u(\lambda s))ds. \tag{10}$$

Due to the Lemma 3, the above equation has a globe solution and by the mathematical induction we obtain $\|x_n(t)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})} \leq x(t)$ for all $t\in J$. Particularly, we have

$$\sup_{n\geq 0}\|x_n(t)\|_{\mathscr{L}_2(\Omega,\mathscr{H})} \leq [x(T)]^{\frac{1}{2}}.$$

Lemma 6.*The sequence of stochastic processes $\{x_n\}_{n\geq 0}$ is a Cauchy sequence.*

Herein, we will prove the existence and uniqueness of the solution of the problem (1)-(2).

Theorem 1.*Under the hypotheses $(H_1)-(H_3)$ hold, then there exists a unique solution of equation (5).*

*Proof.*Existence: Let $x(t)$ by the limit of the sequence $\{x_n(t)\}_{n\geq 0}$ and by using Lemma 6 then we can see that the right hand side in the second Picard's iteration tend to

$$\begin{cases} x_0+\frac{1}{\Gamma(\alpha)}\int_0^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\alpha-1}f(s,x(s),x(\lambda s))ds\\ +\frac{1}{\Gamma(\alpha)}\int_0^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\alpha-1}\sigma(s,x(s),x(\lambda s))dW(s),\end{cases}$$

which is just a solution of equation (5).

Uniqueness: Consider $x(t)$ and $y(t)$ are two solution of equation (5), using Lemma 5, we have

$$\|x(t)-y(t)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})} \leq c_3\int_0^t K\left(s,2\|x(s)-y(s)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})}\right)ds.$$

Due to Lemmas 3 and 4, we can get $\|x(t)-y(t)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})}=0$ for all $t\in J$, which yields that $x(t)=y(t)$.

4 Conclusion

In the last decades, the stochastic pantograph differential equations have been played an important role in application areas, such as physics, biology, economics, and finance. In this paper, we employed the standard Picard's iteration method to study the existence and uniqueness of analytical solution of stochastic pantograph differential equations involving ϑ-Caputo fractional derivative in Hilbert space.

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