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Generalized $\left(\frac{G'}{G}\right)$-Expansion Method For Generalized Fifth Order KdV Equation with Time-Dependent Coefficients

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Abstract: In this work, the generalized $\left(\frac{G'}{G}\right)$-expansion method is presented to seek some new exact solutions for generalized fifth order KdV equation with time-dependent coefficients. As a result, more explicit traveling wave solutions involving arbitrary parameters are found out, which are expressed in terms of hyperbolic functions, the trigonometric functions and rational functions. When the parameters are taken special values, different types of wave solutions can be obtained.

Keywords: Generalized Fifth Order KdV Equation with Time-Dependent Coefficients, Generalized $\left(\frac{G'}{G}\right)$-Expansion method, Exact Solutions.

1 Introduction

Many phenomena in physics and other fields are often described by nonlinear partial differential equations (PDEs). When we want to understand the physical mechanism of natural phenomenon described by nonlinear PDEs, exact solutions for these nonlinear PDEs have to be explored. Thus, investigation of exact solutions of nonlinear PDEs has become one of the most important topics in mathematical physics. Powerful methods that make it possible to generate exact solutions to nonlinear equations have emerged from the literatures in the past decades. Among them are the tanh-sech method [1,2], extended tanh method [3,4], sine-cosine method [5,6], Hirota method [7,8], homogeneous balance method [9,10], Jacobi elliptic function method [11,12], F-expansion method [13,14], homotopy perturbation method [15,16], variational iteration method [17,18], Lie symmetry analysis [19,20,21,22] and so on. With the use of the methods mentioned, many exact solutions, including the solitary wave solutions, shock wave solutions, and periodic wave solutions etc. of NLPDEs, were obtained.

Among the other methods, the generalized $\left(\frac{G'}{G}\right)$-expansion method [23,24,25,26] has been proposed to construct traveling wave solutions for nonlinear partial differential equations with variable coefficients. The method is based on the homogeneous balance principle and linear ordinary differential equation (LODE) theory. Being concise and straightforward, this method has been applied to various nonlinear partial differential equations with variable coefficients to discover some more general solutions with some free parameters [20,27,28,29]. Also, it handles nonlinear partial differential equations in a direct manner with no requirement for initial/boundary conditions or initial trial function at the outset.

The general fifth order KdV equation reads

$$u_t + auu_{xxx} + bu_x u_{xx} + cu^2 u_x + u_{xxxxx} = 0, \qquad (1)$$

where a, b and c are real constants. This includes the Lax [30], Swada-Kotera(SK) [31,32], Kaup-Kupershmidt (KK) [33,34,35] and Ito equations [36].

As the constants a, b and c take different values, The properties of eq (1) drastically change. For instance, the lax equation with $a = 10$, $b = 20$ and $c = 30$, and The SK equation where $a = b = c = 5$, are completely integrable.

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These two equations have N-soliton solutions and an infinite set of conservation laws. The KK equation, with $a = 10, b = 25$ and $c = 20$, is also known to be integrable [34] and to have bilinear representations [37], but the explicit form of its N-soliton solution is apparently not known. A fourth equation in this class (1) is the Ito equation, with $a = 3, b = 6$ and $c = 2$, which is not integrable, but has a limited number of conservation laws [36].

The variable-coefficient version of nonlinear equations can be considered as generalization of the constant coefficients equations as there are choices for parameters. Due to this, much attention has been paid to the study of nonlinear equations with variable coefficients to obtain exact solutions, and some recent contributions can be found in [19,20]. These exact solutions provide much information about nonlinear phenomena and well-described various aspects of the physical phenomena. In the present paper, the general fifth order KdV equation with time-dependent coefficients,

$$u_t + \delta(t)uu_{xxx} + \sigma(t)u_xu_{xx} + \beta(t)u^2u_x + \rho(t)u_{xxxxx} = 0, \tag{2}$$

which is an important mathematical model in nonlinear physics, has been studied for exploring exact solutions by using generalized $\left(\frac{G'}{G}\right)$-expansion method.

Our interest in the present paper is to search for the exact solutions for equation (2). In this direction, the layout of the paper is as follows: In Section 2, we have summarized the generalized $\left(\frac{G'}{G}\right)$-expansion method. Then, this method is applied to variable coefficient version of general fifth order KdV equation in section 3 and a rich variety of exact solutions are obtained which included the hyperbolic functions, the trigonometric functions and rational functions. Finally, the conclusions and remarks are given in last section.

2 Description of The Generalized $\left(\frac{G'}{G}\right)$-Expansion Method [23]

In this section, we have described the generalized $\left(\frac{G'}{G}\right)$-expansion method for finding out some new exact solutions of nonlinear evolution equations.

Consider the nonlinear partial differential equation in the following form:

$$F(u, u_t, u_x, u_y, u_z, \ldots u_{xt}, u_{yt}, u_{zt}, u_{tt}, \ldots) = 0, \tag{3}$$

with independent variables $X = (x, y, z, \ldots, t)$ and dependent variable $u = u(x, y, z, \ldots t)$ is an unknown function, F is a polynomial in $u = u(x, y, z, \ldots t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. There are following three main steps of the generalized $\left(\frac{G'}{G}\right)$-expansion method.

Step 1: Suppose that the solution of Eq. (3) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$u = \alpha_0(X) + \sum_{i=1}^{m} \alpha_i(X)\left(\frac{G'}{G}\right)^i, \quad \alpha_m(X) \neq 0, \tag{4}$$

where $\alpha_0(X), \alpha_i(X), (i = 1, 2, \ldots, m)$ and $\theta = \theta(X)$ are all functions of X, to be determined later and $G = G(\theta)$ satisfies following equation

$$G''(\theta) + \lambda G'(\theta) + \mu G(\theta) = 0, \tag{5}$$

where $\theta = p(t)x + q(t)$, $p(t)$ and $q(t)$ are functions to be determined.

Step 2: In order to determine u explicitly, the positive integer m is determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms of u appearing in Eq. (3).

Step 3: Substitute (4) into Eq. (3) along with Eq. (5) and collect all terms with the same order of $\left(\frac{G'}{G}\right)$ together, the left hand side of Eq. (3) is converted into a polynomial in $\left(\frac{G'}{G}\right)$. Then by setting each coefficient of this polynomial to zero, we derive a set of over-determined partial differential equations for $\alpha_0(X), \alpha_i(X)$ and ζ.

Step 5: The expressions for $\alpha_0(X), \alpha_i(X)$ and ζ can be found by solving the system of partial differential equations obtained in above step and hence the solutions of Eq. (3) can be derived depending on $\left(\frac{G'}{G}\right)$, since the solutions of Eq. (5) have been well known to us depending on the sign of the discriminant $\Delta = \lambda^2 - 4\mu$.

3 Application of Generalized $\left(\frac{G'}{G}\right)$-Expansion Method to Generalized Fifth Order KdV Equation with Time-Dependent Coefficients

In this section, some exact solutions of generalized fifth order KdV equation with time-dependent coefficients (2) have been furnished by generalized $\left(\frac{G'}{G}\right)$-expansion method. As a result, hyperbolic function solutions, trigonometric function solutions and rational solutions with various parameters are obtained.

According to the method described in section 2, the positive integer m is determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms of u in Equation (2) and we found $m = 2$. Thus, the solution of Eq (2) according to Eq (4)is as follows:

$$u = \alpha_0(t) + \alpha_1(t)\left(\frac{G'}{G}\right) + \alpha_2(t)\left(\frac{G'}{G}\right)^2, \tag{6}$$

Substituting (6) into (2) and using (5), collecting all terms with the same order of $\left(\frac{G'}{G}\right)$ together, the left-hand side of Eq. (2) is converted into polynomial in $x^j\left(\frac{G'}{G}\right)$, $(j=0,1)$. Setting each coefficient of this polynomial to zero, we derived a system of overdetermined differential equations for $\alpha_0(t),\alpha_1(t),\alpha_2(t),\alpha_3(t),\alpha_4(t),p(t)$ and $q(t)$. Solving this set of equations, we have following results:

Case 1:

$$\begin{aligned}
&p(t)=c_1,\\
&\alpha_0(t)=c_2,\ \ \alpha_2(t)=c_3,\ \ \alpha_1(t)=c_3\lambda,\ \ \sigma(t)=\tfrac{1}{4}\delta(t),\\
&\beta(t)=-\tfrac{6\delta(t)c_1^2}{c_3},\ \ \rho(t)=-\tfrac{\delta(t)c_3}{48c_1^2},\\
&q(t)=\int\left(\tfrac{\left(\frac{1}{48}c_3^2\lambda^4+\frac{7}{3}c_3^2\mu^2+6c_2^2-8c_3c_2\mu+\frac{5}{6}c_3^2\lambda^2\mu-c_3c_2\lambda^2\right)c_1^3\delta(t)}{c_3}\right)dt\\
&\qquad +c_4,
\end{aligned} \tag{7}$$

where c_1,c_2,c_3 and c_4 are arbitrary constants.
Substituting the general solutions of (5) into (6) and using (7), we have three types of exact solutions of (2) as follows:
When $\lambda^2-4\mu>0$, we have obtained hyperbolic function solution in the form

$$\begin{aligned}
u(x,t)=c_2+c_3\lambda&\left(\frac{1}{2}\frac{\sqrt{(\lambda^2-4\mu)}(a_1\sinh(\zeta)+a_2\cosh(\zeta))}{(a_2\sinh(\zeta)+a_1\cosh(\zeta))}-\frac{\lambda}{2}\right)\\
&+c_3\left(\frac{1}{2}\frac{\sqrt{(\lambda^2-4\mu)}(a_1\sinh(\zeta)+a_2\cosh(\zeta))}{(a_2\sinh(\zeta)+a_1\cosh(\zeta))}-\frac{\lambda}{2}\right)^2,
\end{aligned} \tag{8}$$

where $\zeta=(\frac{1}{2}((c_1x+q(t))\sqrt{(\lambda^2-4\mu)}))$ and $q(t)=\int\left(\frac{\left(\frac{1}{48}c_3^2\lambda^4+\frac{7}{3}c_3^2\mu^2+6c_2^2-8c_3c_2\mu+\frac{5}{6}c_3^2\lambda^2\mu-c_3c_2\lambda^2\right)c_1^3\delta(t)}{c_3}\right)dt$
$+c_4$.

When $\lambda^2-4\mu<0$, we have trigonometric function solution

$$\begin{aligned}
u(x,t)=c_2+c_3\lambda&\left(\frac{1}{2}\frac{\sqrt{(-\lambda^2+4\mu)}(-a_2\sin(\zeta)+a_1\cos(\zeta))}{(a_1\sin(\zeta)+a_2\cos(\zeta))}-\frac{\lambda}{2}\right)\\
&+c_3\left(\frac{1}{2}\frac{\sqrt{(-\lambda^2+4\mu)}(-a_2\sin(\zeta)+a_1\cos(\zeta))}{(a_1\sin(\zeta)+a_2\cos(\zeta))}-\frac{\lambda}{2}\right)^2,
\end{aligned} \tag{9}$$

where $\zeta=(\frac{1}{2}((c_1x+q(t))\sqrt{(-\lambda^2+4\mu)}))$ and $q(t)=\int\left(\frac{\left(\frac{1}{48}c_3^2\lambda^4+\frac{7}{3}c_3^2\mu^2+6c_2^2-8c_3c_2\mu+\frac{5}{6}c_3^2\lambda^2\mu-c_3c_2\lambda^2\right)c_1^3\delta(t)}{c_3}\right)dt$
$+c_4$.

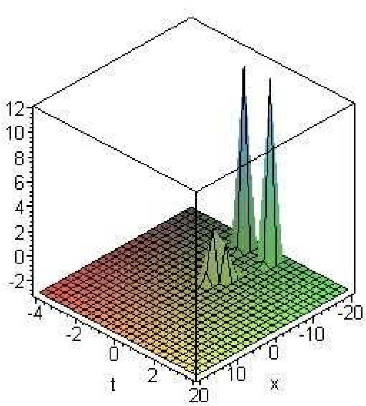

Fig. 1: Graphical representation of solution (8), for $\lambda=4,\mu=2,c_1=1,c_2=1,c_3=2,c_4=4,a_1=1,a_2=2$ and $\delta(t)=t$

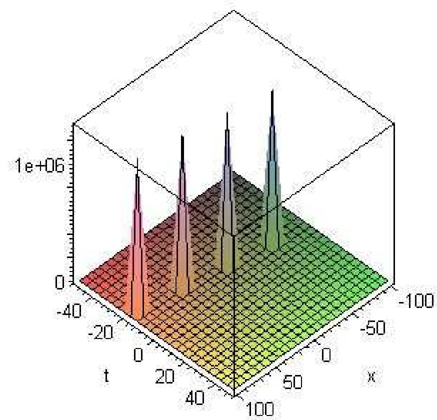

Fig. 2: Graphical representation of solution (9), for $\lambda=2,\mu=4,c_1=2,c_2=2,c_3=1,c_4=2,a_1=1,a_2=2$, and $\delta(t)=1$

When $\lambda^2-4\mu=0$, we get rational solution as follows:

$$\begin{aligned}
u(x,t)=c_2+c_3\lambda&\left(\frac{A_2}{A_1+A_2((c_1x+q(t)))}-\frac{\lambda}{2}\right)+\\
&c_3\left(\frac{A_2}{A_1+A_2((c_1x+q(t)))}-\frac{\lambda}{2}\right)^2,
\end{aligned} \tag{10}$$

where A_1,A_2,a_1,a_2 are arbitrary constants and $q(t)=\int\left(\frac{\left(\frac{3}{8}c_3^2\lambda^4+6c_2^2-3c_3c_2\lambda^2\right)c_1^3\delta(t)}{c_3}dt+c_4\right)$.

Case 2:

$$\begin{aligned}
&p(t)=c_1,\\
&\alpha_0(t)=\tfrac{1}{12}c_2\lambda^2+\tfrac{2}{3}c_2\mu,\ \ \alpha_2(t)=c_2,\ \ \alpha_1(t)=c_2\lambda,\\
&\sigma(t)=\tfrac{1}{2}\tfrac{-3\delta(t)c_2-3\delta(t)c_1^2c_2}{c_2},\\
&\beta(t)=\tfrac{144\rho(t)c_1^4-3\delta(t)c_1^2c_2}{c_2^2},\\
&q(t)=\int\left(-\tfrac{1}{16}c_1^3(4\mu-\lambda^2)^2(32\rho(t)c_1^2+\delta(t)c_2)\right)dt+c_3,
\end{aligned} \tag{11}$$

where c_1,c_2 and c_3 are arbitrary constants.
Consequently, we have the following three types of exact solutions of (2):

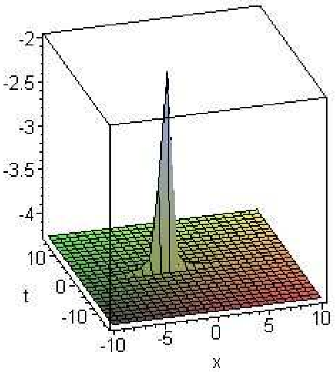

Fig. 3: Graphical representation of solution (10), for $\lambda = 5, c_1 = 2, c_2 = 2, c_3 = 1, c_4 = 2, A_1 = 4, A_2 = 2$ and $\delta(t) = t$

When $\lambda^2 - 4\mu > 0$, we have obtained hyperbolic function solution in the form

$$\begin{aligned} u(x,t) = c_2\lambda &\left(\frac{1}{2}\frac{\sqrt{(\lambda^2-4\mu)}(a_1\sinh(\zeta)+a_2\cosh(\zeta))}{(a_2\sinh(\zeta)+a_1\cosh(\zeta))} - \frac{\lambda}{2}\right) \\ &+c_3\left(\frac{1}{2}\frac{\sqrt{(\lambda^2-4\mu)}(a_1\sinh(\zeta)+a_2\cosh(\zeta))}{(a_2\sinh(\zeta)+a_1\cosh(\zeta))} - \frac{\lambda}{2}\right)^2 \\ &+\frac{2}{3}c_2\mu + \frac{1}{12}c_2\lambda^2, \end{aligned} \tag{12}$$

where $\zeta = \left(\frac{1}{2}((c_1x + q(t))\sqrt{(\lambda^2-4\mu)})\right)$ and $q(t) = \int\left(-\frac{1}{16}c_1^3(4\mu-\lambda^2)^2(32\rho(t)c_1^2+\delta(t)c_2)\right)dt + c_3$.
When $\lambda^2 - 4\mu < 0$, we have trigonometric function solution

$$\begin{aligned} u(x,t) = c_2\lambda &\left(\frac{1}{2}\frac{\sqrt{(-\lambda^2+4\mu)}(-a_2\sin(\zeta)+a_1\cos(\zeta))}{(a_1\sin(\zeta)+a_2\cos(\zeta))} - \frac{\lambda}{2}\right) \\ &+c_3\left(\frac{1}{2}\frac{\sqrt{(-\lambda^2+4\mu)}(-a_2\sin(\zeta)+a_1\cos(\zeta))}{(a_1\sin(\zeta)+a_2\cos(\zeta))} - \frac{\lambda}{2}\right)^2, \\ &+\frac{2}{3}c_2\mu + \frac{1}{12}c_2\lambda^2, \end{aligned} \tag{13}$$

where $\zeta = \left(\frac{1}{2}((c_1x + q(t))\sqrt{(-\lambda^2+4\mu)})\right)$ and $q(t) = \int\left(-\frac{1}{16}c_1^3(4\mu-\lambda^2)^2(32\rho(t)c_1^2+\delta(t)c_2)\right)dt + c_3$.

When $\lambda^2 - 4\mu = 0$, we get rational solution as follows:

$$\begin{aligned} u(x,t) = \frac{1}{4}c_2\lambda^2 + c_2\lambda\left(\frac{A_2}{A_1+A_2((c_1x+c_3))} - \frac{\lambda}{2}\right) + \\ c_2\left(\frac{A_2}{A_1+A_2((c_1x+c_3))} - \frac{\lambda}{2}\right)^2, \end{aligned} \tag{14}$$

where A_1, A_2, a_1, a_2 and c_3 are arbitrary constants.
Case 3:

$$\begin{aligned} q(t) = &\int\left(\frac{c_1^3}{c_3}\left(6\delta(t)c_2^2 - 8\delta(t)\mu c_2c_3 + 2\delta(t)c_3^2\mu^2\right)\right)dt \\ &+\int\left(\frac{c_1^3}{c_3}\left(-16\rho(t)c_1^2c_3\mu^2 + 8\rho(t)c_1^2c_3\lambda^2\mu\right)\right)dt \\ &+\int\left(-\rho(t)c_1^2c_3\lambda^4 + \delta(t)c_3^2\lambda^2\mu - \delta(t)c_2c_3\lambda^2\right)dt + c_4, \\ p(t) = &\, c_1,\ \ \alpha_0(t) = c_2,\ \ \alpha_2(t) = c_3,\ \ \alpha_1(t) = c_3\lambda, \\ \sigma(t) = &\, \frac{-\delta(t)c_3-60\rho(t)c_1^2}{c_3},\ \ \beta(t) = -\frac{6\delta(t)c_1^2}{c_3}, \end{aligned} \tag{15}$$

where c_1, c_2 and c_3 are arbitrary constants.
Again, substituting Equations (15) together with the general solution Equation (5) into the Equation (6), yields the following exact solutions of Equation (2):
When $\lambda^2 - 4\mu > 0$, we have obtained hyperbolic function solution in the form

$$\begin{aligned} u(x,t) = c_2 + c_3\lambda &\left(\frac{1}{2}\frac{\sqrt{(\lambda^2-4\mu)}(a_1\sinh(\zeta)+a_2\cosh(\zeta))}{(a_2\sinh(\zeta)+a_1\cosh(\zeta))} - \frac{\lambda}{2}\right) \\ &+c_3\left(\frac{1}{2}\frac{\sqrt{(\lambda^2-4\mu)}(a_1\sinh(\zeta)+a_2\cosh(\zeta))}{(a_2\sinh(\zeta)+a_1\cosh(\zeta))} - \frac{\lambda}{2}\right)^2, \end{aligned} \tag{16}$$

where $\zeta = \left(\frac{1}{2}((c_1x + q(t))\sqrt{(\lambda^2-4\mu)})\right)$ and
$q(t) = \int\left(\frac{c_1^3}{c_3}\left(6\delta(t)c_2^2 - 8\delta(t)\mu c_2c_3 + 2\delta(t)c_3^2\mu^2\right)\right)dt$
$+\int\left(\frac{c_1^3}{c_3}\left(-16\rho(t)c_1^2c_3\mu^2 + 8\rho(t)c_1^2c_3\lambda^2\mu\right)\right)dt$
$+\int\left(-\rho(t)c_1^2c_3\lambda^4 + \delta(t)c_3^2\lambda^2\mu - \delta(t)c_2c_3\lambda^2\right)dt + c_4$.
When $\lambda^2 - 4\mu < 0$, we have trigonometric function solution

$$\begin{aligned} u(x,t) = c_2 + c_3\lambda &\left(\frac{1}{2}\frac{\sqrt{(-\lambda^2+4\mu)}(-a_2\sin(\zeta)+a_1\cos(\zeta))}{(a_1\sin(\zeta)+a_2\cos(\zeta))} - \frac{\lambda}{2}\right) \\ &+c_3\left(\frac{1}{2}\frac{\sqrt{(-\lambda^2+4\mu)}(-a_2\sin(\zeta)+a_1\cos(\zeta))}{(a_1\sin(\zeta)+a_2\cos(\zeta))} - \frac{\lambda}{2}\right)^2, \end{aligned} \tag{17}$$

where $\zeta = \left(\frac{1}{2}((c_1x + q(t))\sqrt{(-\lambda^2+4\mu)})\right)$ and
$q(t) = \int\left(\frac{c_1^3}{c_3}\left(6\delta(t)c_2^2 - 8\delta(t)\mu c_2c_3 + 2\delta(t)c_3^2\mu^2\right)\right)dt$
$+\int\left(\frac{c_1^3}{c_3}\left(-16\rho(t)c_1^2c_3\mu^2 + 8\rho(t)c_1^2c_3\lambda^2\mu\right)\right)dt$
$+\int\left(-\rho(t)c_1^2c_3\lambda^4 + \delta(t)c_3^2\lambda^2\mu - \delta(t)c_2c_3\lambda^2\right)dt + c_4$.

When $\lambda^2 - 4\mu = 0$, we get rational solution as follows:

$$\begin{aligned} u(x,t) = c_2 + c_3\lambda &\left(\frac{A_2}{A_1+A_2((c_1x+q(t)))} - \frac{\lambda}{2}\right) \\ &+c_3\left(\frac{A_2}{A_1+A_2((c_1x+q(t)))} - \frac{\lambda}{2}\right)^2, \end{aligned} \tag{18}$$

where A_1, A_2, a_1, a_2 and c_3 are arbitrary constants and $q(t) = \int\left(\frac{c_1^3(6\delta(t)c_2^2-3\delta(t)c_2c_3\lambda^2+\frac{3}{8}\delta(t)c_3^2\lambda^4)}{c_3}\right) + c_4$.

Case 4:

$$q(t)=\int\left(-\frac{c_1}{144}(4\mu-\lambda^2)^2(144\rho(t)c_1^4+12\delta(t)c_1^2c_2+\beta(t)c_2^2)\right)dt+c_3$$
$$p(t)=c_1,\quad \alpha_0(t)=\frac{1}{12}c_2\lambda^2+\frac{2}{3}c_2\mu,\quad \alpha_2(t)=c_2,$$
$$\alpha_1(t)=c_2\lambda,\quad \sigma(t)=\frac{1}{6}\frac{-12\delta(t)c_1^2c_2-360\rho(t)c_1^4-\beta(t)c_2^2}{c_1^2c_2}, \tag{19}$$

where c_1,c_2 and c_3 are arbitrary constants.
By using the general solutions of (5) and (19) into (6), we found three types of exact solutions of (2) as follows:
When $\lambda^2-4\mu>0$, we have obtained hyperbolic function solution in the form

$$u(x,t)=c_2\lambda\left(\frac{1}{2}\frac{\sqrt{(\lambda^2-4\mu)}(a_1\sinh(\zeta)+a_2\cosh(\zeta))}{(a_2\sinh(\zeta)+a_1\cosh(\zeta))}-\frac{\lambda}{2}\right)+c_2\left(\frac{1}{2}\frac{\sqrt{(\lambda^2-4\mu)}(a_1\sinh(\zeta)+a_2\cosh(\zeta))}{(a_2\sinh(\zeta)+a_1\cosh(\zeta))}-\frac{\lambda}{2}\right)^2+\frac{1}{12}c_2\lambda^2+\frac{2}{3}c_2\mu, \tag{20}$$

where $\zeta=(\frac{1}{2}((c_1x+q(t))\sqrt{(\lambda^2-4\mu)}))$ and $q(t)=\int\left(-\frac{c_1}{144}(4\mu-\lambda^2)^2(144\rho(t)c_1^4+12\delta(t)c_1^2c_2+\beta(t)c_2^2)\right)dt+c_3$.
When $\lambda^2-4\mu<0$, we have trigonometric function solution

$$u(x,t)=c_2\lambda\left(\frac{1}{2}\frac{\sqrt{(-\lambda^2+4\mu)}(-a_2\sin(\zeta)+a_1\cos(\zeta))}{(a_1\sin(\zeta)+a_2\cos(\zeta))}-\frac{\lambda}{2}\right)+c_2\left(\frac{1}{2}\frac{\sqrt{(-\lambda^2+4\mu)}(-a_2\sin(\zeta)+a_1\cos(\zeta))}{(a_1\sin(\zeta)+a_2\cos(\zeta))}-\frac{\lambda}{2}\right)^2+\frac{1}{12}c_2\lambda^2+\frac{2}{3}c_2\mu, \tag{21}$$

where $\zeta=(\frac{1}{2}((c_1x+q(t))\sqrt{(-\lambda^2+4\mu)}))$ and $q(t)=\int\left(-\frac{c_1}{144}(4\mu-\lambda^2)^2(144\rho(t)c_1^4+12\delta(t)c_1^2c_2+\beta(t)c_2^2)\right)dt+c_3$.

When $\lambda^2-4\mu=0$, we get rational solution as follows:

$$u(x,t)=c_2+c_3\lambda\left(\frac{A_2}{A_1+A_2((c_1x+k))}-\frac{\lambda}{2}\right)+c_3\left(\frac{A_2}{A_1+A_2((c_1x+k))}-\frac{\lambda}{2}\right)^2, \tag{22}$$

where A_1,A_2,a_1,a_2,k and c_3 are arbitrary constants.

Case 5:

$$q(t)=\int\left(-\frac{7}{192}\frac{\delta(t)c_2c_1^3(4\mu-\lambda^2)^2\left(-25\delta(t)c_2+817c_2\delta(t)\left(\frac{41}{416}-\frac{\sqrt{849}}{416}\right)\right)}{-\delta(t)c_2+41c_2\delta(t)\left(\frac{41}{416}-\frac{\sqrt{849}}{416}\right)}\right)dt+c_3$$
$$p(t)=c_1,\quad \alpha_1(t)=2c_2\left(\frac{\lambda}{2}-\frac{\sqrt{(-4\mu+\lambda^2)}}{2}\right),\quad \alpha_2(t)=c_2,$$
$$\alpha_0(t)=\frac{-4\delta(t)c_2^2\mu+\delta(t)c_2^2\lambda^2+116c_2^2\mu\delta(t)\left(\frac{41}{416}-\frac{\sqrt{849}}{416}\right)-41c_2^2\lambda^2\delta(t)\left(\frac{41}{416}-\frac{\sqrt{849}}{416}\right)}{48\left(\frac{41}{416}-\frac{\sqrt{849}}{416}\right)c_2\delta(t)}+\frac{48c_2^2\lambda\delta(t)\left(\frac{41}{416}-\frac{\sqrt{849}}{416}\right)\left(\frac{\lambda}{2}-\frac{\sqrt{(-4\mu+\lambda^2)}}{2}\right)}{48\left(\frac{41}{416}-\frac{\sqrt{849}}{416}\right)c_2\delta(t)},$$
$$\sigma(t)=\frac{1}{2}\frac{-3\delta(t)c_2-56c_2\delta(t)\left(\frac{41}{416}-\frac{\sqrt{849}}{416}\right)}{c_2},$$
$$\beta(t)=\frac{48c_2\delta(t)c_1^2\left(\frac{41}{416}-\frac{\sqrt{849}}{416}\right)-3\delta(t)c_1^2c_2}{c_2^2},$$
$$\rho(t)=\frac{1}{3}\frac{\left(\frac{41}{416}-\frac{\sqrt{849}}{416}\right)c_2\delta(t)}{c_1^2}, \tag{23}$$

where c_1,c_2 and c_3 are arbitrary constants.
The following three types of solutions of (2) are found by substituting the general solutions of (5) into (6) and using (19):
When $\lambda^2-4\mu>0$, we have obtained hyperbolic function solution in the form

$$u(x,t)=c_2(\lambda-\sqrt{(-4\mu+\lambda^2)})\left(\frac{1}{2}\frac{\sqrt{(\lambda^2-4\mu)}(a_1\sinh(\zeta)+a_2\cosh(\zeta))}{(a_2\sinh(\zeta)+a_1\cosh(\zeta))}-\frac{\lambda}{2}\right)+c_2\left(\frac{1}{2}\frac{\sqrt{(\lambda^2-4\mu)}(a_1\sinh(\zeta)+a_2\cosh(\zeta))}{(a_2\sinh(\zeta)+a_1\cosh(\zeta))}-\frac{\lambda}{2}\right)^2+\alpha_0(t), \tag{24}$$

where $\zeta=(\frac{1}{2}((c_1x+q(t))\sqrt{(\lambda^2-4\mu)}))$ and $q(t)=\int\left(-\frac{7}{192}\frac{\delta(t)c_2c_1^3(4\mu-\lambda^2)^2\left(-25\delta(t)c_2+817c_2\delta(t)\left(\frac{41}{416}-\frac{\sqrt{849}}{416}\right)\right)}{-\delta(t)c_2+41c_2\delta(t)\left(\frac{41}{416}-\frac{\sqrt{849}}{416}\right)}\right)dt+c_3$
When $\lambda^2-4\mu<0$, we have trigonometric function solution

$$u(x,t)=c_2(\lambda-\sqrt{(-4\mu+\lambda^2)})\left(\frac{1}{2}\frac{\sqrt{(-\lambda^2+4\mu)}(-a_2\sin(\zeta)+a_1\cos(\zeta))}{(a_1\sin(\zeta)+a_2\cos(\zeta))}-\frac{\lambda}{2}\right)+c_2\left(\frac{1}{2}\frac{\sqrt{(-\lambda^2+4\mu)}(-a_2\sin(\zeta)+a_1\cos(\zeta))}{(a_1\sin(\zeta)+a_2\cos(\zeta))}-\frac{\lambda}{2}\right)^2+\alpha_0(t), \tag{25}$$

where $\zeta=(\frac{1}{2}((c_1x+q(t))\sqrt{(-\lambda^2+4\mu)}))$ and $q(t)=\int\left(-\frac{7}{192}\frac{\delta(t)c_2c_1^3(4\mu-\lambda^2)^2\left(-25\delta(t)c_2+817c_2\delta(t)\left(\frac{41}{416}-\frac{\sqrt{849}}{416}\right)\right)}{-\delta(t)c_2+41c_2\delta(t)\left(\frac{41}{416}-\frac{\sqrt{849}}{416}\right)}\right)dt+c_3$.
When $\lambda^2-4\mu=0$, we get rational solution as follows:

$$u(x,t)=\frac{1}{4}c_2\lambda^2+c_2\lambda\left(\frac{A_2}{A_1+A_2((c_1x+k))}-\frac{\lambda}{2}\right)+c_2\left(\frac{A_2}{A_1+A_2((c_1x+k))}-\frac{\lambda}{2}\right)^2, \tag{26}$$

where A_1,A_2,a_1,a_2,k and c_3 are arbitrary constants.

4 Conclusions

The generalized $\left(\frac{G'}{G}\right)$-expansion method has been successfully used to obtain some new exact solutions of generalized fifth order KdV equation with time-dependent coefficients. We found a rich variety of exact solutions which include hyperbolic, trigonometric, and rational functions involving arbitrary parameters. Also, We have presented the graphical representation of obtained solutions, so that they can depict the importance of each obtained solution and physically interpret their importance. The free parameters c_1, c_2, c_3, c_4, k and especially the arbitrary function $\delta(t), \beta(t), \rho(t)$ in various solutions, can make us discuss the behaviors of solutions and also provide us enough freedom to construct solutions that may be related to real physical problem. Note that the nonlinear evolution equation proposed in the present paper is difficult and more general. Therefore, the solutions of generalized fifth order KdV equation with time-dependent coefficients equation in this paper have many potential applications in physics.

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