Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.18576/msl/070101

Modified Active Controller for the Complete Synchronization of Hyperchaotic Systems

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Received: 10 Feb. 2016, Revised: 4 Sep. 2017, Accepted: 6 Sep. 2017
Published online: 1 Jan. 2018

Abstract: The synchronization speed, quality, robustness and cost of the controller design are the four important factors in chaotic synchronization phenomena. This paper presents improved results when two identical hyperchaotic systems are synchronized using a modified active control approach. The complete synchronization objective is achieved by means of only two stabilizing controllers with a single feedback linear controller gain that reduce the controller and synchronization cost significantly, and guarantee the globally exponential stability of the closed-loop in a short transient time. The advantages of the proposed modified active control approach are revealed by analytically and numerically comparing the amplitude of the synchronized error signals, synchronization transient speed and cost of the designed controller with the past published works in the literature concerned. The robust synchronization of two Lorenz-Stenflo hyperchaotic systems is taken as an example to verify the theoretical findings.

Keywords: Chaos Synchronization, Routh-Hurwitz criterion, Active control, Hyperchaotic system

1 Introduction

The concept of synchronization between two nearly identical chaotic systems under different initial conditions was proposed by Pecora and Carroll [1]. Since, then, the chaos synchronization phenomenon has been extensively developed in the last two decades. Many possible applications of the chaos synchronization have been discussed theoretically as well as practically [2-4], among others. To this end, certain effective control methods and techniques have been developed to achieve chaotic synchronization. For example, the linear state feedback control [5], sliding mode control [6], active control [7], adaptive control [8], projective synchronization [9], lag synchronization [10] and nonlinear control techniques [11] etc. Among the reported techniques, the active control algorithm has been widely accepted as one of the effective control strategy in synchronizing chaotic systems. In recent decades, the active control method has been applied successfully for synchronization of various physical systems, such as the electric circuit that exhibits chaos [12], nonlinear gyros [13], RCL-shunted Josephson junction [14], and recently the problem of Enceladus [15], etc. Using the active control algorithm and the Routh-Hurwitz criterion, [16] studied the complete synchronization (CS) of two identical Lorenz-Stenflo hyperchaotic (LSH) systems. Using the active control method based on the Lyapunov direct method, the CS of two identical LSH systems is further investigated [17].

However, in the finding of these results [16-17], there are three main limitations. Firstly, the CS problem on hyperchaotic systems with control inputs numerically equal to the number of error states is addressed. This imposes an extra burden on the controller design and somehow difficult to implement in practical applications. Secondly, the closed-loop stability [16-17] has been achieved by simply assigning the eigenvalues of the coefficient matrix to the left half of the complex plane, which obeys the Routh-Hurwitz criterion. In selecting the high controller gain(s) for fast synchronization transient speed may create signal saturation and the coupled systems may loss the synchronization stability. Thirdly, the CS objective is achieved without considering the effect of unknown external disturbance due to the environmental changes. These results [16-17] would have been more interesting if it had come up with less control effort and fast synchronization speed with the presence of unknown time varying external disturbance. Likewise, the

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computation of suitable linear controller gain(s) is (are) still of great interest, both from a theoretical as well a practical viewpoint.

To address the aforesaid issues, a modified active control approach is focused. Using the active control strategy and Routh-Hurwitz criterion [18], a corresponding frame work for the computation of a suitable linear controller gain is set up to achieve the globally exponential synchronization between two identical LSH [16] systems. The feedback controllers are designed in a way that it minimize the number of controllers, synchronization cost and ensure the globally exponential stability of the closed-loop. In comparison with the known results [16-17], the presented study does not only improved the synchronization speed and quality but also reduced the number of controllers significantly. Effects of the feedback controllers, synchronization transient speed and quality with the presence of unknown time varying external disturbance are the key features of this paper.

The rest of the paper is organized as follows. In Section 2, problem statement and a frame work for the active controller design are given. In Section 3, description of the LSH system is given and solved the CS problem of two identical LSH systems. Finally, the concluding remarks are given in Section 4.

2 A theory of the active control strategy

2.1 Problem statement

Consider a general class of hyperchaotic (chaotic) system is described by:

$$\dot{X}(t) = AX(t) + F(X(t)), \tag{1}$$

where $X(t) = [x_1(t), \ldots, x_n(t)]^T \in R^n$, represents the state vectors, $A_{(n\times n)}$ is the constant matrix that contains the parameter vector and F is the nonlinear continuous function of the system (1).

In order to discuss the CS behavior of two identical hyperchaotic (chaotic) systems (1), let us consider a master-slave system synchronization for the two coupled hyperchaotic (chaotic) systems (1) that is described by:

$$\begin{cases} Master\ system : \dot{X}(t) = A_1X(t) + F_1(X(t)) \\ Slave\ system : \dot{Y}(t) = A_2Y(t) + F_2(Y(t)) + u(t), \end{cases} \tag{2}$$

where $X(t) = [x_1(t), \ldots, x_n(t)]^T \in R^n$ and $Y(t) = [y_1(t), \ldots, y_n(t)]^T \in R^n$ are the states vectors, $A_{1(n\times n)}$ and $A_{2(n\times n)}$ are the constant systems matrices, which contain the parameter vectors, and F_1 and F_2 are the nonlinear continuous functions in the master and slave systems (2), alternatively, where $u(t) \in R^n$ represents the control input, which will be determined later.

Mathematically, the synchronization error dynamics can be defined as the difference between the master and slave systems (2), given as follows:

$$e(t) = Y(t) - X(t), e(t) \in R^n.$$

Thus, from (2):

$$\begin{aligned} \dot{e}(t) &= A_2Y(t) - A_1X(t) + F_2(Y(t)) - F_1(X(t)) + u(t) \\ &= A_2(Y(t) - X(t)) + (A_2 - A_1)X(t) + F_2(Y(t)) - \\ &\quad F_1(X(t)) + u(t) \\ &= A_2e(t) + A_3X(t) + H(F_1(X(t)),\ F_2(Y(t))) + u(t) \end{aligned}, \tag{3}$$

where $H(F_1(X(t)),\ F_2(Y(t))) = F_2(Y(t)) - F_1(X(t))$ and $A_3 = A_1 - A_2$.

The CS objective is accomplished in the sense that:

$$\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|Y(t) - X(t)\| = 0,\ for\ e(0) \in R^n.$$

Remark 2.1. $\|\ .\ \|$ denotes the Euclidian norm.

Theorem 2.1 (Sylvesters' theorem). The necessary and sufficient condition for a matrix $A \in R^{n\times n}$ being a positive definite matrix (PDM), is that all of its principle minors are strictly positive.

Remark 2.2. The proof of Theorem 2.1 can be found in [19], and the details are omitted here.

Theorem 2.2. The CS scheme (2) is established, if the control input $u(t) \in R^{n\times 1}$, is synthesized by using the following active feedback controller:

$$u(t) = -H(F_1(X(t)),\ F_2(Y(t))) - A_3X(t) + \eta(t), \tag{4}$$

where $\eta(t) \in R^{n\times 1}$ is the sub-controller matrix which is a function of $e(t)$, and is defined as follows:

$$\eta(t) = -K[e(t)]^T, \tag{5}$$

where $K = diag[k,\ i = 1\ldots n]$, is a feedback linear controller gain matrix.

Proof. The first part of the feedback controller (4) eliminates the nonlinear and un-common terms from (3) and the second part $\eta(t) \in R^{n\times 1}$ acts as an external impute to stabilize the error system (3) at the origin. Using Eqs. (3), (4) and (5), the closed-loop is given by the following:

$$\begin{aligned} e^{\cdot}(t) &= A_3e(t) + \eta(t), \\ e^{\cdot}(t) &= -Ae(t), \end{aligned} \tag{6}$$

where $A = (K - A_3) \in R^{n\times n}$.

At this stage, the problem is reduced to show that if the linear controller gain matrix $K = diag[k_i,\ i = 1,\ldots,n]$, is properly constructed such that the coefficients matrix $A_{(n\times n)}$ in (6), satisfies the Routh-Hurwitz criterion, then, by the Lyapunov stability theory [19], the closed-loop system (6) is globally exponentially stable and hence, the CS scheme (2) is globally exponentially established.

Remark 2.3. According to the published works [12-17], many possible selections for the construction of a feedback controller gain matrix $K = diag[k_i, i = 1,\dots,n]$, are available such that the matrix $A_{(n\times n)}$, must have all of its eigenvalues with positive real parts. But with this hypothesis, the message signal can be easily extracted from the communications channel during the transmission and the same signal could be reproduced with any possible choice of the controller gain matrix $K = diag[k_i,\ i = 1,\dots,n]$. This may lead to a security issue. Therefore, it is necessary and typically significant from a theoretical as well a practical point of view to compute a suitable controller gain matrix that secure the transmission message and guarantees the globally exponential stability of the closed-loop system, which will be presented in the subsection 3.2.

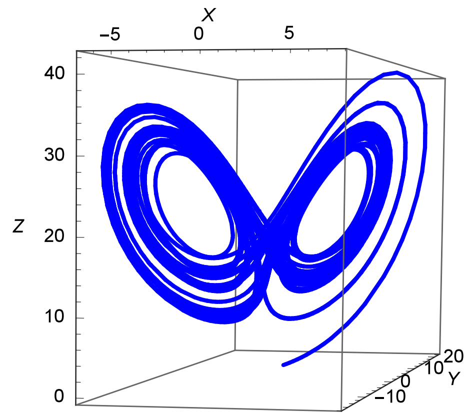

Fig. 1: 3D view of the LSH hyperchaotic system in the (x, y, z) space.

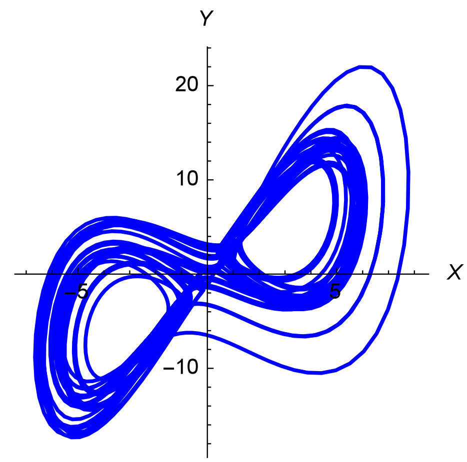

Fig. 2: 2D view of the LSH hyperchaotic system in the (x, y) plane.

3 Numerical example

In this section, we apply the proposed modified active control approach developed in Section 2, to achieve the CS behavior between two nearly identical Lorenz-Stenflo hyperchaotic [16] systems.

3.1 System description

A system of differential equations that describes the LSH system [16] is given as follows:

$$\begin{cases} \dot{x}(t) = a(y(t) - x(t)) + cw(t) \\ \dot{y}(t) = dx(t) - y(t) - x(t)z(t) \\ \dot{z}(t) = -bz(t) + x(t)y(t) \\ \dot{w}(t) = -x(t) - aw(t) \end{cases} \tag{7}$$

where $[x(t),\ y(t),\ z(t),\ w(t)]^T \in R^4$ are the state variables and $a > 0,\ b > 0,\ c > 0,\ and\ d > 0$ are the controlled parameters of the LSH system (7). With the parameters values $a = 1,\ b = 0.7,\ c = 1.5$, and $d = 26$, the LSH system exhibits a hyperchaos as shown in Figs. 1-2.

3.2 Problem statement

To study the CS of the LSH system, let us consider two nearly identical LSH systems, where the master LSH system is denoted by the subscript 1 and the slave LSH is represented by the subscript 2. Although, the initial conditions of the two LSH systems are different. Thus, the CS for the two coupled identical LSH systems is described as follows:

$$\begin{cases} (Master\ system) \\ \dot{x}_1(t) = a(y_1(t) - x_1(t)) + cw_1(t) \\ \dot{y}_1(t) = dx_1(t) - y_1(t) - x_1(t)z_1(t) \\ \dot{z}_1(t) = -bz_1(t) + x_1(t)y_1(t) \\ \dot{w}_1(t) = -x_1(t) - aw_1(t) \\ (Slave\ system) \\ \dot{x}_2(t) = a(y_2(t) - x_2(t)) + cw_2(t) + u_1(t) \\ \dot{y}_2(t) = dx_2(t) - y_2(t) - x_2(t)z_2(t) + u_2(t) \\ \dot{z}_2(t) = -bz_2(t) + x_2(t)y_2(t) + u_3(t) \\ \dot{w}_2(t) = -x_2(t) - aw_2(t) + u_4(t) \end{cases} \tag{8}$$

where $[x_1(t),\ y_1(t),\ z_1(t),\ w_1(t)]^T \in R^4$ and $[x_2(t),\ y_2(t),\ z_2(t),\ w_2(t)]^T \in R^4$ are the state variables and a, b, c and d are the controlled parameters of the master and slave systems (8), respectively and $u(t) = [0,\ u_2(t),\ u_3(t),\ 0]^T \in R^4$ is the control input. The error dynamics for the CS scheme (8) can be described as follows:

$$\begin{cases} \dot{e}_1(t) = a(e_2(t) - e_1(t)) + ce_4(t) + u_1(t) \\ \dot{e}_2(t) = de_1(t) - e_2(t) - x_2(t)z_2(t) + \\ \qquad x_1(t)z_1(t) + u_2(t) \\ \dot{e}_3(t) = -be_3(t) + x_2(t)y_2(t) - x_1(t)y_1(t) + u_3(t) \\ \dot{e}_4(t) = -e_1(t) - ae_4(t) + u_4(t) \end{cases} \tag{9}$$

Theorem 3.1. The CS (8) will achieve the globally exponential synchronization, if the following active controllers are used:

$$\begin{cases} u_2(t) = x_2(t)z_2(t) - x_1(t)z_1(t) + \eta_2(t) \\ u_3(t) = -x_2(t)y_2(t) - x_1(t)y_1(t) + \eta_3(t) \\ u_1(t) = u_4(t) = 0 \end{cases} \tag{10}$$

where the sub-controller $\eta_i(t)$, is defined as follows: $\eta_i(t) = -K[e_i(t)]^T$ and $K = diag[k_i]$, $for\ i = 2,3$.

Proof. Using Eqs. (9) and (10), the closed-loop system is given by:

$$\begin{cases} \dot{e}_1(t) = a(e_2(t) - e_1(t)) + ce_4(t) \\ \dot{e}_2(t) = de_1(t) - e_2(t) - k_2e_2(t) \\ \dot{e}_3(t) = -be_3(t) + k_3e_3(t) \\ \dot{e}_4(t) = -e_1(t) - ae_4(t) \end{cases} \tag{11}$$

The CS (8) is accomplished in a sense that: $\lim_{t\to\infty} ||e_i(t)|| = 0$, $(i = 1,\ldots,4)$.

Since, $u_1(t) = u_4(t) = 0$, and considering $k_3 = 0$, Eq. (11) yields:

$$\begin{bmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \\ \dot{e}_3(t) \\ \dot{e}_4(t) \end{bmatrix} = -\begin{bmatrix} a & -a & 0 & -c \\ -d & 1+k_2 & 0 & 0 \\ 0 & 0 & b & 0 \\ 1 & 0 & 0 & a \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{bmatrix} \tag{12}$$

$$\dot{e}(t) = -Ae(t), \tag{13}$$

where $e(t) = [e_1(t),\ e_2(t),\ e_3(t),\ e_4(t)]^T$ and

$$A = \begin{bmatrix} a & -a & 0 & -c \\ -d & 1+k_2 & 0 & 0 \\ 0 & 0 & b & 0 \\ 1 & 0 & 0 & a \end{bmatrix} \tag{14}$$

At this stage, the problem is reduced to show that A is PDM. Since $a > 0$, $b > 0$, $c > 0$, and $d > 0$, thus, the matrix A will be PDM, if the controller gain k_2 satisfies the following condition:

$$k_2 > max(p_1,\ p_2), \tag{15}$$

where

$$p_1 = (d-1) > 0\ \ and\ \ p_2 = (d - \frac{c}{a^2b} - 1) > 0, \tag{16}$$

then, the closed-loop (12) is globally exponentially stable. This completes the proof of Theorem 3.1.

3.3 Numerical simulation and results discussion

Numerical simulation results are provided to verify the efficiency and effectiveness of the proposed synchronization approach by using *mathematica 10.0 v*. The parameters for the LSH system [16] are set as $a = 1$, $b = 0.7$, $c = 1.5$, and $d = 26$ with initial conditions are taken as $[x_1(0),\ y_1(0),\ z_1(0),\ w_1(0)]^T = [0.028, 0.02, 0.03, 0.04]^T$, and $[x_2(0), y_2(0), z_2(0), w_2(0)] = [0.02, 0.037, 0.059, 0.048]^T$, respectively. According to the condition (15), the controller gain k_2 is selected as $k_2 = 26$.

Case 1. Let us assume a particular case when two coupled LSH systems (8) are disturbance free systems. The corresponding numerical results are given as follows:

For the two coupled LSH systems (8), convergence of the error signals (12) under the control action (10), are depicted in Fig. 3, in which the dotdashed line represents the error signal $e_1(t)$, the thin line indicates $e_2(t)$, the dotted line denotes $e_3(t)$ and the thick line represents $e_4(t)$. It is observed that the error signals reach to the zero state in the range of $[-0.02,\ 0.02]$ within $t \approx 7s$, with small amplitude of the oscillations of the error signals. The synchronizations in [16, 17] are achieved at $t = 7s$ and $t = 100s$, in the range of $[-0.02,\ 0.02]$ and $[-7,\ 7]$, respectively. Furthermore, only two feedback controllers and a single linear controller gain are designed to synchronize two identical LSH systems, while in [16, 17], natural controllers are utilized to achieve the synchronization.

Fig. 4, demonstrates the time series of the control inputs, which are dependent on $e_2(t)$ and $e_3(t)$, only.

Case 2. Consider a practical environment, where the time varying external disturbances perturb the two coupled hyperchaotic systems (8). In order to construct the control inputs developed in Theorem 3.1, it is assumed that the two coupled LSH systems are perturbed by the following external disturbances.

$$\begin{aligned} \psi_i^m(t) &= -0.01cos(180t),\ i = 2,3 \\ \psi_i^s(t) &= -0.02sin(270t),\ i = 2,3, \end{aligned} \tag{17}$$

where, $\psi_i^m(t)$ and $\psi_i^s(t)$, denote the time varying external disturbances present in the master and slave systems (8), respectively.

Fig. 5, illustrates the time series of the synchronized error states with the presence of time varying external disturbances. It is observed that the effect of the external disturbances is eliminated and the error signals are converged to the zero state with the same rate and quality as discussed in Case 1. This feature of the proposed controller approach (10) shows the requirement of ensuring robustness property against the time varying external disturbances.

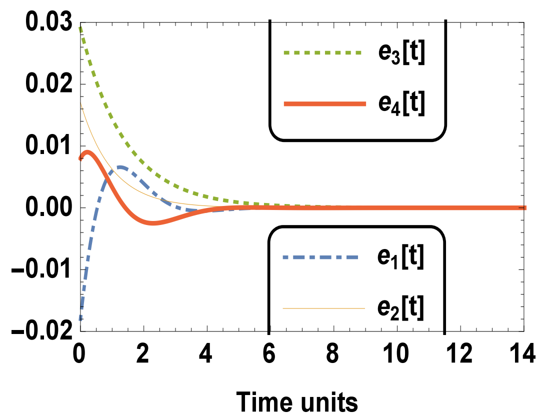

Fig. 3: Time series of the synchronized error states without the presence of time varying external disturbance signals

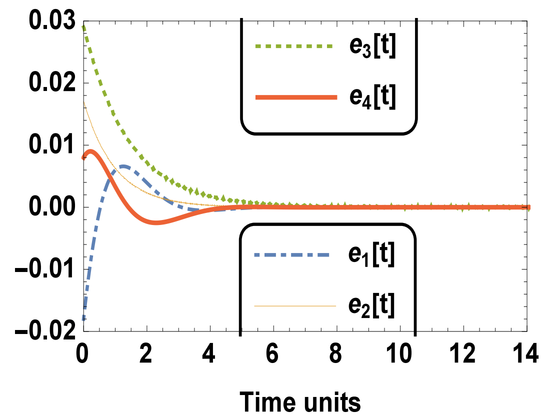

Fig. 5: Time series of the synchronized error states with the presence of time varying external disturbance signals

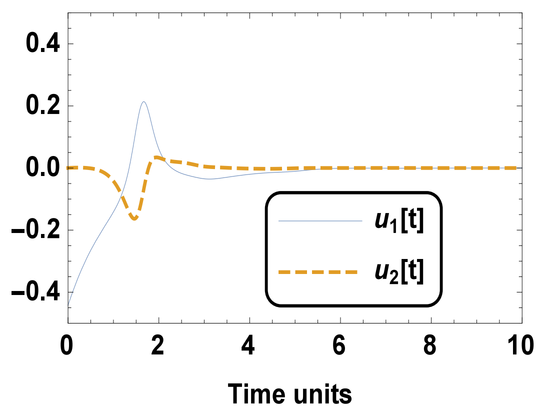

Fig. 4: Time series of the control inputs

4 Conclusion

In this paper, a modified active control strategy is presented. Suitable area for the feedback controller gain is identified that established the globally exponential complete synchronization. Numerical simulation results further verified the robustness of the proposed modified active controller approach. The proposed controller approach could be employed for the complete synchronization of a class of chaotic as well as hyperchaotic systems.

Acknowledgement

The authors would like to extend their gratitude to valuable comments of the respected reviewers.

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