Math. Sci. Lett. 5, No. 2, 181-187 (2016)

Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.18576/msl/050211

Using Homo-Separation of Variables for Pricing European Option of the Fractional Black-Scholes Model in Financial Markets

Mohammad Ali Mohebbi Ghandehari[1] *and Mojtaba Ranjbar*[2,*]

Department of Applied Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran

Received: 25 Feb. 2015, Revised: 21 Sep. 2015, Accepted: 23 Sep. 2015
Published online: 1 May 2016

Abstract: In this study, we present the exact solution of the option pricing problems based on the fractional Black-Scholes equation by using a modified homotopy perturbation method (MHPM). The new method is a combination of two well-established mathematical methods, namely, the homotopy perturbation method (HPM) and the separation of variables method. The proposed method is introduced an efficient tool for solving Black-Scholes equation of frctional order. The results show that this scheme is accurate and efficient.

Keywords: Mittag-leffler functions, Green function, Caputo derivative, Fractional Black- Scholes equation, American and European option pricing problems.

1 Introduction

A financial derivative is an instrument whose price depends on, or is derived from, the value of another asset [1]. Often, this underlying asset is a stock. The concept of financial derivatives is not new. In 1973, Fischer Black and Myron Scholes [2] derived the famous theoretical valuation formula for options. The main conceptual idea of Black and Scholes lie in the construction of a riskless portfolio taking positions in bonds (cash), option and the underlying stock. The Black-Scholes model (BS) for pricing stock options has been applied to many different commodities and payoff structures. The Black-Scholes model for value of an option is described by the equation

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 x^2\frac{\partial^2 V}{\partial x^2}+r(t)x\frac{\partial V}{\partial x}-r(t)V=0, \quad (x,t)\in R^+\times(0,T), \tag{1}$$

where $V(x,t)$ is the European option price at asset price x and at time t, T is the maturity, $r(t)$ is the risk free interest rate and $\sigma(x,t)$ represents the volatility function of underlying asset. It is well-known that problem (1) has a closed-form solution obtained for the price of a European call or European put option after several changes of variables and solving certain related diffusion equations [1]. We denoted the payoff functions $c(x,t)$ and $p(x,t)$ for the European call and put options, respectively. Thus

$$c(x,t)=max(x-E,0) \quad , \quad p(x,t)=max(E-x,0),$$

where E is the exercise price. The Black-Scholes equation has been increasingly attracting interest over the last two decades since it provides effectively the values of options. But the classical Black-Scholes equation was established under some strict assumptions. Therefore, some improved models have been proposed to weaken these assumptions, such as stochastic interest model [3], Jump-diffusion model [4], stochastic volatility model [5], and models with transactions costs [6,7]. With the discovery of the fractal structure for financial market, the fractional Black-Scholes models [8,9,10,11] are derived by replacing the standard Brownian motion involved in the classical model with fractional Brownian motion. Option pricing in fractional Black-Scholes markets was proposed in [12,13,14].

Fractional differential equations are increasingly used to model problems in acoustics and thermal systems, rheology and mechanical systems, signal processing and systems identification, control and robotics and other areas of applications (see [15,16]). The interdisciplinary applications show the importance and necessity of fractional calculus. Some promising approximate

* Corresponding author e-mail: m_ranjbar@azaruniv.edu

analytical solutions are proposed, such as handle these problems such as Backlunds transformation [17], Hirotas bilinear method [18,19], the tanh method [20], the Laplace transform method, and the Mellin transform method [21], Differential transform method [22] and homotopy perturbation method [23]. Lately, the fractional differential equations have been solved with converting it into an NLP problem [24,25].
The standard homotopy analysis method (HAM), which was proposed by Liao in his Ph.D. thesis, is the most effective and convenient one for both linear and nonlinear equations. Different from perturbation techniques; the (HAM) doesnt depend upon any small or large parameter. This method has been successfully applied to solve many types of nonlinear [26,27,28] differential equations. (HAM) is different from all analytical methods; it provides us with a simple way to adjust and control the convergence region of the series solution by introducing the auxiliary para- meter h and the auxiliary function. In fact, it is the auxiliary parameter h that provides us, for the first time, a simple way to ensure the convergence of the series solution. Due to this reason, it seems reasonable to rename h the convergence-control parameter. It should be emphasized that, without the use of the convergence-parameter, one had to assume that the homotopy series is convergent. However, with the use of the convergence-parameter h, such an assumption is unnecessary; because it seems that one can always choose a proper value of h to obtain convergent homotopy-series solution. Since then, the homotopy analysis method has been developing greatly and more generalized zeroth-order deformation Equations are suggested by Liao [26, 27]. The homotopy perturbation method (HPM) is a series expansion method used in the solution of nonlinear partial differential equations. The HPM was introduced by Ji-Huan He in 1998 [30]. In general is proved the homotopy perturbation method (HPM) is a special case of the homotopy analysis method (HAM) by Sajid and et al [29]. The HPM is a universal approach which can be used to solve both fractional ordinary differential equations as well as fractional partial differential equations. Various combinations of the methods mentioned previously have been proposed recently to solve fractional partial differential equations. Examples of such combination methods are the Homotopy Analysis Transform Method, the homotopy perturbation Sumudu transform method, Laplace homotopy perturbation method, the Variational Homotopy Perturbation Method and the Homotopy Perturbation Transformation Method. Recently, Karbalaie et al. [31] found the exact solution of one-dimensional FPDEs by using truncated versions of modified HPM. Therefore, by getting inspiration of the ideas, methods, and tools of previous works, the novel approach called the homo-separation of variables method is developed and utilized to find the exact solutions of systems of FPDE. The similar idea can be found in [32,33]. This new approach is constructed by a smart combination of HPM and the separation of variables method. By using this method, FPDE to be solved is changed into FODE. In this paper, the MHPM is used to derive the exact solution of European option pricing problems for the fractional Black-Scholes model.

2 Preliminaries

Definition 2.1. The single parameter and the two parameters variants of the Mittag-leffler function are denoted by $E_\alpha(z)$, and $E_{\alpha,\beta}(z)$, respectively, which are relevant for their connection with fractional calculus, and are defined as,

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \quad \alpha > 0, \quad z \in \mathbb{C}.$$

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} \quad \alpha,\beta > 0, \quad z \in \mathbb{C}.$$

Their k-th derivatives are

$$E_\alpha^{(k)}(z) = \frac{d^k}{dz^k} E_\alpha(z) \sum_{n=0}^{\infty} \frac{(k+n)!z^n}{n!\Gamma(\alpha n+\alpha k+1)}, \quad k = 0,1,..., \tag{2}$$

$$E_{\alpha,\beta}^{(k)}(z) = \frac{d^k}{dz^k} E_{\alpha,\beta}(z) \sum_{n=0}^{\infty} \frac{(k+n)!z^n}{n!\Gamma(\alpha n+\alpha k+\beta)}, \quad k = 0,1,..., \tag{3}$$

other properties of the Mittag-leffler functions can be found in [34].
Definition 2.2. A real function $y(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p(> \mu)$, such that $y(t) = t^p y_1(t)$, where $y_1(t) \in C[0,\infty]$, and it is said to be in the space C_μ^m iff $y^{(m)} \in C_\mu, m \in \mathbb{N}$.

The Riemann-Liouville fractional integral and Caputo derivative are defined as follows.
Definition 2.3. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $y \in C_\mu,\ \mu \geq -1$, is defined as:

$$J^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} y(\tau) d\tau,\ \alpha > 0,\ t > 0,$$
$$J^0 y(t) = y(t).$$

Some of the most important properties of operator J^α for $y \in C_\mu$,
$\mu \geq -1,\ \alpha,\ \beta \geq 0$ and $\gamma > -1$, are as follows [35]:
1. $J^\alpha J^\beta y(t) = J^{(\alpha+\beta)} y(t)$;
2. $J^\alpha J^\beta y(t) = J^\beta J^\alpha y(t)$;
3. $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

Definition 2.4. The Riemann-Liouville fractional derivative of y is defined as:

$$_{RL}D^{\alpha}y(t)=\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t(t-\tau)^{m-\alpha-1}y(\tau)d\tau,$$

for $m-1<\alpha\leq m, m\in\mathbb{N},\ t>0,\ y\in C_{-1}^m$.

Definition 2.5. The fractional derivative of $y(t)$ in the Caputo sense is defined as:

$$\begin{aligned}_{C}D^{\alpha}y(t)&=J^{m-\alpha}D^my(t)\\&=\frac{1}{\Gamma(m-\alpha)}\int_0^t(t-\tau)^{m-\alpha-1}y^{(m)}(\tau)d\tau,\end{aligned}$$

for $m-1<\alpha\leq m,\ m\in\mathbb{N},\ t>0,\ y\in C_{-1}^m$. Note that the relation between Riemann-Liouville fractional integral operator and modified Riemann-Liouville fractional differential operator is given by fractional Leibnitz formulation as follows

$$J_t^{\alpha}D_t^{\alpha}f(t)=D_t^{-\alpha}D_t^{\alpha}f(t)=f(t)-\sum_{k=0}^{m-1}\frac{t^k}{k!}f^k(0),$$
$$m-1<\alpha\leq m.$$

Theorem 2.6. Consider the following n-term linear fractional differential equation:

$$(a_nD_t^{\beta_n}+a_n-1D_t^{\beta_{n-1}}+...+a_0D_t^{\beta_0})u(t)=f(t),\qquad(4)$$

with the constant initial condition:

$$u^{j_i}(0)=c_{ij_i},\quad i=0,1,...,n,\quad j_i=1,2,...,l_i,\qquad(5)$$

where $a_i,\ c_{ij_i}\in\mathbb{R},\ n_i-1<\beta_i<n_i,\ n_i\in\mathbb{N}\bigcup\{0\}$ and

$$\beta_0<\beta_1<...<\beta_{n-1}<n\leq\beta_n<n+1.\qquad(6)$$

Then, we see that the analytical general solution of Eq. (4) is

$$u(t)=\int_0^tG_n(t-\varsigma)f(\varsigma)dt+\sum_{i=0}^{\infty}\sum_{j_i=0}^{l_i-1}a_ic_{ij_i}G_n^{\beta_i-j_i-1}(t),\qquad(7)$$

where $G_n(t)$ is the Green function and it is defined as

$$\begin{aligned}G_n(t)=&\frac{1}{a_n}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\times\sum_{k_0+k_1+...+k_{n-2}=m}(m;k_0,k_1,...,k_{n-2})\\&\times\prod_{p=0}^{n-2}(\frac{a_p}{a_n})^{k_p}t^{(\beta_n-\beta_{n-1})m+\beta_n+\sum_{j=0}^{n-2}(\beta_{n-1}-\beta_j)k_j-1}\\&\times E^{(m)}_{\beta_n-\beta_{n-1},\beta_n+\sum_{j=0}^{n-2}(\beta_{n-1}-\beta_j)k_j-1}(-\frac{a_{n-1}}{a_n}D^{\beta_n-\beta_{n-1}}),\end{aligned}\qquad(8)$$

where

$$(m;k_0,k_1,...,k_{n-2})=\frac{m!}{k_0!k_1!...k_{n-2}!}\qquad(9)$$

and $E^{(m)}_{(.),(.)}$ is the m-th derivative of the Mittag-Leffler function. In a special case of the latter theorem, the following relaxation-oscillation equation is solved:

$$\begin{aligned}&D_t^{\alpha}u(t)+Au(t)=f(t),\quad t>0,\\&u^i(0)=b_i,\quad i=1,2,...,n-1,\end{aligned}\qquad(10)$$

where b_i are real constants and $n-1<\alpha\leq n$. By utilizing theorem 2.6, we obtain the solution of Eq. (10) as follows:

$$\int_0^tG_2(t-\varsigma)f(\varsigma)dt+\sum_{j=0}^{n-1}b_jD_t^{\alpha-j-1}G_2(t),\qquad(11)$$

where $G_2(t)=t^{\alpha-1}E_{\alpha,\alpha}(-At^{\alpha})$. It is easy to see that if $0<\alpha\leq1$; then the solution of Eq. (10) becomes as follows:

$$u(t)=\int_0^tG_2(t-\varsigma)f(\varsigma)dt+b_0D_t^{\alpha-1}G_2(t).\qquad(12)$$

3 The Homo-Separation of Variables Method

The Homotopy Perturbation Method (HPM) is an especial case of the standard homotopy analysis method (HAM). The Homotopy Perturbation Method (HPM) is a combination of the Homotopy technique and the classical Perturbation Method. In this section, the algorithm of this method is briefly illustrated. To achieve our goal, we consider the nonlinear partial differential equation:

$$\begin{aligned}&L(u)+N(u)-f(r)=0,\quad r\in\Omega,\\&B(u,\frac{\partial u}{\partial n})=0,\quad r\in\Gamma,\end{aligned}\qquad(13)$$

where L is a linear operator, N is a nonlinear operator, B is a boundary operator, Γ is the boundary of the domain Ω, and $f(r)$ is a known analytical function. By using the homotopy perturbation technique, we construct a homotopy $v(r,p):\Omega\times[0,1]\to\mathbb{R}$ which satisfies:

$$\begin{aligned}&H(v,p)=L(v)-L(u_0)+pL(u_0)+p[N(v)-f(r)]=0,\\&0\leq p\leq1,\end{aligned}\qquad(14)$$

where $r\in\Omega$ and u_0 is an initial approximation for Eq. (13) and p is an embedding parameter. When the value of p is changed from $p=0$ to $p=1$, we can easily see that

$$H(v,0)=L(v)-L(u_0)=0,\qquad(15)$$

$$H(v,1)=L(v)+N(v)-f(r)=0.\qquad(16)$$

This changing process is called deformation, and Eq. (15) and (16) are called homotopic in field of topology. We can assume that the solution of Eq. (14) can be expressed as a power series in p, as given below

$$v=\sum_{i=0}^{\infty}p^iv_i=v_0+pv_1+p^2v_2+...,\qquad(17)$$

In case the p-parameter is considered as small, the best approximation for the solution of Eq. (13) is:

$$u = \lim_{p\to 1} v = \sum_{i=0}^{\infty} v_i = v_0 + v_1 + v_2 + \dots, \quad (18)$$

Now we are able to apply the HPM to solve the class of time fractional partial differential equations defined as follows:

$$D_t^{\alpha} u(x,t) = L(u(x,t)) + N(u(x,t)) + f(x,t), \quad (19)$$

subject to the initial condition $u(x,0) = g(x)$, and $0 < \alpha \leq 1$ in D_t^{α}, which is identical to the Caputo fractional derivative of order α. According to the homotopy perturbation technique (HPM), we can construct the following homotopy

$$\begin{aligned} &H(v(x,t),p) = (1-p)[D_t^{\alpha}(v(x,t)) - D_t^{\alpha}(u_0(x,t))] + \\ &p[D_t^{\alpha}(v(x,t)) - L(v(x,t)) - N(v(x,t)) - f(x,t)] = 0, \end{aligned} \quad (20)$$

where $p \in [0,1]$ and $u_0(x,t)$ is an initial approximation of the solution of Eq.(19) which also satisfies the initial condition. By simplifying Eq. (20) we get

$$\begin{aligned} &D_t^{\alpha}(v(x,t)) = D_t^{\alpha}(u_0(x,t)) + \\ &p[D_t^{\alpha}(u_0(x,t)) - L(v(x,t)) - N(v(x,t)) - f(x,t)], \end{aligned} \quad (21)$$

where the embedding parameter p is considered to be small and applied to the classical perturbation technique. The next step is to use this homotopy parameter p to expand the solution into the following form:

$$v(x,t) = v_0(x,t) + pv_1(x,t) + p^2 v_2(x,t) + \dots, \quad (22)$$

eventually, at $p = 1$, we will obtain the approximate solution of Eq. (19). By substituting (22) into (21) and equating the terms with identical powers of p, a set of equations is obtained as follows:

$$\begin{aligned} p^0 &: D_t^{\alpha}(v_0(x,t)) = D_t^{\alpha}(u_0(x,t)), \\ p^1 &: D_t^{\alpha}(v_1(x,t)) = D_t^{\alpha}(v_0(x,t)) - L(v_0(x,t)) \\ &\qquad - N(v_0(x,t)) - f(x,t), \\ &\vdots \end{aligned} \quad (23)$$

Applying the operator J_t^{α}, which is the Riemann- Liouville fractional integral of order $\alpha \geq 0$, on both sides of all cases of Eq.(24), the solution can be given by

$$\begin{aligned} v_0(x,t) &= u_0(x,t), \\ v_1(x,t) &= J_t^{\alpha}[D_t^{\alpha}(v_0(x,t)) - L(v_0(x,t)) \\ &\qquad - N(v_0(x,t)) - f(x,t)], \\ &\vdots \end{aligned} \quad (24)$$

By utilizing the results in Eq. (24), and substituting them into Eq. (22), we get an accurate n^{th} approximation of the exact solution as follows

$$u_n(x,t) = v_0 + v_1 + \dots + v_n = \sum_{i=0}^{n} v_i. \quad (25)$$

In Eq. (25), if there exists some $v_n = 0,\ n \geq 1$; then the exact solution can be written in the following form

$$u(x,t) = v_0 + v_1 + \dots + v_{n-1} = \sum_{i=0}^{n-1} v_i. \quad (26)$$

For simplicity, we assume that $v_1(x,t) \equiv 0$ in Eq. (26), which means that the exact solution in Eq. (13) is $u(x,t) = v_0(x,t)$, and solving Eq. (13), we obtain the result

$$u_0(x,t) = v_0(x,t). \quad (27)$$

Therefore, we have $u(x,t) = u_0(x,t) = v_0(x,t)$. Now we can introduce the core of the work in this paper. At first, we consider the initial approximation of Eq.(19) as follows

$$u_0(x,t) = u(x,0)c_1(t) + u(x)c_2(t), \quad (28)$$

where $u(x,0)$ is the initial condition of Eq.(19), and $u(x) = \frac{\partial u(x,0)}{\partial x}$. The task now is to find the terms $c_1(t)$ and $c_2(t)$ to obtain the exact solution of the FPDE in (19). Since $u(x,t)$ satisfies the initial condition, we get

$$u(x,0) = v_0(x,0) = u(x,0)c_1(0) + u(x)c_2(0) = g(x), \quad (29)$$

therefore

$$c_1(0) = 1, \quad c_2(0) = 0. \quad (30)$$

On the other hand, we have

$$\begin{aligned} D_t^{\alpha}(v_1(x,t)) &= D_t^{\alpha}(u_0(x,t)) - L(v_0(x,t)) \\ &\quad - N(v_0(x,t)) - f(x,t) \equiv 0. \end{aligned} \quad (31)$$

By substituting Eq. (28) and (27) into Eq. (31), we obtain

$$\begin{aligned} &u(x,0)D_t^{\alpha}(c_1(t)) + u(x)D_t^{\alpha}(c_2(t)) = \\ &L(u(x,0)c_1(t) + u(x)c_2(t)) \\ &+ N(u(x,0)c_1(t) + u(x)c_2(t)) + f(x,t). \end{aligned} \quad (32)$$

In this case, the partial differential equation is changed into an ODE, which simplifies the problem at hand. The exact solution of the PDE is found when the target unknowns $c_1(t)$ and $c_2(t)$ are computed; by utilizing Eq.(32) and the initial conditions in Eq. (30).

4 Numerical Examples

In the section, we use our approach and investigate it's accuracy through the Option pricing models based on the time-fractional differential equations.

Example 4.1. Consider the following fractional Black-Scholes option pricing equation [36]

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + (k-1)\frac{\partial u}{\partial x} - ku, \qquad (33)$$

where $0 < \alpha \leq 1$, with initial condition $u(x,0) = max(e^x - 1, 0)$.
Note that this system of equations contains just two dimensionless parameters $k = \frac{2r}{\sigma^2}$, where k represents the balance between the rate of interests and the variability of stock returns and the dimensionless time to expiry $\frac{\sigma^2 T}{2}$, even though there are four dimensional parameters, E, T, σ^2, and r, in the original statements of the problem.
To solve (33) by using the proposed homo-separation of variables method, we choose the initial approximation as follows:

$$\begin{aligned} u_0(x,t) &= u(x,0)c_1(t) + u(x)c_2(t) \\ &= max(e^x - 1, 0)c_1(t) + max(e^x, 0)c_2(t) \end{aligned} \qquad (34)$$

then

$$\begin{aligned} D_t^\alpha(v_1(x,t)) =& D_t^\alpha(max(e^x - 1, 0)c_1(t) + max(e^x, 0)c_2(t)) \\ & - max(e^x, 0)c_1(t) - max(e^x, 0)c_2(t) \\ & (-k+1)(max(e^x, 0)c_1(t) + max(e^x, 0)c_2(t)) \\ & + k(max(e^x - 1, 0)c_1(t) + max(e^x, 0)c_2(t)) \\ & \equiv 0, \end{aligned} \qquad (35)$$

$$\begin{aligned} D_t^\alpha(v_1(x,t)) =& D_t^\alpha(max(e^x - 1, 0)c_1(t) + max(e^x, 0)c_2(t)) \\ & - kmax(e^x, 0)c_1(t) + kmax(e^x - 1, 0)c_1(t) \\ & \equiv 0. \end{aligned} \qquad (36)$$

We obtain the fractional differential system

$$\begin{cases} D_t^\alpha c_1(t) + kc_1(t) = 0, \\ c_1(0) = 1, \end{cases} \qquad (37)$$

$$\begin{cases} D_t^\alpha c_2(t) - kc_1(t) = 0, \\ c_2(0) = 0. \end{cases} \qquad (38)$$

Solving Eq. (37) and (38) by applying Eq. (12), we obtain

$$\begin{aligned} c_1(t) &= E_\alpha(-kt^\alpha), \\ c_2(t) &= 1 - E_\alpha(-kt^\alpha), \end{aligned} \qquad (39)$$

and the exact solution is

$$\begin{aligned} u(x,t) =& max(e^x - 1, 0)E_\alpha(-kt^\alpha) \\ & + max(e^x, 0)(1 - E_\alpha(-kt^\alpha)), \end{aligned} \qquad (40)$$

where $E_\alpha(z)$ is Mittag-Leffler function in one parameter. The analytical solution of this problem is consistent with the result obtained by Kumar and et al. [36]. For case $\alpha = 1$, we have

$$u(x,t) = max(e^x - 1, 0)e^{-kt} + max(e^x, 0)(1 - e^{-kt}), \quad (41)$$

which is an exact solution of the given classic Black-Scholes equation.

Example 4.2. Consider the following generalized fractional Black-Scholes equation as follows [37]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + 0.08(2 + sin(x))^2 x^2 \frac{\partial^2 u}{\partial x^2} + 0.06x\frac{\partial u}{\partial x} - 0.06u = 0, \qquad (42)$$

with $0 < \alpha \leq 1$ and initial condition $u(x,0) = max(x - 25e^{-0.06}, 0)$.
if we choose the initial approximation (28) for this problem; then we obtain $c_1(t) = c_2(t) \equiv 0$. In this case, we earn the trivial solution $u(x,t) \equiv 0$. Therefore, we choose the initial approximation as follows

$$\begin{aligned} u_0(x,t) &= u(x,0)c_1(t) + xu(x)c_2(t) \\ &= max(x - 25e^{-0.06}, 0)c_1(t) + xc_2(t), \end{aligned} \qquad (43)$$

then

$$\begin{aligned} D_t^\alpha(v_1(x,t)) =& D_t^\alpha(max(x - 25e^{-0.06}, 0)c_1(t) + xc_2(t)) \\ & + 0.06x(c_1(t) + c_2(t)) \\ & - 0.06(max(x - 25e^{-0.06}, 0)c_1(t) + xc_2(t)) \\ & \equiv 0. \end{aligned} \qquad (44)$$

Now, we obtain the fractional differential system

$$\begin{cases} D_t^\alpha c_1(t) - 0.06c_1(t) = 0, \\ c_1(0) = 1, \end{cases} \qquad (45)$$

$$\begin{cases} D_t^\alpha c_2(t) + 0.06c_1(t) = 0, \\ c_2(0) = 0, \end{cases} \qquad (46)$$

Solving Eq. (45) and (46) by applying Eq. (12), we obtain

$$\begin{aligned} c_1(t) &= E_\alpha(0.06t^\alpha), \\ c_2(t) &= 1 - E_\alpha(0.06t^\alpha), \end{aligned} \qquad (47)$$

and the exact solution is

$$\begin{aligned} u(x,t) =& max(x - 25e^{-0.06}, 0)E_\alpha(0.06t^\alpha) \\ & + x(1 - E_\alpha(0.06t^\alpha)), \end{aligned} \qquad (48)$$

which is the exact solution of the given fractional Black-Scholes equation, for pricing the European option.
The exact solution of the given option pricing equation for $\alpha = 1$ is

$$u(x,t) = max(x - 25e^{-0.06}, 0)e^{0.06t} + x(1 - e^{0.06t}). \quad (49)$$

5 Conclusion

In this paper, we have proposed a new analytical method based on the homotopy perturbation method (HPM) for pricing European option of the fractional Black-Scholes model. This method is intuitive and very easy to understand. new approach converts the fractional Black-Scholes equation into a system of ordinary differential equations (ODEs) and after that proceeds to solve the resulting ODE. Finally, the resulting homo-separation of variables method, which is analytical, can be used to solve equations with fractional and integer order with respect to time.

Acknowledgement

Research of the authors was supported by Azarbaijan Shahid Madani University.
The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] J. C. Hull, Options future and other derivatives. Sixth edition, Pearson prentice Hall, Toronto, (2006).
[2] F. Black and M. S. Scholes, The pricing of options and corporate liabilities, J. Polit. Econ. **81**, 637-654 (1973).
[3] R. C. Merton, On the pricing of corporate debt: the risk structure of interest rates, Journal of Finance **29**, 449- 470 (1974).
[4] R. C. Merton, Option pricing when underlying stock returns are discontinuous, Journal of Financial Economics. **3**(1-2), 125-144 (1976).
[5] J. C. Hull and A. D. White, Thepricingof options on assets with stochastic volatilities, Journal of Finance **42**, 281-300 (1987).
[6] M. H. A. Davis, V. G. Panas, and T. Zariphopoulou, European option pricing with transaction costs, SIAM Journal on Control and Optimization **31**(2), 470-493 (1993).
[7] G. Barles and H. M. Soner, Option pricing with transaction costs and a nonlinear Black-Scholes equation, Finance and Stochastics **2**(4), 369-397 (1998).
[8] T. Bjork and H. Hult, A note on Wick products and the fractional Black-Scholes model, Finance and Stochastics **9**(2), 197-209 (2005).
[9] X.-T. Wang, Scaling and long-range dependence in option pricing I: Pricing European option with transaction costs under the fractional Black-Scholes model, Physica A. **389**(3), 438-444 (2010).
[10] J.-R. Liang, J. Wang, W.-J. Zhang, W.-Y. Qiu, and F.-Y. Ren, Option pricing of a bi-fractional Black-Merton-Scholes model with the Hurst exponent H in [1/2,1], Applied Mathematics Letters **23**(8), 859-863 (2010).
[11] J. Wang, J.-R. Liang, L.-J. Lv, W.-Y. Qiu, and F.-Y. Ren, Continuous time Black-Scholes equation with transaction costs in subdiffusive fractional Brownian motion regime, Physica A. **391**(3), 750-759 (2012).
[12] M. A. M. Ghandehari and M. Ranjbar, European option pricing of fractional Black-Scholes model with new Lagrange multipliers, Computational Methods for Differential Equations **2**, 1-10 (2014).
[13] M. A. M. Ghandehari and M. Ranjbar, European option pricing of fractional version of the Black- Scholes model: Approach via expansion in series, International Journal of Nonlinear Science. **17**, 105-110 (2014).
[14] M. A. M. Ghandehari and M. Ranjbar, Barrier options pricing of fractional version of the Black-Scholes model. **7**(2), 171-178 (2015).
[15] D. Baleanu. K. Diethelm, E. Scalas and J.J. Trujillo, Fractional Calculus Models and Numerical Methods, World Scientific, Singapore, (2009).
[16] G. M. Zaslavsky, Chaos, fractional kenetics, and anomalous transport.Phys.Rep. **371**, 461-580 (2002).
[17] M. R. Miura, Backlund Transformation, Berlin: Springer-Verlag, Berlin: Springer-Verlag.
[18] R. Hirota, Direct Methods in Soliton Theory, in: Bullogh, R.K., Caudrey (Eds.), Solitons, P.J. Berlin: Springer, BulloghRK, CaudreyPJ, editors, Solitons, Berlin: Springer, (1980).
[19] R. Hirota, Exact enve lope-soliton solutions of a non linear wave, J. Math. Phys. **14**(7), 805-809 (1973).
[20] W. Maliet, The tanh method: a tool for solving certain classes of nonlinear evolution and wave equations, J. Comput. Appl. Math. **164**, 529-541 (2004).
[21] I. Podlubny, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, vol. 198 of Mathematics in Science and Engineering, Academic Press, San Diego, Calif, USA, (1999).
[22] Z. Odibat, C. Bertelle, M. A. Aziz-Alaoui, G. Duchamp, A multi-step diffrential transform method and application to non-chaotic or chaotic systems, Computers and Mathematics with Applications **59**(4), 1462-1472 (2010).
[23] T. Ozis, A. YildIrIm, Comparison between Adomians method and Hes homotopy perturbation method, Comput. Math. Appl. **56**(5), 1216-1224 (2008).
[24] M. A. M. Ghandehari and M. Ranjbar, A numerical method for solving a fractional partial differential equation through converting it into an NLP problem. Computers and Mathematics with Applications **65**, 975-982 (2013).
[25] M. A. M. Ghandehari and M. Ranjbar, Solving the fractional Volterra integro-differential equations by an extremum problem, Journal of Advanced Research in Scientific Computing. **7**, 38-49 (2015).
[26] S. J. Liao, Notes on the Homotopy Analysis Method: Some Definitions and Theorems, Communications in Nonlinear Science and Numerical Simulation, **14**(4), 983-997 (2009).
[27] S. J. Liao, The Proposed Homotopy Analysis Technique for the Solution of Nonlinear Problems, Ph.D. Thesis, Shanghai Jiao Tong University, Shanghai, (1992).
[28] M. Zurigat, S. Momani, Z. Odibat and A. Alawneh, The Homotopy Analysis Method for Handling Systems of Fractional Differential Equations, Applied Mathematical Modelling **34**(1), 24-35 (2010).
[29] J. H. He, An approximation sol. technique depending upon an artificial parameter, Comm. Nonlinear Sci. Num. Simul.**3**, 92-97(1998).
[30] M. Sajid, T. Hayat and S. Asghar, Comparison between the HAM and HPM solutions of thin film flows of non-Newtonian fluids on a moving belt, Nonlinear Dynamic **50**, 27-35 (2007).

[31] A. Karbalaie, M. M. Montazeri, and H. H. Muhammed, New approach to find the exact solution of fractional partial differential equation, WSEAS Transactions on Mathematics **11**(10), 908-917 (2012).

[32] A. Karbalaie, H. H. Muhammed, B. E. Erlandsson, Using Homo-Separation of Variables for Solving Systems of Nonlinear Fractional Partial Differential Equations, International Journal of Mathematics and Mathematical Sciences, Article ID 421378, (2013).

[33] A. Karbalaie, H. H. Muhammed, M. Shabani, M. M. Montazeri, Exact Solution of Partial Differential Equation Using Homo-Separation of Variables, International Journal of Nonlinear Science **17**(1), 84-90 (2014).

[34] A. A. M. Arafa, S. Z. Rida, and H. Mohamed, Homotopy analysis method for solving biological population model, Communications in Theoretical Physics **56**(5), 797- 800 (2011).

[35] I. Podlubny. Fractional Differential equations, 1st Edn., Academic Press, New York, (1999).

[36] S. Kumar, A. Yildirim, Y. Khan, H. Jafari, K. Sayevand and L. Wey, Analytical solution of fractional Black-Scholes European option pricing equation by using laplace transform. Journal of Fractional Calculus and Applications. **2**, 1-9 (2012).

[37] Z. Cen and A. Le, A robust and accurate finite difference method for a generalized Black-Scholes equation. J. Comput. Appl. Math. **235**, 3728-3733 (2011).

Mohammad Ali Mohebbi Ghandehari Mohammad Ali Mohebbi Ghandehari started his bachelor and master courses in applied mathematics at Ferdowsi University of Mashhad. He received Ph.D degree in financial mathematics from Azarbaijan Shahid Madani University in 2015. His research interests include: Mathematical Finance, Fractional Differential Equations, Stochastic Differential Equations, Fractional Brownian Motion. He has published research articles in international journals of Mathematics and financial mathematics.

Mojtaba Ranjbar Mojtaba Ranjbar was born in Tabriz-Iran in 1975. He received B.Sc degree in pure mathematics from Tabriz University, and his M.Sc and Ph.D degree in applied mathematics from Amirkabir University of Technology, Tehran, Iran in 2002 and 2008 respectively. He is an Assistant Prof. in the department of mathematics at Azarbaijan Shahid Madani University, Tabriz, Iran. His research interests include: numerical solution of ordinary differential equation, partial differential equation, stochastic differential equation and numerical methods with applications to finance.