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Solution of Generalized Abel's Integral Equations by Homotopy Perturbation Method with Adaptation in Laplace Transformation

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Abstract: In this paper we present a new and simple algorithm to solve the Abel's second kind integral equation using the concept of homotopy perturbation method with a modern adaptation in Laplace transformation. By this method, analytical solution and good approximate solutions can be obtained with only a few iterations. Several numerical examples are presented to illustrate the method, and the results show the proposed method is very effective.

Keywords: Generalized Abel's integral equations, Homotopy perturbation method, Laplace transformations, Analytical and approximate solutions.

1 Introduction

In recent years, the homotopy perturbation method (HPM) has been studied and successfully applied by many engineers, scientists and researchers, and used to solve the integral equations and differential equations [1, 2, 3, 4, 5, 6, 7, 8, 9]. This HPM was introduced first by Dr. Ji Huan He in 1998 [10, 11]. HPM is a coupling of the homotopy method, a basic concept of topology, and the classical perturbation technique. This coupling will provide with a suitable way to obtain approximate or analytic solution for different problems arising in various scientific fields. Some advantages of HPM are: good approximate solutions can be obtained with only a few iterations, and it yields a very fast convergence of the solution series in many cases. This method has been used to solve many problems [2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20], specially in integral equations [1, 4, 8, 13, 14]. There are several algorithms created to solve differential equation with maple implementation, see for example, [21, 22, 23, 24, 25, 26, 27, 28, 29]. In this paper, we propose a new method for finding the analytic solution or an approximate solution of generalized Abel's integral equations of the second kind using the concept of HPM via Laplace transformation, and the approach to analytic solution of this method is very simple and computationally winsome.

Frequently, we come across Abel's integral equations in mathematical physics, biology, electronics and mechanics, many applications of chemistry such as crystal growth, heat conduction, and electro-chemistry etc. (see [30]), and the generalized Abel's integral equation often looks in two forms namely first kind and second kind. The first one is as follows

$$f(x) = \int_0^x \frac{u(\xi)}{(x-\xi)^\alpha}\, d\xi, \tag{1}$$

and the second kind is as follows

$$u(x) = f(x) + \int_0^x \frac{u(\xi)}{(x-\xi)^\alpha}\, d\xi, \tag{2}$$

where $f(x)$ is continuous function, $0 < \alpha < 1$ and $0 \le x, \xi \le \rho$ and ρ is constant.

The main aim of this paper is to find the analytic solution or approximate solutions of the generalized Abel's integral equations of the second kind of the form (2) using the concept of HPM via Laplace transformation. The main advantage of this proposed method is the combination of two powerful methods, namely HPM and Laplace transformation, for obtaining

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speedy convergent series for generalized Abel's integral equations of the second kind. The paper is organized as follows: in Section 2, we described the proposed method; and in Section 3, we discussed several examples to illustrate the exactness and stability of the proposed method.

2 Description of the method

In this section, we illustrate the basic idea of the proposed method for solution of the generalized Abel's integral equation of the second kind via Laplace transform as follows.

Recall the equation (2),

$$u(x) = f(x) + \int_0^x \frac{u(\xi)}{(x-\xi)^\alpha}\, d\xi. \tag{3}$$

Take the Laplace transform on both sides in equation (3) using convolution property of the Laplace transform, we have

$$L[u(x)] = L[f(x)] + \frac{\Gamma(1-\alpha)}{s^{1-\alpha}} L[u(x)]. \tag{4}$$

Now operate the inverse Laplace transform in equation (4) on both sides, we have

$$u(x) = f(x) + L^{-1}\left[\frac{\Gamma(1-\alpha)}{s^{1-\alpha}} L[u(x)]\right]. \tag{5}$$

According to HPM (see, for example, [2,7,9,10,11,20]) , we can write the solution of the equation (3) as a power series in p:

$$v(x) = \sum_{n=0}^{\infty} p^n v_n(x). \tag{6}$$

To obtain an approximate solution of equation (6), put $p = 1$:

$$u(x) = \lim_{p\to 1} v(x) = \sum_{n=0}^{\infty} v_n(x). \tag{7}$$

Here the functions $v_n(x)$, for $n = 0, 1, \ldots$, are determined by using the convex homotopy iterative scheme (see [1,2, 3,4,5,7,8,9,10,11,12,13,14,20,31]) as follows:

$$\sum_{n=0}^{\infty} p^n v_n(x) = f(x) + p\left(L^{-1}\left[\frac{\Gamma(1-\alpha)}{s^{1-\alpha}} L\left[\sum_{n=0}^{\infty} p^n v_n(x)\right]\right]\right). \tag{8}$$

The functions $v_n(x)$ are obtained by equating the corresponding coefficients of powers of p on both sides, for $n = 1, 2, 3, \ldots$,

$$\begin{aligned} p^0 &: v_0(x) = f(x) \\ p^n &: v_n(x) = L^{-1}\left[\frac{\Gamma(1-\alpha)}{s^{1-\alpha}} L[v_{n-1}(x)]\right]. \end{aligned} \tag{9}$$

Here we are mainly concerned about the perturbation equation, that can be constructed easily in several ways by homotopy, and an initial guess $v_0(x)$, that can also be selected freely by HPM. Once if we choose these parts, then the homotopy equation is determined easily, because the other part is the original equation and hence less changes are required.

In the following section, we shall provide several example by the homotopy perturbation technique and comparisons are made to show the efficiency of the proposed method.

3 Numerical examples

In Example 1, we find the analytical solution of the generalized Abel's integral equation of second kind going through the procedure as described in Section 2, and other examples are presented to show the efficiency of the proposed method. All the results in examples are calculated using the computer algebraic system Maple 13.

Example 1[32] *Consider the generalized Abel's integral equation*

$$u(x) = x^2 + \frac{27}{40}x^{\frac{8}{3}} - \int_0^x \frac{u(\xi)}{(x-\xi)^{\frac{1}{3}}}\, d\xi. \tag{10}$$

The exact solution of (10) *is* $u(x) = x^2$.

Applying Laplace transform to equation (10) *on both sides, we get*

$$L[u(x)] = L\left[x^2 + \frac{27}{40}x^{\frac{8}{3}}\right] - \frac{\Gamma(\frac{2}{3})}{s^{\frac{2}{3}}} L[u(x)]. \tag{11}$$

Operating the inverse Laplace transform on both sides in equation (11)*, we get*

$$u(x) = x^2 + \frac{27}{40}x^{\frac{8}{3}} - L^{-1}\left[\frac{\Gamma(\frac{2}{3})}{s^{\frac{2}{3}}} L[u(x)]\right]. \tag{12}$$

By HPM, we have the following convex homotopy

$$\sum_{n=0}^{\infty} p^n v_n(x) = x^2 + \frac{27}{40}x^{\frac{8}{3}} - p\left(L^{-1}\left[\frac{\Gamma(\frac{2}{3})}{s^{\frac{2}{3}}} L\left[\sum_{n=0}^{\infty} p^n v_n(x)\right]\right]\right). \tag{13}$$

Now equating the corresponding coefficients of powers of p *in equation* (13)*, we have the following iterates* $v_n(x)$*, for* $n = 0, 1, 2, \ldots$,

$$p^0 : v_0(x) = x^2 + \frac{27}{40}x^{\frac{8}{3}},$$
$$p^1 : v_1(x) = -L^{-1}\left[\frac{\Gamma(\frac{2}{3})}{s^{\frac{2}{3}}}L[v_0(x)]\right],$$
$$= -\frac{27}{40}x^{\frac{8}{3}} - \frac{81\sqrt{3}}{280}\frac{\Gamma(\frac{2}{3})^3x^{\frac{10}{3}}}{\pi},$$
$$p^2 : v_2(x) = \frac{1}{840}\frac{\Gamma(\frac{2}{3})^3(243\sqrt{3}x^{\frac{10}{3}} + 70x^4\pi)}{\pi},$$
$$p^3 : v_3(x) = -\frac{1}{18480}\left(1540x^4 + 729x^{\frac{14}{3}}\right)\Gamma\left(\frac{2}{3}\right)^3,$$
$$p^4 : v_4(x) = \frac{243}{6160}\Gamma\left(\frac{2}{3}\right)^3 x^{\frac{14}{3}} + \frac{729}{58240}\frac{\Gamma(\frac{2}{3})^6x^{\frac{16}{3}}\sqrt{3}}{\pi},$$
$$p^5 : v_5(x) = -\frac{1}{524160}\frac{\Gamma(\frac{2}{3})^6(6561\sqrt{3}x^{\frac{16}{3}} + 1456x^6\pi)}{\pi},$$

and so on. In this manner, one can obtain the rest of components of the homotopy perturbation iterations. Now, we can approximate the analytical solution $u(x)$ by the shortened series as

$$u(x) = \sum_{i=0}^{n} v_i(x) \to x^2 \text{ as } n \to \infty.$$

We show the comparison between the exact solution and the approximate solution obtained by the proposed method graphically in Figure 1. One can observe that the solution obtained by the present method almost identical to the exact solution. Better approximation of the results can be obtained by perform more number of iterations. The numerical results are given in Table 1 with absolute errors due to the approximation, which shows the approximate solution is strongly agreed with the exact solution. In Figure 2, we show the absolute error $E_{20} = |u_{exact}(x) - u_{appox}(x)|$ between approximate solution and exact solution due to the approximation at level $n = 20$ which is very small.

Example 2[33] *Consider a second kind Abel's integral equation as follow*

$$u(x) = x + \frac{4}{3}x^{\frac{3}{2}} - \int_0^x \frac{u(\xi)}{(x-\xi)^{\frac{1}{2}}}\,d\xi, \tag{14}$$

with exact solution x.

Now apply the homotopy perturbation method to (14), we have the following convex homotopy

$$\sum_{n=0}^{\infty} p^n v_n(x) = x + \frac{4}{3}x^{\frac{3}{2}} - p\left(L^{-1}\left[\sqrt{\frac{\pi}{s}}L\left[\sum_{n=0}^{\infty} p^n v_n(x)\right]\right]\right). \tag{15}$$

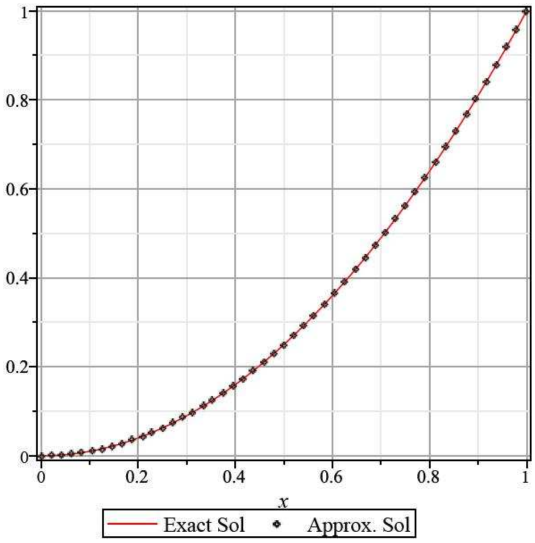

Fig. 1: Graphical comparison for example 1.

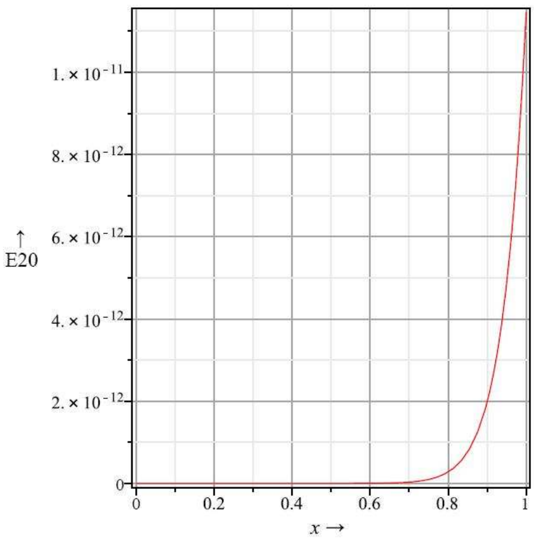

Fig. 2: The graph of absolute error E_{20} for example 1.

By equating the corresponding coefficients of powers of p in equation (15), we get the following iterates $v_n(x)$ for $n = 0, 1, 2, \ldots$:

Table 1: Numerical results for example 1.

x	Exact value	Approx. value	Absolute errors
0.10	0.0100	0.0100	0
0.25	0.0625	0.0625	0
0.47	0.2209	0.2209	0
0.58	0.3364	0.3364	0
0.92	0.8464	0.8464	0
1.20	1.4400	1.4400	0
1.28	1.6384	1.638399999	1.0×10^{-9}
1.38	1.9044	1.904399998	2.0×10^{-9}
1.44	2.0736	2.073599995	5.0×10^{-9}
1.50	2.2500	2.249999990	1.0×10^{-8}

$$
\begin{aligned}
p^0 &: v_0(x) = x + \frac{4}{3}x^{\frac{3}{2}},\\
p^1 &: v_1(x) = -L^{-1}\left[\sqrt{\frac{\pi}{s}}L[v_0]\right] = -\frac{4}{3}x^{\frac{3}{2}} - \frac{1}{2}\pi x^2,\\
p^2 &: v_2(x) = -L^{-1}\left[\sqrt{\frac{\pi}{s}}L[v_1]\right] = \frac{\pi}{30}(15x^2 + 16x^{\frac{5}{2}}),\\
p^3 &: v_3(x) = -L^{-1}\left[\sqrt{\frac{\pi}{s}}L[v_2]\right] = -\frac{8}{15}x^{\frac{5}{2}}\pi - \frac{1}{6}\pi^2 x^3,\\
p^4 &: v_4(x) = \frac{\pi^2}{210}(35x^3 + 31x^{\frac{7}{2}}),\\
p^5 &: v_5(x) = -\frac{16}{105}\pi^2 x^{\frac{7}{2}} - \frac{1}{24}\pi^3 x^4,\\
p^6 &: v_6(x) = \frac{\pi^3}{7560}(315x^4 + 256x^{\frac{9}{2}}),
\end{aligned}
$$

Now, an approximate solution of the analytical solution $u(x)$ is obtained by the shortened series, i.e.,

$$
u(x) = \sum_{i=0}^{n} v_i(x) \to x \text{ as } n \to \infty. \tag{16}
$$

We show the graphical comparison between the exact solution and approximate solutions in Figure 3, and the graph of absolute error at level $n = 30$ shown in Figure 4. The numerical results are given in Table 2 with absolute errors due to the approximation, which show that the approximate solution is strongly agreed with the exact solution.

Table 2: Numerical results for example 2.

x	Exact value	Approx. value	Absolute errors
0.20	0.20	0.20	0
0.40	0.40	0.40	0
0.60	0.60	0.60	0
0.80	0.80	0.7999999943	5.7×10^{-9}
1.00	1.00	0.9999997469	2.531×10^{-7}

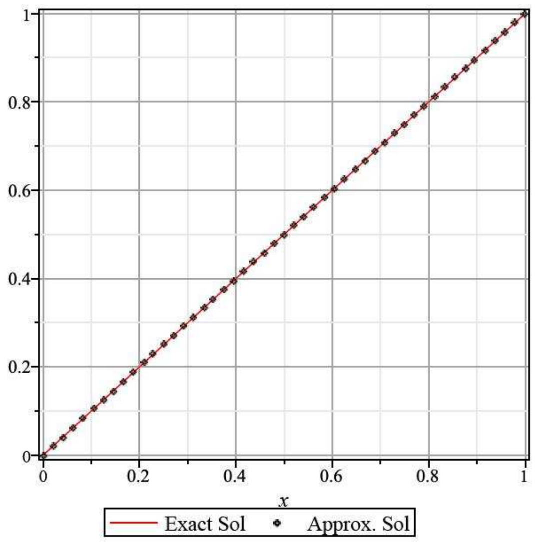

Fig. 3: Graphical comparison for example 2.

Example 3 *[34] We consider the weakly singular Volterra integral equation of second kind*

$$
u(x) - \int_0^x \frac{1}{\sqrt{x-\xi}}u(\xi)\,d\xi = x^7\left(1 - \frac{4096}{6435}x^{\frac{1}{2}}\right), \tag{17}
$$

with exact solution is $u(x) = x^7$. Using the HPM to equation (17), we have the following approximate

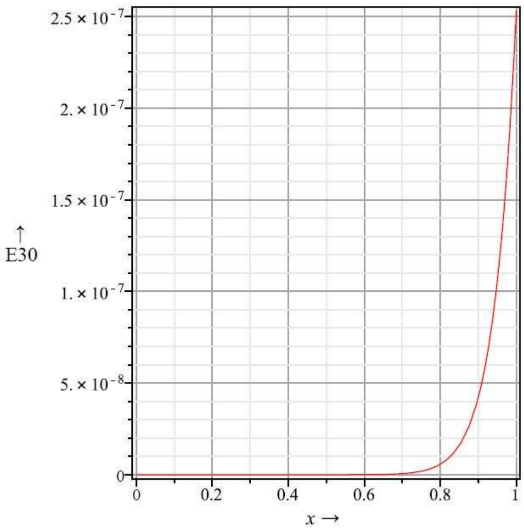

Fig. 4: The graph of absolute error E_{20} for example 2.

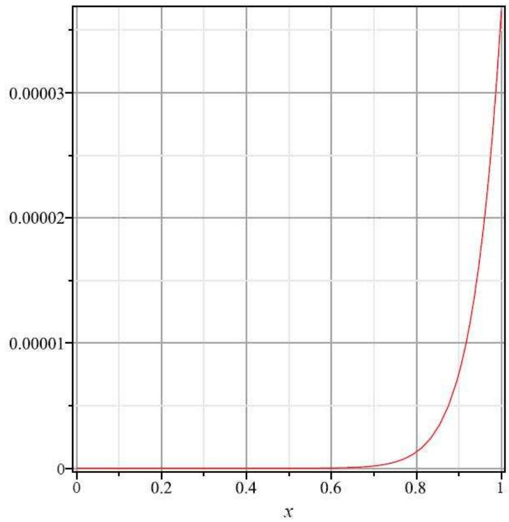

Fig. 6: Absolute error at level $n = 15$ for example 3.

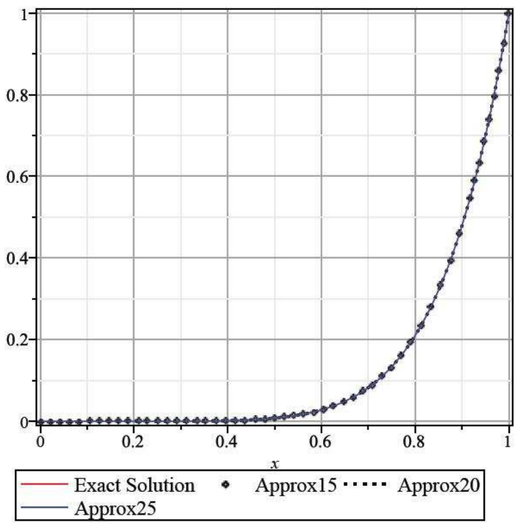

Fig. 5: Graphical comparison for example 3.

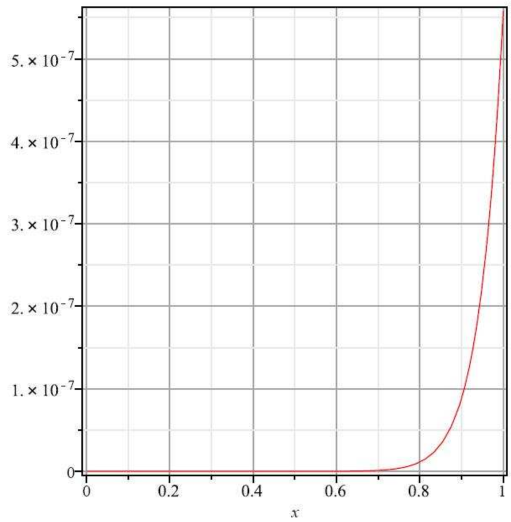

Fig. 7: Absolute error at level $n = 20$ for example 3.

solutions:

$$\begin{aligned}
For\ \ n = 15,\ \ u(x) &= \sum_{i=0}^{n} v_i(x) \\
&= x^7 - \frac{\pi^8 x^{15}}{259459200}, \\
For\ \ n = 20,\ \ u(x) &= \sum_{i=0}^{n} v_i(x) \\
&= x^7 - \frac{\pi^{10} x^{\frac{35}{2}}}{703628874529205625}, \\
For\ \ n = 25,\ \ u(x) &= \sum_{i=0}^{n} v_i(x) \\
&= x^7 - \frac{\pi^{13} x^{\frac{41}{2}}}{41628795103771392391875}, \\
As\ n \to \infty\ \ u(x) &= \sum_{i=0}^{n} v_i(x) \to x^7.
\end{aligned}$$

Using HPM, we have the numerical results in Table 3. From Table 3, the error decreases when the integer n increases until $n = 25$. The Figure 5 gives the graphical comparison between the approximate solution and the exact solution at different levels, $n = 15, 20, 25$, and the corresponding absolute errors are also shown in Figure 6, 7, 8 for $n = 15, 20, 25$ respectively.

Example 4*Consider a second kind Abel's integral equation [33]*

$$u(x) = x^2 + \frac{16}{15} x^{\frac{5}{2}} - \int_0^x \frac{u(\xi)}{\sqrt{x - \xi}}\, d\xi, \tag{18}$$

with exact solution $\frac{1}{x+1}$.

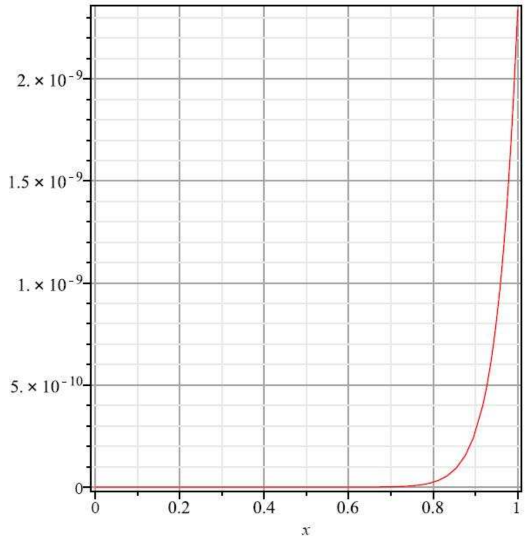

Fig. 8: Absolute error at level $n = 25$ for example 3.

Now apply the homotopy perturbation method to (18), we have the following convex homotopy

$$\sum_{n=0}^{\infty} p^n v_n(x) = x^2 + \frac{16}{15}x^{\frac{5}{2}} - p\left(L^{-1}\left[\sqrt{\frac{\pi}{s}}L\left[\sum_{n=0}^{\infty} p^n v_n(x)\right]\right]\right). \tag{19}$$

By equating the corresponding coefficients of powers of p in equation (19), we get the following iterates $v_n(x)$ for $n = 0, 1, 2, \ldots$:

$$\begin{aligned}
p^0 &: v_0(x) = x^2 + \frac{16}{15}x^{\frac{5}{2}},\\
p^1 &: v_1(x) = -\frac{16}{15}x^{\frac{5}{2}} - \frac{1}{3}\pi x^3,\\
p^2 &: v_2(x) = \frac{1}{105}\pi(35x^3 + 32x^{\frac{7}{2}}),\\
p^3 &: v_3(x) = -\frac{32}{105}\pi x^{\frac{7}{2}} - \frac{1}{12}\pi^2 x^4,\\
p^4 &: v_4(x) = \frac{1}{3780}\pi^2(315x^4 + 256x^{\frac{9}{2}}),\\
p^5 &: v_5(x) = -\frac{64}{945}\pi^2 x^{\frac{9}{2}} - \frac{1}{60}\pi^3 x^5,\\
p^6 &: v_6(x) = \frac{1}{41580}\pi^3(690x^5 + 512x^{\frac{11}{2}}),
\end{aligned}$$

Now, the approximate solution of $u(x)$ for $n = 30$ is obtained by the shortened series, as follows

$$u(x) = \sum_{i=0}^{30} v_i(x) = x^2 - \frac{\pi^6 x^{18}}{3201186852864000} \approx x^2.$$

Therefore, we have

$$u(x) = \sum_{i=0}^{n} v_i(x) \to x^2 \text{ as } n \to \infty.$$

Table 3: Absolute errors at different levels for example 3.

x	Absolute errors for $n = 15$	Absolute errors for $n = 20$	Absolute errors for $n = 25$
0.10	3.65×10^{-20}	1.76×10^{-24}	7.40×10^{-30}
0.20	1.19×10^{-15}	3.27×10^{-19}	1.09×10^{-23}
0.30	5.24×10^{-13}	3.94×10^{-16}	1.46×10^{-20}
0.40	3.92×10^{-11}	6.06×10^{-14}	1.62×10^{-17}
0.50	1.11×10^{-9}	3.01×10^{-12}	1.57×10^{-15}
0.60	1.71×10^{-8}	7.31×10^{-11}	6.62×10^{-14}
0.70	1.73×10^{-7}	1.08×10^{-9}	1.56×10^{-12}
0.80	1.28×10^{-6}	1.12×10^{-8}	2.41×10^{-11}
0.90	7.52×10^{-6}	8.83×10^{-8}	2.69×10^{-10}
1.00	3.65×10^{-5}	5.58×10^{-7}	2.34×10^{-9}

4 Conclusion

In this paper, we presented a new homotopy perturbation method via Laplace transformations to obtain approximate and accurate solution of the generalized Abel's integral equations of the second kind. We discussed several numerical examples; and comparisons of the results, obtained by the proposed method, reveals that the proposed method is very effective and convenient.

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