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On the Stability of Commensurate Fractional-Order Lorenz System

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Abstract: In the presented work, with the help of using Adomian decomposition nethod (ADM), two particular situations connected to the equilibria's stability of the nonlinear fractional-order Lorenz system (FoLS) are examined and confirmed numerically. Such situations can be extremely valuable for discerning between several other situations that might be employed to explore the stability of Lorenz system. In this study, all numerical simulations are carried out by using MATLAB software package.

Keywords: Fractional calculus, fractional-order Lorenz system, stability.

1 Introduction

More than three hundred years back, fractional calculus came out as a generalisation of the traditional calculus. It lured considerable research activities due to nearly all systems are good addressed by fractional-order dynamics [1,2,3,4,5]. Numerous systems in multidisciplinary scopes can be precisely outlined through implementing fractional-order derivatives. Several systems are determined to study the impact of the fractional-order values on the system's dynamics, and within this area of research some fractional-order systems were explored to exhibit their chaotic modes [6,7,8,9].

Grigorenko et al. [10] are the first mathematicians who analyzed the fractional-order Lorenz System (FoLS). Lately, the dynamics of such system were analyzed by Jia et al., and then applied to analog circuit through utilizing the frequency domain approach [11]. In order to solve the fractional-order chaotic systems, another method can be also performed namely the Adams-Bashforth-Moulton algorithm. It could be furthermore employed to analyze their dynamics [12]. In the same regard, the Adomian decomposition method (ADM) [13] can be typically implemented to gain a high-accuracy numerical solution of a well-defined fractional-order chaotic system with fast speed of convergence [14]. In reference [15], the ADM was proposed for handling such system and the outcomes of the this scheme were paralleled with the outcomes of the fractional Adams method. When Shaobo et al. [16] compared the results of the ADM with the results of the Adams-Bashforth-Moulton algorithm in solving the FoLS, they concluded that the ADM implies more precise outcomes and demands less calculating than the other method. Such method is even more precise than Runge-Kutta method when the traditional Lorenz system is addressed [2]. In reference [17], Shaobo et al. returned again to present a study on the implementation of digital circuit associated with the fractional-order Lorenz system by employing ADM. In the conclusion of their work, they showed that this system has a highly complex behavior and it is a feasible model suited for several implementations. Before one year of that work, Huihai et al. [18] solved the simplified Lorenz system of fractional-order, and applied it on the processor of digital signal by adopting the ADM. They proved that the phase portraits produced on this processor are paralleled with the findings that were yielded by certain numerical simulations.

Several criteria have been proposed for fractional-order systems, including the Matignon stability theorem [19,20]. The goal of this theorem is to verify the stability of the considered systems by assessing the locations of their eigenvalues generated from their the dynamic matrices in the complex plane. Besides, the LMI scheme [19,21] and the Lyapunov approach [19,22] can also be employed to study the stability of fractional-order linear-time invariant systems. At a future

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date, many efforts and endeavors were dedicated to reveal more properties of fractional-order chaotic systems [23]. The obtained results of these studies are, indeed, difficult to be verified without numerical simulations[24,25].

The Lorenz system is a reduction of another more complex system backs to Saltzman. This complicated system is used commonly to describe the buoyancy driven convection patterns in the conventional rectangular Rayleigh Bénard dilemma implemented on thermal convection between two plates perpendicular to the direction of the Earth's gravitational force [26]. The traditional Lorenz system might generate chaotic and non-chaotic modes in accordance to different values of its parameters. This system has the form [27]:

$$\begin{aligned} \frac{d}{dt}x(t) &= \sigma(y(t)-x(t)), \\ \frac{d}{dt}y(t) &= Rx(t)-x(t)z(t)-y(t), \\ \frac{d}{dt}z(t) &= x(t)z(t)-bz(t). \end{aligned} \tag{1a}$$

with the following initial conditions:

$$x(0)=v_1, y(0)=v_2, z(0)=v_3, \tag{1b}$$

where $x(t)$, $y(t)$, $z(t)$ are the states of the system, and σ, b, R are the parameters of the system in which they are positive real constants [28]. The FoLS was introduced and takes the following form [28]:

$$\begin{aligned} D^{\alpha}x(t) &= \sigma(y(t)-x(t)), \\ D^{\alpha}y(t) &= Rx(t)-x(t)z(t)-y(t), \\ D^{\alpha}z(t) &= x(t)z(t)-bz(t). \end{aligned} \tag{2a}$$

with the following initial conditions:

$$x(0)=v_1, y(0)=v_2, z(0)=v_3 \tag{2b}$$

where D^{α} is the Caputo fractional-order operator of order $0<\alpha\leq 1$ [5].

System (2) outlines a FoLS, which has hereditary and memory impact on several processes and materials [29]. It has been implemented on numerous areas in engineering and science, like communication, control theory, acoustics, biology, chemistry, fluid mechanics, anomalous diffusion, viscoelasticity, and others [30]. The major aim of this work is to propose some new results connected with the equilibria's stability of the nonlinear FoLS by employing the ADM. However, for a complete overview about such method, refer to the references [13,31,32]. Anyhow, the remaining of this research work is arranged in the following manner as follows. In Section 2, some preliminaries and necessary definitions are recalled, while Section 3 discusses two particular cases associated with the system's stability together with some simulation results, followed by the conclusion of this work.

2 Preliminaries

Fractional calculus is a generalization of differentiation and integration of the classical operator. Its operator is of the form D_a^{α}, where a and t are the lower and upper bounds of the operation, respectively. There are various definitions for the fractional-order derivatives and integrals; the primary ones are the definitions proposed by Riemann-Liouville and Caputo. Next, we state some major facts connected with fractional calculus together with some concepts related to the matrix analysis.

Definition 1. *Let $f(t)$ be an integrable piecewise continuous function on any finite subinterval of $(0,+\infty)$, then the fractional integral of $f(t)$ of order α is defined as convolution [5,33]:*

$$J^{\alpha}f(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} * f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau,\ t>0,\ \alpha>0, \tag{3}$$

where $*$ denotes the convolution product in which

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau \tag{4}$$

Based on (3), one can deduce the following useful equality:

$$J^{\alpha}t^{\mu}=\frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}t^{\mu+\alpha},\ t>0,\ \alpha>0,\ \mu>-1. \tag{5}$$

Definition 2. *The Caputo fractional-order derivative is defined as [5,33]:*

$$D^{\alpha}f(t)=\frac{1}{\Gamma(M-\alpha)}\int_0^t\frac{f^M(\tau)}{(t-\tau)^{\alpha+1-M}}d\tau,\ f^M(\tau)=\frac{d^Mf(\tau)}{d\tau^M}, \tag{6}$$

where $\Gamma(\cdot)$ is the Gamma function and $M-1\leq\alpha<M$, $M\in\mathbb{N}$.

Theorem 1. *(Sylvester criterion) Let H be an $n\times n$ symmetric matrix. Then, H is negative definite if and only if its n leading principal minors alternate in sign as follows [34]:*

$$|H_1|<0,|H_2|>0,|H_3|<0,... \tag{7}$$

Theorem 2. *(Schur complement) Given a symmetric matrix $H=\begin{pmatrix}M & N\\ N^T & L\end{pmatrix}$, with L negative definite. The matrix H is negative definite if and only if the Schur complement $M-NL^{-1}N^T$ is thus [35].*

3 Stability analysis and simulation results

Only through numerical simulations, some researchers investigated the stability analysis and the chaotic behaviours of the FoLS [6,10,23,36,37]. In this section, two particular cases associated with the stability of the FoLS will be discussed and verified by some numerical simulations implemented using ADM. In order to proceed with this subject; let us, firstly, begin with the so-called nonlinear autonomous fractional-order differential equations which is outlined as [23]:

$$D^{\alpha}\boldsymbol{x}(t)=\mathbf{F}(\mathbf{x}(t)) \tag{8a}$$

subject to the initial conditions:

$$\boldsymbol{x}(0)=[v_1,v_2,...,v_n]^T, \tag{8b}$$

where

$$\mathbf{F}(\mathbf{x}(t))=\begin{bmatrix}f_1(x_1(t),x_2(t),...,x_n(t))\\ f_2(x_1(t),x_2(t),...,x_n(t))\\ \vdots\\ f_n(x_1(t),x_2(t),...,x_n(t))\end{bmatrix}, \tag{8c}$$

and where $\mathbf{x}(t)=[x_1,x_2,...,x_n]^T$, $\alpha=[\alpha_1,\alpha_2,...,\alpha_n]$ and $0<\alpha_i\leq 1$; $i=1,2,...,n$. We find the following two theorems about the stability analysis of the fractional-order differential equations very useful to derived some of the primary outcomes of this work.

Theorem 3. *Let $\hat{\boldsymbol{x}}=[\hat{x}_1,\hat{x}_2,...,\hat{x}_n]^T$ be the equilibrium of system (8), i.e. $D^{\alpha}\hat{\boldsymbol{x}}=\boldsymbol{F}(\hat{\boldsymbol{x}})=0$, and $H=\frac{\partial\boldsymbol{F}}{\partial\boldsymbol{x}}|_{\boldsymbol{x}=\hat{\boldsymbol{x}}}$ be the Jacobian matrix at the point $\hat{\boldsymbol{x}}$, if $|arg(eig(H))|>\frac{\alpha_m\pi}{2}$, then the point $\hat{\boldsymbol{x}}$ is asymptotically stable, where $\alpha_m=max\{\alpha_i\};1\leq i\leq n$, (i.e when all eigenvalues of H are negative) [23].*

Theorem 4. *System (8) is asymptotically stable if and only if the Jacobian matrix $H=\frac{\partial\boldsymbol{F}}{\partial\boldsymbol{x}}|_{\boldsymbol{x}=\hat{\boldsymbol{x}}}$ has k-multiple zero eigenvalues corresponding to the Jordan block $diag(J_1,J_2,...,J_i)$, where J_l is a Jordan canonical form with order $n_l,\sum_{l=1}^{i}n_i=k$, and $n_l\alpha<1,1\leq l\leq i$ [38].*

Based on the previous two theorems, the following new propositions for system (2) can be deduced easily through considering that the three parameters (σ, R and b) of such system can be taken to be positive due to their physical origins [39]. Anyhow, such system has three equilibria stated in reference [23] as: $O(0,0,0)$, $q_+(\sqrt{b(R-1)},\sqrt{b(R-1)},R-1)$ and $q_-(-\sqrt{b(R-1)},-\sqrt{b(R-1)},R-1)$.

Proposition 1. *Regarding to system* (2)*; the equilibrium origin O is asymptotically stable if $\sigma = R < 1$, and this assumption can be hold if the Jacobian matrix, H_o, is symmetric-negative definite.*

Proof. At the equilibrium origin O, system (2) has the Jacobian matrix [23]:

$$H_o = \begin{bmatrix} -\sigma & \sigma & 0 \\ R & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} \tag{9}$$

Now, H_o will be symmetric if and only if $\sigma = R$, and then:

$$H_o = \begin{bmatrix} -R & R & 0 \\ R & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} \tag{10}$$

Note that the leading principal minors of H_o alternate in sign if and only if $R < 1$. That is;

$$|H_o(1)| = -R < 0, |H_o(2)| = R(1-R) > 0, |H_o(3)| = Rb(R-1) < 0 \text{ iff } R < 1 \tag{11}$$

Thus, by Sylvester Criterion Theorem (1), H_o is negative definite which yields that all eigenvalues of H_o will be negative. Consequently, $|arg(eig(H_o))| > \frac{\alpha_m \pi}{2}$, where $\alpha_m = max\{\alpha_i\}; 1 \le i \le n$, and so the equilibrium O is asymptotically stable.

Remark. Observe that proposition (1) can be proved also by using Schur Complement Theorem given in (2). To see this; we should note that H_o in (10) can be partitioned as:

$$H_o = \begin{bmatrix} M & N \\ N^T & L \end{bmatrix} \tag{12}$$

where $M = \begin{pmatrix} -R & R \\ R & -1 \end{pmatrix}$ and $N = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, with $L = \begin{pmatrix} -b \end{pmatrix}$ is, clearly, negative definite. Now, H_o is negative definite if and only if the Schur complement $M - NL^{-1}N^T = M$ is so. But, M is negative definite if and only if $R(1-R) < 0$, (i.e $R < 1$).

With the aim of showing the efficiency of the above result, the following numerical simulation for the FoLS (2) has been performed. Of course, the ADM has been employed to solve this system. By taking $\sigma = R = 0.4, b = 8/3$ and $\alpha = \mathbf{0.7}$, and by setting $\boldsymbol{x}(0) = [-15.8 \ \ -17.48 \ \ 35.64]^T$; one can observe, from Figure 1, that all states of system (2) are asymptotically decreasing towards zero.

Proposition 2. *Regarding to system* (2)*; the equilibria q_+ and q_- are asymptotically stable if $\sigma = R = 1$ and $b \neq 2$.*

Proof. At the equilibria q_+ and q_-, the Jacobian matrices of system (2) are, respectively [23]:

$$H_{q+} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b(R-1)} \\ \sqrt{b(R-1)} & \sqrt{b(R-1)} & -b \end{bmatrix} \tag{13}$$

and

$$H_{q-} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \sqrt{b(R-1)} \\ -\sqrt{b(R-1)} & -\sqrt{b(R-1)} & -b \end{bmatrix}. \tag{14}$$

Now, H_{q+} and H_{q-} will be symmetric if and only if $\sigma = R = 1$, and then:

$$H_{q+} = H_{q-} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} = H_q. \tag{15}$$

Consequently, the eigenvalues of H_q are: $\lambda_1 = 0$, $\lambda_2 = -2$ and $\lambda_3 = -b$, (i.e., H_q has one zero eigenvalue). Since all eigenvalues of H_q are distinct; then H_q is similar to the following Jordan canonical form:

$$J = diag(J_1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -b \end{bmatrix}. \tag{16}$$

As $b \neq 2$; then H_q has one zero eigenvalue corresponding to one Jordan block $J = diag(J1)$. Thus; q_+ and q_- are asymptotically stable by Theorem (4), and this complete the proof.

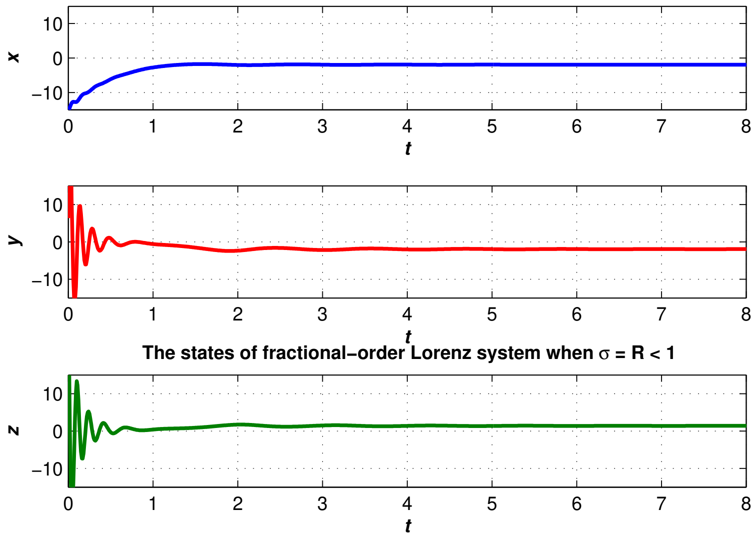

Fig. 1: The states of FoLS when $\sigma = R < 1$.

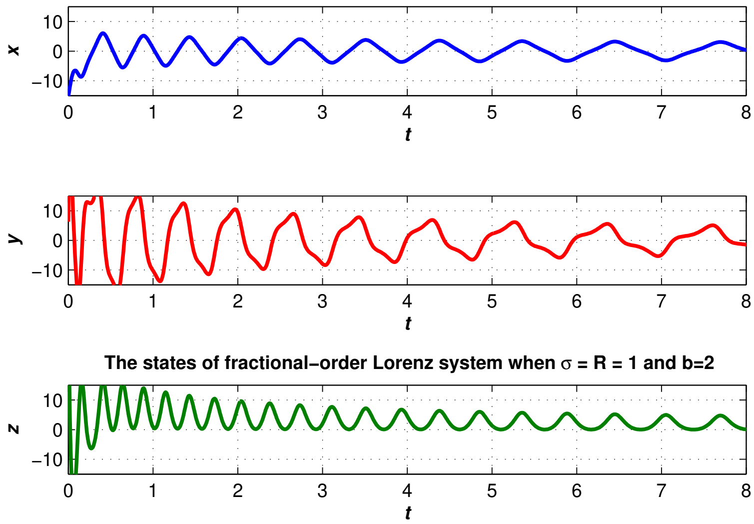

Fig. 2: The states of the FoLS when $\sigma = R = 1$ and $b = 2$.

For the purpose of confirming the effectiveness of the latest generated result, two numerical simulations for system (2) have been shown using ADM. These two simulations consider the following two cases:

- When $\sigma = R = 1$ and $b = 2$. This case has been shown in Figure 2, which demonstrates that these states are still unstable in the case of $\alpha = \mathbf{0.7}$ and $\boldsymbol{x}(0) = [-15.8 \;\; -17.48 \;\; 35.64]^T$.

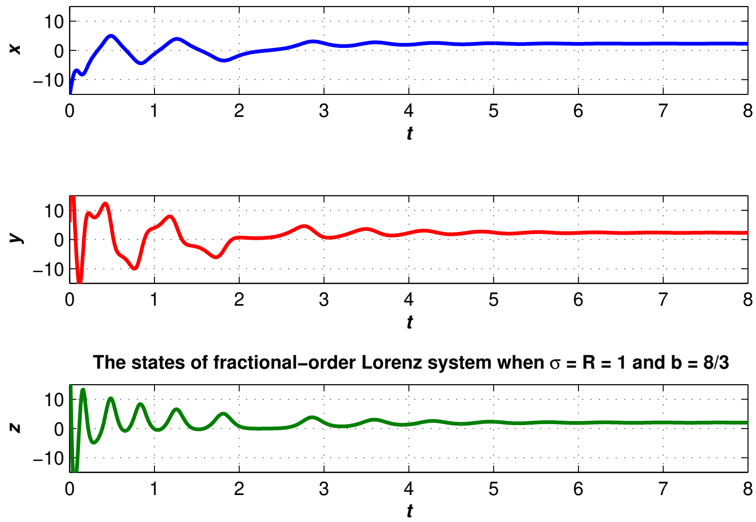

Fig. 3: The states of the FoLS when $\sigma = R = 1$ and $b = 8/3$.

- When $\sigma = R = 1$ and $b \neq 2$. This case has been shown in Figure 3, which demonstrates that the states of system (2) $(x(t);\ y(t)$ and $z(t))$ are all asymptotically decreasing towards zero in the case of $b = 8/3$, $\alpha = \mathbf{0.7}$ and $\boldsymbol{x}(0) = [-15.8\ \ -17.48\ \ 35.64]^T$.

4 Conclusion

In this work, two particular results associated with the equilibria's stability of the nonlinear fractional-order Lorenz system have been examined and confirmed graphically using the Adomain decomposition method. These results are deemed really helpful for discerning between various cases that can be used to explore the stability of Lorenz system of fractional-order.

References

[1] R. B. Albadarneh, M. Zerqat and I. M. Batiha, Numerical solution for linear and non-linear fractional differential equations, *Int. J. Pure App. Math.* **106**, 859–871 (2016).
[2] I. M. Batiha, R. El-Khazali and S. Momani, A powerful scheme for handling the fractional-order Lorenz system, *Proc. Int. Conf. Frac. Differen. App. (ICFDA)*, Amman, Jordan July (2018).
[3] R. Hilfer, *Applications of fractional calculus in physics*, World Scientific, New Jersey, 2000.
[4] R. B. Albadarneh, I. M. Batiha and M. Zurigat, Numerical solutions for linear fractional differential equations of order $1 < \alpha < 2$ using finite difference method (FFDM), *J. Math. Comp. Sci.* **1**, 103-111 (2016).
[5] I. Podlubny, *Fractional differential equations*, Academic Press, New York, 1999.
[6] J. G. Lu, Chaotic dynamics of the fractional-order Lü system and its synchronization, *Phys. Lett. A* **354**, 305–311 (2006).
[7] A. Ouannas, I. M. Batiha, A-A. Khennaoui, A. Zraiqat and A. A. Al-Nana, Novel control law for the fractional-order chaotic Duffing map, *2021 Int. Conf. Infor. Tech. (ICIT)*, Amman, Jordan July (2021).
[8] A. Ouannas, I. M. Batiha, A-A. Khennaoui and I.H. Jebril, On the 0-1 test for chaos applied to the generalized fractional-order Arnold map, *2021 International Conference on Information Technology (ICIT)*, Amman, Jordan July (2021).
[9] I. M. Batiha, R. B. Albadarneh, S. Momani and I. H. Jebril, Dynamics analysis of fractional-order Hopfield neural networks, *Int. J. Biomath.* **13**, (2020).
[10] I. Grigorenko and E. Grigorenko, Chaotic dynamics of the fractional Lorenz system, *Phys. Rev. Lett.* **91**, 34101–34104 (2003).

[11] H. Y. Jia, Z. Q. Chen and W. Xue, Analysis and circuit implementation for the fractional-order Lorenz system, *Acta Phys. Sin.* **62**, 373–452 (2013).
[12] K. H. Sun, X. Wang and C. Sprott, Bifurcations and chaos in fractional order simplified Lorenz system, *Int. J. Bifurc. Chaos* **20**, 1209–1219 (2010).
[13] G. Adomian, A review of the decomposition method and some recent results for nonlinear equations, *Math. Comp. Model.* **13**, 17–43 (1990).
[14] V. Daftardar-Gejji and H. Jafari, An iterative method for solving nonlinear functional equations, *J. Math. Anal. Appl.* **316**, 753–763 (2006).
[15] C. Li and Y. Wang, Numerical algorithm based on Adomian decomposition for fractional differential equations, *Comp. Math. App.* **57**, 1672–1681 (2009).
[16] S. He, K. Sun and H. Wang, Solution of the fractional-order chaotic system based on Adomian decomposition algorithm and its complexity analysis, *Acta Phys. Sin.* **63**, 1–8 (2014).
[17] S. He, K. Sun and H. Wang, Dynamics of the fractional-order Lorenz system based on Adomian decomposition method and its DSP implementation, *IEEE/CAA J. Autom. Sinica*, 1–6 (2016).
[18] H. Wang, K. Sun and S. He, Characteristic analysis and DSP realization of fractional-order simplified Lorenz system based on Adomian decomposition method, *Int. J. Bifur. Chaos* **25**, 1209–1219 (2015).
[19] H. Li, J. Cheng, H. B. Li and S. M. Zhong, Stability analysis of a fractional-order linear system described by the Caputo–Fabrizio derivative, *Math.* **7**, 200 (2019).
[20] D. Matignon, Stability results on fractional differential equations to control processing, *Proc. Comput. Eng. Sys. App. Multiconf.* **2**, 963–968 (1996).
[21] J. Sabatier, M. Moze and C. Farges, LMI stability conditions for fractional order systems, *Comput. Math. Appl.* **59**, 1594–1609 (2010).
[22] H. S. Ahn, Y. Q. Chen and I. Podlubny, Robust stability test of a class of linear time-invariant interval fractional-order system using Lypunov inequality, *Appl. Math. Comput.* **187**, 27–34 (2007).
[23] y. Yu, H-X Li, S. Wanga and J. Yu, Dynamic analysis of a fractional-order Lorenz chaotic system, *Chaos Solit. Fract.* **42**, 1181–1189 (2009).
[24] M. Abu Hammad, I. H. Jebril, I. M. Batiha and A. M. Dababneh, Fractional frobenius series solutions of confluent α-hypergeometric differential equation, *Prog. Frac. Differ. App.* **8**, 297–304 (2022).
[25] I. M. Batiha, J. Oudetallah, A. Ouannas, A. A. Al-Nana and I. H. Jebril, Tuning the fractional-order PID-controller for blood glucose level of diabetic patients, *Int. J. Adv. Soft Comput. App.* **13**, 1–10 (2021).
[26] R. Barrio and S. Serrano, A three-parametric study of the Lorenz model, *Physica D* **229**, 43–51 (2007).
[27] M. Merdan, Variational iteration and homotopy perturbation method for solving Lorenz system, *J. Balikesir Univ.* **18**, 3–14 (2016).
[28] A. A. M. Arafa, S. Z. Rida and H. M. Ali, Generalized Mittag-Leffler function method for solving Lorenz system, *Int. J. Innov. App. Stu.***3**, 105–111 (2013).
[29] E. Hairer and G. Wanner, *Solving ordinary differential equations II: stiff and differential-algebraic problems*, Springer-Verlag, Berlin, 1991.
[30] R. El-Khazali, W. Ahmad and Y. Al-Assaf, Sliding mode control of generalized fractional chaotic systems, *Int. J. Bifurc. Chaos* **16**, 1–13 (2006).
[31] G. Adomian, *Nonlinear stochastic systems theory and applications to physics*, Kluwer Academic Publishers, Dordrecht, Netherlands, 2006.
[32] G. Adomian and R. Rach, Generalization of Adomian polynomials of functions of several variables, *Comp. Math. Applic.* **24**, 11–24 (1992).
[33] I. M. Batiha, R. El-Khazali, A. Alsaedi and S. Momani, The general solution of singular fractional-order linear time-invariant continuous systems with regular pencils, *Entropy* **20**, 400 (2018).
[34] G. Giorgi, Various proofs of the sylvester criterion for quadratic forms, *J. Math. Res.* **9**, 55–66 (2017).
[35] A. Khare, *Matrix analysis and positivity*, Indian Institute of Science, Bangalore, India, 2018.
[36] A. El-Sayed, F. Gaafar and H. Hashem, On the maximal and minimal solutions of arbitrary orders nonlinear functional integer and differential equations, *Math. Sci. Res. J.* **8**, 336–348 (2004).
[37] J. Wang and Y. Zhang, Designing synchronization schemes for chaotic fractional order unified systems, *Chaos Soliton. Fract.* **30**, 1265–1272 (2006).
[38] C. P. Li and F. R. Zhang, A survey on the stability of fractional differential equations, *Eur. Phys. J. Spec. Top.* **193**, 27–47 (2011).
[39] A. V. Holden, *Chaos*, Princeton University Press, Princeton, New Jersey, 2014.