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Global Solution to a Nonlinear Fractional Differential Equation for the Caputo–Fabrizio Derivative

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Abstract: In this article we prove existence and uniqueness of global solution to an initial value problem for a nonlinear fractional differential equation with a Caputo–Fabrizio (CF) derivative. We provide a new compact formula for the computation of the CF derivative to power functions (which is given in terms of Mittag–Leffler functions). We also give the convergence to classical derivatives for a regular class of functions when the order of the CF derivative tends to one, as well as some other useful properties.

Keywords: Fractional ordinary differential equations, Caputo-Fabrizio derivative, Mittag-Lefler function, global solution.

1 Introduction

Fractional calculus has been developed in the last fifty years cutting across almost all areas of mathematics, both pure and applied. In the field of ordinary fractional differential equations, the fractional derivatives in the Riemann–Liouville sense or in the Caputo sense where hardly studied. See for example the books [1,2,3] where properties and applications are treated in detail. The Caputo derivative of order $\alpha \in (0,1)$ was defined by Caputo in 1967 [4] as

$$ {}_{a}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau. \tag{1}$$

The Caputo derivative is usually considered for modeling process involving memory effects, diffusion in non-homogeneous domains, or in the study of anomalous diffusion, which is closely linked to non-Brownian motions. Works in this direction are e.g. [5,6,7,8,9].
Clearly, from definition (1), the Caputo derivative is an integro–differential operator involving a singular kernel, given by the function

$$K(t) = \begin{cases} \frac{t^{-\alpha}}{\Gamma(1-\alpha)} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0 \end{cases}. \tag{2}$$

We can observe that the fractional derivative in the Caputo sense is a generalized weighted backward sum where the kernel (2) assigns more weight ("importance") to the nearest rates of changes of function f.
In the aim to avoid the singular kernel (2), and motivated by physical situations related to the need of an exponential kernel in some constitutive equations (see for example the works [10,11]), Caputo and Fabrizio defined in 2015 [12] a new fractional derivative with no singular kernel. This fractional derivative named Caputo-Fabrizio derivative (CF), is defined as

$$ {}_{a}^{CF}D^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(\tau) e^{-\frac{\alpha(t-\tau)}{1-\alpha}} \mathrm{d}\tau \tag{3}$$

for every $f \in W^1(a,b) = \{f \in \mathscr{C}[a,b] / f' \in L^1(a,b)\}$, $-\infty \leq a < b \leq +\infty$, $\alpha \in (0,1)$, where $M(\cdot)$ is a normalized function such that $M(0) = M(1) = 1$.

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The above definition was already treated in different areas of applied mathematics. In Gómez–Aguilar et al. [13] a representation of the fractional diffusion equation and a fractional diffusion–advection equation by applying the CF derivative is developed. In [14] a problem associated to a plate that is oscillating in its own plane with isotherm boundary condition is modelled by using the CF derivative. Also, in [15], the Dirichlet problem and source problem for the fractional advection–diffusion equation with time fractional CF derivative is studied in the half-plane.

Integro–differential operators with non-singular kernels has been widely studied in the area of mathematical analysis, and clearly, (3) corresponds to an integro–differential operator. In this paper, we want to develop some properties, examples and even the main objective, which is the demonstration of an existence and uniqueness theorem, focusing on the integro-differential operator (3) viewed as a fractional derivative, paying special attention to the notation, the spaces of functions considered and the type of convergence given in each case.

Following this purpose, we will provide new formulas for the computation of the fractional derivative (3) to powers and trigonometric functions, being these formulas more compact than the previously given in the literature. Furthermore, the most important result is to give global existence and uniqueness of a solution to an initial value problem for a nonlinear fractional differential equation for the CF derivative. The proof is based on the existence for short times given by Lozada and Nieto in [16] and a translation formula (which will be enunciated later in Proposition 2 item 2). It is worth noting that, due to the exposed results are purely mathematical, we will consider that the normalized function M defined in (3) is given by

$$M(\alpha)=1, \qquad \text{for every } \alpha\in(0,1).$$

The paper is organized as follows: In Section 2 some useful properties of the Caputo-Fabrizio derivative (3) are presented: the convergence to the classical derivatives, the translation formula, the analysis of the inverse operator, the fractional derivation of power functions in terms of the Mittag–Leffler functions, among others. In Section 3, an initial value problem for the governing equation

$${}^{CF}_{\ a}Df(t)=\varphi(t,f(t))$$

is considered, and the existence and uniqueness of a global solution is proved by using a previous result of existence for short times given by Losada an Nieto in [16].

2 Basic definitions and calculations

Hereinafter we denote by ${}^{CF}D^{\alpha}$ to the fractional derivative of Caputo–Fabrizio with lower limit $a=0$.

Definition 1. *For every $n\in\mathbb{N}_0$ and $\alpha\in(0,1)$, the fractional Caputo Fabrizio derivative of order $n+\alpha$ is defined as*

$${}^{CF}_{\ a}D^{(n+\alpha)}f(t):={}^{CF}_{\ a}D^{\alpha}\left(\frac{d^n}{dt^n}f(t)\right) \tag{4}$$

for every $f\in W^{(n+1)}(a,b)=\{f\in\mathscr{C}^{(n)}[a,b]/f^{(n+1)}\in L^1(a,b)\}$, $-\infty\leq a<b\leq+\infty$.

Note 1. The case $n=0$ is the one given in (3).

Proposition 1. *Let $\alpha\in(0,1)$, $n\in\mathbb{N}$ and $f\in W^{(n+1)}(a,b)$. Then*

1. For every $t\in[a,b]$, $\lim_{\alpha\searrow 0}{}^{CF}_{\ a}D^{(\alpha+n)}f(t)=\int_a^t f^{(n+1)}(\tau)\mathrm{d}\tau$.

2. If $f^{(n+1)}$ is a sectional continuous function with a finite number of roots in (a,b), then $\lim_{\alpha\nearrow 1}{}^{CF}_{\ a}D^{(\alpha+n)}f(t)=f^{(n+1)}(t)$ a. e. $t\in(a,b)$.

In particular:

1'. If $f\in\mathscr{C}^{(n+1)}[a,b]$ then $\lim_{\alpha\searrow 0}{}^{CF}_{\ a}D^{(\alpha+n)}f(t)=f^{(n)}(t)-f^{(n)}(a)$ for all $t\in[a,b]$.

2'. If $f\in\mathscr{C}^{(n+2)}[a,b]$ then $\lim_{\alpha\nearrow 1}{}^{CF}_{\ a}D^{(\alpha+n)}f(t)=f^{(n+1)}(t)$ for all $t\in(a,b]$.

Proof.1. According to definition 1, it is sufficient to consider the case $n=0$. Let $f \in W^1(a,b)$ be. Note that

$$\left|\frac{1}{1-\alpha}\int_a^t f'(\tau)e^{-\frac{\alpha(t-\tau)}{1-\alpha}}\mathrm{d}\tau - \int_a^t f'(\tau)\mathrm{d}\tau\right| = \left|\frac{1}{1-\alpha}\int_a^t f'(\tau)\left[e^{-\frac{\alpha(t-\tau)}{1-\alpha}} - (1-\alpha)\right]\mathrm{d}\tau\right|.$$

Then, taking the limit when α tends to zero and applying Lebesgue Convergence theorem, limit 1 holds for every $t \in [a,b]$.

2. Now we take the L^1 norm, given by

$$||f||_{L^1(a,b)} = \int_a^b |f(t)|\,\mathrm{d}t.$$

Let $g(t) = \dfrac{1}{1-\alpha}\displaystyle\int_a^t f'(\tau)e^{-\frac{\alpha(t-\tau)}{1-\alpha}}\mathrm{d}\tau - f'(t)$ be. Being f' a sectional continuous function in (a,b) it follows that g is a continuous function in (a,b). Also, taking into account that f' has a finite number of roots, we can divide the interval (a,b) in M subintervals where g conserves its sign in every subinterval (a_i,a_{i+1}) for all $i=0,\dots,M-1$, $a_0=a$ and $a_M=b$. The L^1 norm becomes

$$||g||_{L^1(a,b)} = \int_a^b |g(t)|\,\mathrm{d}t = \sum_{i=0}^{M-1}\int_{a_i}^{a_{i+1}}|g(t)|\mathrm{d}t = \sum_{i=0}^{M-1}(-1)^{k_i}\int_{a_i}^{a_{i+1}} g(t)\mathrm{d}t, \tag{5}$$

where $k_i = \begin{cases} 0, & \text{if } g(t) \geq 0 \text{ in } (a_i,a_{i+1}) \\ 1, & \text{if } g(t) < 0 \text{ in } (a_i,a_{i+1}) \end{cases}$, for every $i=0,\dots,M-1$.

Applying Fubini's Theorem in each subinterval we have

$$\begin{aligned}
&\int_{a_i}^{a_{i+1}} g(t)\mathrm{d}t \\
&= \int_{a_i}^{a_{i+1}}\left[\frac{1}{1-\alpha}\int_a^t f'(\tau)e^{-\frac{\alpha(t-\tau)}{1-\alpha}}\mathrm{d}\tau - f'(t)\right]\mathrm{d}t \\
&= \int_{a_i}^{a_{i+1}}\int_a^t \frac{1}{1-\alpha}f'(\tau)e^{-\frac{\alpha(t-\tau)}{1-\alpha}}\mathrm{d}\tau\mathrm{d}t - \int_{a_i}^{a_{i+1}} f'(t)\mathrm{d}t \\
&= \int_a^{a_i}\int_{a_i}^{a_{i+1}}\frac{1}{1-\alpha}f'(\tau)e^{-\frac{\alpha(t-\tau)}{1-\alpha}}\mathrm{d}t\mathrm{d}\tau + \\
&\quad + \int_{a_i}^{a_{i+1}}\int_\tau^{a_{i+1}}\frac{1}{1-\alpha}f'(\tau)e^{-\frac{\alpha(t-\tau)}{1-\alpha}}\mathrm{d}t\mathrm{d}\tau - \int_{a_i}^{a_{i+1}} f'(\tau)\mathrm{d}\tau
\end{aligned}$$

$$\begin{aligned}
&= \int_a^{a_i} f'(\tau)\left[-\frac{1}{\alpha}\left(e^{-\frac{\alpha(a_{i+1}-\tau)}{1-\alpha}} - e^{-\frac{\alpha(a_i-\tau)}{1-\alpha}}\right)\right]\mathrm{d}\tau + \\
&\int_{a_i}^{a_{i+1}} f'(\tau)e^{\frac{\alpha\tau}{1-\alpha}}\left(\int_\tau^{a_{i+1}}\frac{1}{1-\alpha}e^{-\frac{\alpha t}{1-\alpha}}\mathrm{d}t\right)\mathrm{d}\tau - \int_{a_i}^{a_{i+1}} f'(\tau)\mathrm{d}\tau \\
&= \int_a^{a_i} f'(\tau)\left[-\frac{1}{\alpha}\left(e^{-\frac{\alpha(a_{i+1}-\tau)}{1-\alpha}} - e^{-\frac{\alpha(a_i-\tau)}{1-\alpha}}\right)\right]\mathrm{d}\tau + \\
&+ \int_{a_i}^{a_{i+1}} f'(\tau)\left(-\frac{1}{\alpha}e^{-\frac{\alpha(a_{i+1}-\tau)}{1-\alpha}} + \frac{1}{\alpha} - 1\right)\mathrm{d}\tau.
\end{aligned} \tag{6}$$

Also, for every $\alpha \geq \frac{1}{2}$ it holds that

$$\begin{aligned}
\left|-\frac{1}{\alpha}e^{-\frac{\alpha(a_{i+1}-\tau)}{1-\alpha}} + \frac{1}{\alpha} - 1\right| &\leq \left|-\frac{1}{\alpha}e^{-\frac{\alpha(a_{i+1}-\tau)}{1-\alpha}}\right| + \left|\frac{1}{\alpha} - 1\right| \leq \\
&\leq 2e^{-\frac{\alpha(a_{i+1}-\tau)}{1-\alpha}} + 1 \leq 3
\end{aligned} \tag{7}$$

and

$$\left|-\frac{1}{\alpha}\left(e^{-\frac{\alpha(a_{i+1}-\tau)}{1-\alpha}} - e^{-\frac{\alpha(a_i-\tau)}{1-\alpha}}\right)\right| \leq 4. \tag{8}$$

Then, replacing (6) in (5) and applying the Lebesgue Convergence Theorem to each part of the finite sum (5) (due to inequalities (7) and (8)), it follows that

$$\begin{aligned}
\lim_{\alpha\nearrow 1}||{}^{CF}_{\ a}D^{\alpha}f-f'||_{L^1(a,b)} &= \lim_{\alpha\nearrow 1}\int_a^b\left|{}^{CF}_{\ a}D^{\alpha}f(t)-f'(t)\right|\mathrm{d}t=\\
&=\lim_{\alpha\nearrow 1}\int_a^b\left|\frac{1}{1-\alpha}\int_a^t f'(\tau)e^{-\frac{\alpha(t-\tau)}{1-\alpha}}\mathrm{d}\tau-f'(t)\right|\mathrm{d}t=\\
&=\sum_{i=0}^{M-1}(-1)^{k_i}\int_a^{a_i}f'(\tau)\lim_{\alpha\nearrow 1}\left[-\frac{1}{\alpha}\left(e^{-\frac{\alpha(a_{i+1}-\tau)}{1-\alpha}}-e^{-\frac{\alpha(a_i-\tau)}{1-\alpha}}\right)\right]\mathrm{d}\tau+\\
&\quad+\int_{a_i}^{a_{i+1}}f'(\tau)\lim_{\alpha\nearrow 1}\left(-\frac{1}{\alpha}e^{-\frac{\alpha(a_{i+1}-\tau)}{1-\alpha}}+\frac{1}{\alpha}-1\right)\mathrm{d}\tau\\
&=0.
\end{aligned} \tag{9}$$

Then

$$\lim_{\alpha\nearrow 1}{}^{CF}_{\ a}D^{\alpha}f(t)=f'(t),\quad \text{a.e. in } (a,b).$$

The limit in 1' follows by applying the Fundamental Theorem of Calculus in 1, after assuming that $f\in\mathscr{C}^{(1)}[a,b]$. Finally, integrating by parts under the assumption that $f\in\mathscr{C}^2[a,b]$ in (3) gives

$${}^{CF}_{\ a}D^{\alpha}f(t)=\frac{1}{\alpha}\left[f'(t)-f'(a)e^{-\frac{\alpha(t-a)}{1-\alpha}}-\int_a^t f''(\tau)e^{-\frac{\alpha(t-\tau)}{1-\alpha}}\mathrm{d}\tau\right]. \tag{10}$$

Note that $\lim_{\alpha\nearrow 1}f'(a)e^{-\frac{\alpha(t-a)}{1-\alpha}}=0$ for every $t>a$. Therefore, taking the limit when $\alpha\nearrow 1$ in (10) the limit in 2' holds.

*Note 2.*The previous proposition enables us to redefine the fractional Caputo–Fabrizio derivative given in Definition 3 for every $\alpha\in(0,1]$. Roughly speaking, we can say that the fractional Caputo–Derivative is a left–continuous operator at any positive integer.

*Remark.*We would like to highlight that the convergence given in Proposition 1 item 2', does not necessary holds at the lower extreme $t=a$. It will be shown in Example 2 that

$${}^{CF}D^{\alpha}\sin t=\frac{1}{(1-\alpha)^2+\alpha^2}\left(\alpha\cos t+(1-\alpha)\sin t-\alpha e^{\frac{-\alpha t}{1-\alpha}}\right). \tag{11}$$

From (11) it follows that ${}^{CF}D^{\alpha}\sin 0=0$ for every $\alpha\in(0,1)$, whereas that $\lim_{\alpha\nearrow 1}{}^{CF}D^{\alpha}\sin t=\cos t$ which tends to 1 when t tends to 0.

Proposition 2.*The following properties for the Caputo-Fabrizio derivative hold:*

1.If $u\in W^1(a,b)$ and $f(t)={}^{CF}_{\ a}D^{\alpha}u(t)$, then $f(a)=0$.
2.Let $g\in W^1(a,b)$ be and $\alpha\in(0,1)$. Then for every $a>0$, the following translation formula is valid:

$${}^{CF}_{\ a}D^{\alpha}g(t)={}^{CF}D^{\alpha}g(t)-\exp\left\{\frac{-\alpha(t-a)}{1-\alpha}\right\}{}^{CF}D^{\alpha}g(a). \tag{12}$$

*Proof.*1. Being u a function in $W^1(a,b)$, it yields that $u\in\left\{v\in L^1(a,b):\ v'\in L^1(a,b)\right\}$. Also we have that $h(\cdot)=e^{-\frac{\alpha(t-\cdot)}{1-\alpha}}$ is a continuous and hence en bounded function in $[a,b]$. Then $u(\cdot)h(\cdot)\in\left\{v\in L^1(a,b):\ v'\in L^1(a,b)\right\}$ and from Theorem 8.1 of Chapter 8 of Brezis [17] , we have that

$$\int_a^t\left(u(\tau)e^{-\frac{\alpha(t-\tau)}{1-\alpha}}\right)'\mathrm{d}\tau=u(\tau)e^{-\frac{\alpha(t-\tau)}{1-\alpha}}\Big|_a^t. \tag{13}$$

Using (13) in definition (3) it holds that

$$f(t)={}^{CF}_{\ a}D^{\alpha}u(t)=\frac{1}{1-\alpha}\left[u(t)-u(a)e^{-\frac{\alpha(t-a)}{1-\alpha}}-\int_a^t u(\tau)e^{-\frac{\alpha(t-\tau)}{1-\alpha}}\frac{\alpha}{1-\alpha}\mathrm{d}\tau\right].$$

Taking the limit when $t\searrow a$ we get that $f(a)=0$.

2. Relation (12) is due to the property of the integral over adjacent intervals.

Let us make the inverse reasoning. Suppose that we want to calculate a "Caputo–Fabrizio primitive" of some given function f. That is, we want to find a function u such that

$$ {}_{a}^{CF}D^{\alpha}u(t) = f(t). \tag{14}$$

Following the procedure described in [16] (that is, differentiating (14) respect on time to both sides and integrating later), from Proposition 2-1 we have

$$u(t)-u(a) = \alpha\int_a^t f(\tau)\mathrm{d}\tau + (1-\alpha)[f(t)-f(a)] = \alpha\int_a^t f(\tau)\mathrm{d}\tau + (1-\alpha)f(t). \tag{15}$$

Calling ${}_{a}^{CF}I^{\alpha}f(t)$ to the right side in (15), the Barrow's rule for the fractional integral of Caputo–Fabrizio holds:

$$u(t)-u(a) = {}_{a}^{CF}I^{\alpha}f(t) \tag{16}$$

and the following definition becomes natural.

Definition 2. *For every $\alpha \in (0,1]$ and $f \in L^1(a,b)$ the fractional integral of Caputo-Fabrizio of f is defined by*

$$ {}_{a}^{CF}I^{\alpha}f(t) = (1-\alpha)f(t) + \alpha\int_a^t f(\tau)\mathrm{d}\tau, \qquad t \geq a. \tag{17}$$

Proposition 3. *Let f be a function in $L^1(a,b)$ or $W^1(a,b)$ as required. Then*

1. *The fractional integral of Caputo-Fabrizio is an inverse operator of the fractional derivative of Caputo-Fabrizio if and only if $f(a) = 0$. That is,*
$$ {}_{a}^{CF}I^{\alpha}\left({}_{a}^{CF}D^{\alpha}f(t)\right) = f(t) \Leftrightarrow f(a) = 0.$$
2. *The fractional derivative of Caputo-Fabrizio is an inverse operator of the fractional integral of Caputo-Fabrizio if and only if $f(a) = 0$.*
$$ {}_{a}^{CF}D^{\alpha}\left({}_{a}^{CF}I^{\alpha}f(t)\right) = f(t) \Leftrightarrow f(a) = 0.$$

Proof. By using Proposition 2-1 and Fubini's theorem it holds that

$$ {}_{a}^{CF}I^{\alpha}\left({}_{a}^{CF}D^{\alpha}f(t)\right) = f(t) - f(a),$$

and then 1. holds.
Integration by parts yields that

$$ {}_{a}^{CF}D^{\alpha}\left({}_{a}^{CF}I^{\alpha}f(t)\right) = f(t) - f(a)\exp\left\{-\frac{\alpha t}{1-\alpha}\right\}, \tag{18}$$

and then 2. holds.

Note 3. It is interesting the fact that the fractional derivative ${}_{a}^{CF}D^{\alpha}$, in general, is not a left inverse operator of the fractional integral ${}_{a}^{CF}I^{\alpha}$, which is not the case when we consider fractional derivatives in the Caputo and Riemann–Liouville sense. In fact, these derivatives are both left inverse operators of the fractional integral of Riemann–Liouville (see for example [1]).
However, when $\alpha \nearrow 1$ we hope to recover, as we know that $D^1(I^1 f) = f$ for every integrable function f. Making α tends to 1 in equation (18) it holds that, for every $t \in [a,b]$

$$\lim_{\alpha\nearrow 1} {}_{a}^{CF}D^{\alpha}\left({}_{a}^{CF}I^{\alpha}f(t)\right) = \lim_{\alpha\nearrow 1}\left[f(t) - f(a)\exp\left\{-\frac{\alpha t}{1-\alpha}\right\}\right] = f(t).$$

Proposition 4. *Let $\alpha \in (0,1)$ and $\beta > 0$ be. Then*

$$ {}_{a}^{CF}D^{\alpha}(t-a)^{\beta} = \frac{\beta}{\alpha}(t-a)^{\beta-1}\left[1-\Gamma(\beta)E_{1,\beta}\left(-\frac{\alpha}{1-\alpha}(t-a)\right)\right], \tag{19}$$

where $E_{\alpha,\beta}(\cdot)$ is the Mittag–Leffler function defined for every $t \in \mathbb{R}$ by

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+\beta)}$$

and $\Gamma(\cdot)$ is the Gamma function.

Proof. Recall the Beta function defined by

$$B(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}\mathrm{d}t, \qquad z>0,\, w>0. \tag{20}$$

A known property of this function (see p. 10 of [18]) is that

$$B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \tag{21}$$

From (20) and (21) it easily follows that

$$\int_a^t(\tau-a)^{z-1}(t-\tau)^{w-1}\mathrm{d}\tau=B(z,w)(t-a)^{z+w-1}=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}(t-a)^{z+w-1}. \tag{22}$$

Now, by using the uniform convergence of the series we have

$$\begin{aligned}
{}_a^{CF}D^{\alpha}(t-a)^{\beta}&=\frac{1}{1-\alpha}\int_a^t\beta(\tau-a)^{\beta-1}e^{-\frac{\alpha}{1-\alpha}(t-\tau)}\mathrm{d}\tau\\
&=\frac{\beta}{1-\alpha}\int_a^t(\tau-a)^{\beta-1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\alpha}{1-\alpha}\right)^k(t-\tau)^k\mathrm{d}\tau\\
&=\frac{\beta}{1-\alpha}\sum_{k=0}^{\infty}\int_a^t(\tau-a)^{\beta-1}\frac{(-1)^k}{k!}\left(\frac{\alpha}{1-\alpha}\right)^k(t-\tau)^k\mathrm{d}\tau.
\end{aligned} \tag{23}$$

Taking $z=\beta$ and $w=k+1$ in (22) and replacing then in (23) we get

$$\begin{aligned}
{}_a^{CF}D^{\alpha}(t-a)^{\beta}&=\frac{\beta}{1-\alpha}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\alpha}{1-\alpha}\right)^k\int_a^t(\tau-a)^{\beta-1}(t-\tau)^k\mathrm{d}\tau\\
&=\frac{\beta}{1-\alpha}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\alpha}{1-\alpha}\right)^k(t-a)^{\beta+k}\frac{\Gamma(\beta)k!}{\Gamma(\beta+k+1)}\\
&=\frac{\beta\Gamma(\beta)}{\alpha}(t-a)^{\beta-1}\left[-\sum_{k=1}^{\infty}\frac{\left(-\frac{\alpha}{1-\alpha}(t-a)\right)^k}{\Gamma(k+\beta)}\right]\\
&=\frac{\beta\Gamma(\beta)}{\alpha}(t-a)^{\beta-1}\left[\frac{1}{\Gamma(\beta)}-E_{1,\beta}\left(-\frac{\alpha}{1-\alpha}(t-a)\right)\right]\\
&=\frac{\beta}{\alpha}(t-a)^{\beta-1}\left[1-\Gamma(\beta)E_{1,\beta}\left(-\frac{\alpha}{1-\alpha}(t-a)\right)\right].
\end{aligned}$$

Remark. From Eq. (7) of Chapter 18.1 in Erdélyi [19] we deduce that

$$\lim_{x\to\infty}E_{1,\beta}(-x)=0, \quad \forall\beta>0.$$

Then

$$\lim_{\alpha\nearrow 1}\frac{\beta}{\alpha}(t-a)^{\beta-1}\left[1-\Gamma(\beta)E_{1,\beta}\left(-\frac{\alpha}{1-\alpha}(t-a)\right)\right]=\beta(t-a)^{\beta-1}.$$

Remark. Proposition 4 can be used to give an example of a function f which is not differentiable (in the classical sense) at $t=a$ but it is "Caputo-Fabrizio differentiable" at $t=a$. Taking $a=0$ and $\beta=\alpha/2$ in (19) we have

$$\begin{aligned}
{}^{CF}D^{\alpha}t^{\alpha/2}&=\frac{\alpha/2}{\alpha}t^{\alpha/2-1}\left[1-\Gamma(\alpha/2)E_{1,\alpha/2}\left(-\frac{\alpha}{1-\alpha}t\right)\right]\\
&=\frac{1}{2}t^{\alpha/2-1}\left[1-\Gamma(\alpha/2)\sum_{k=0}^{\infty}\frac{\left(-\frac{\alpha}{1-\alpha}t\right)^k}{\Gamma\left(k+\frac{\alpha}{2}\right)}\right]\\
&=\frac{1}{2}t^{\alpha/2-1}\left[-\Gamma(\alpha/2)\sum_{k=1}^{\infty}\frac{\left(-\frac{\alpha}{1-\alpha}t\right)^k}{\Gamma\left(k+\frac{\alpha}{2}\right)}\right]\\
&=\frac{-\Gamma(\alpha/2)}{2}\left(\frac{1-\alpha}{\alpha}\right)^{\alpha/2-1}\sum_{k=1}^{\infty}\frac{(-1)^k\left(\frac{\alpha}{1-\alpha}t\right)^{k+\alpha/2-1}}{\Gamma\left(k+\frac{\alpha}{2}\right)}.
\end{aligned} \tag{24}$$

And this function can be defined by 0 at $t = 0$ because $k + \alpha/2 - 1 > 0$ for every $k \geq 1$.

Corollary 1.*If $\alpha \in (0,1)$ and $m \in \mathbb{N}$, then*

$$ {}^{CF}_{a}D^{\alpha}(t-a)^m = \frac{m}{\alpha}(t-a)^{m-1} + \frac{m!}{\alpha}\sum_{k=0}^{m-2}\left(-\frac{1-\alpha}{\alpha}\right)^{m-k-1}\frac{(t-a)^k}{k!} - \frac{m!}{\alpha}\left(-\frac{1-\alpha}{\alpha}\right)^{m-1}\exp\left\{-\frac{\alpha}{1-\alpha}(t-a)\right\}. \tag{25} $$

In the right side of (25) we see that the first term (which is the dominant one) does not tends to zero when α tends to 1. The second addend is closely related to the memory effect of the operator, and the third term is the "exponential perturbation" which is a natural consequence of the considered operator.

*Proof.*Taking into account that the Mittag–Leffler function verifies that $E_{1,m}(t) = \frac{1}{t^{m-1}}\left[\exp\{t\} - \sum_{k=0}^{m-2}\frac{t^k}{k!}\right]$, and replacing it in (19) we have

$$ \begin{aligned} {}^{CF}_{a}D^{\alpha}(t-a)^m &= \frac{m}{\alpha}(t-a)^{m-1}\left[1 - \Gamma(m)E_{1,m}\left(-\frac{\alpha}{1-\alpha}(t-a)\right)\right] \\ &= \frac{m}{\alpha}(t-a)^{m-1}\left\{1 - \frac{\Gamma(m)}{\left[-\frac{\alpha}{1-\alpha}(t-a)\right]^{m-1}}\left[\exp\left\{-\frac{\alpha}{1-\alpha}(t-a)\right\} - \sum_{k=0}^{m-2}\frac{\left[-\frac{\alpha}{1-\alpha}(t-a)\right]^k}{k!}\right]\right\} \\ &= \frac{m}{\alpha}(t-a)^{m-1} + \frac{1}{\alpha}\sum_{k=0}^{m-2}\frac{m!}{k!}\left(-\frac{1-\alpha}{\alpha}\right)^{m-k-1}(t-a)^k - \frac{m!}{\alpha}\left(-\frac{1-\alpha}{\alpha}\right)^{m-1}\exp\left\{-\frac{\alpha}{1-\alpha}(t-a)\right\} \end{aligned} $$

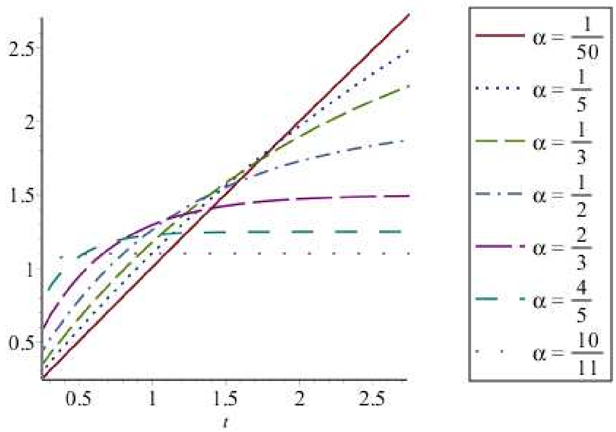

Fig. 1: ${}^{CF}D^{\alpha}t$ for some values of α.

Let us see other examples related to some classical functions.

*Example 1.*The exponential function
It is easy to see that

$$ {}^{CF}D^{\alpha}e^{ct} = \begin{cases} \frac{c}{c(1-\alpha)+\alpha}\left[e^{ct} - e^{-\frac{\alpha t}{1-\alpha}}\right] & \text{if } c(1-\alpha)+\alpha \neq 0, \\ \frac{ct}{1-\alpha}e^{\frac{-\alpha t}{1-\alpha}} & \text{if } c(1-\alpha)+\alpha = 0. \end{cases} \tag{26} $$

Taking $c = 1$ in (26)

$$ {}^{CF}D^{\alpha}e^{t} = e^{t} - e^{-\frac{\alpha t}{1-\alpha}}, $$

and the expected limit holds:

$$ \lim_{\alpha \nearrow 1} {}^{CF}D^{\alpha}e^{t} = e^{t}. $$

Another interesting result is when $a = -\infty$ and $c > -\frac{\alpha}{1-\alpha}$. In fact:

$$ \begin{aligned} {}^{CF}_{-\infty}D^{\alpha}e^{ct} &= \frac{1}{1-\alpha}\int_{-\infty}^{t} ce^{c\tau}e^{-\frac{\alpha(t-\tau)}{1-\alpha}}d\tau = \\ &= \frac{c}{c(1-\alpha)+\alpha}\left(e^{\frac{c(1-\alpha)t}{1-\alpha}} - \lim_{s\to-\infty} e^{\frac{(c-c\alpha+\alpha)s-\alpha t}{1-\alpha}}\right) \\ &= \frac{c}{c(1-\alpha)+\alpha}e^{ct} \end{aligned} $$

which gives the following special result for $c = 1$:

$$ {}^{CF}_{-\infty}D^{\alpha}e^{t} = e^{t}. $$

*Example 2.*The fractional derivative of the sine function.
Integrating by parts it holds

$$ {}^{CF}D^{\alpha}\sin(t) = \frac{1}{1-\alpha}\sin(t) + \frac{\alpha}{(1-\alpha)^2}\cos(t) - \frac{\alpha}{(1-\alpha)^2}e^{-\frac{\alpha t}{1-\alpha}} - \frac{\alpha^2}{(1-\alpha)^2}{}^{CF}D^{\alpha}\sin(t). \tag{27} $$

Then we get

$$ {}^{CF}D^{\alpha}\sin(t) = \frac{1}{(1-\alpha)^2+\alpha^2}\left(\alpha\cos(t) + (1-\alpha)\sin(t) - \alpha e^{-\frac{\alpha t}{1-\alpha}}\right). \tag{28} $$

Noting that $e^{-\frac{\alpha t}{1-\alpha}}$ tends to 0 when $\alpha \nearrow 1$, it follows that

$$ \lim_{\alpha\nearrow 1} {}^{CF}D^{\alpha}\sin t = \cos(t). $$

Analogously, we have

$$ {}^{CF}D^{\alpha}\cos(t) = \frac{1}{(1-\alpha)^2+\alpha^2}\left(-\alpha\sin(t) + (1-\alpha)\cos(t) - (1-\alpha)e^{-\frac{\alpha t}{1-\alpha}}\right) $$

and

$$ \lim_{\alpha\nearrow 1} {}^{CF}D^{\alpha}\cos(t) = -\sin(t). $$

See Figure 2 where some graphics related to this compute are exhibited.

*Remark.*Finally, let us present three computational examples which proves that the changes in a time interval in the past makes consequences in the output function given by the CF operator. Consider the three functions defined in $\mathbb{R}_0^+$ as

$$ f_1(t) = t^2, \tag{29} $$

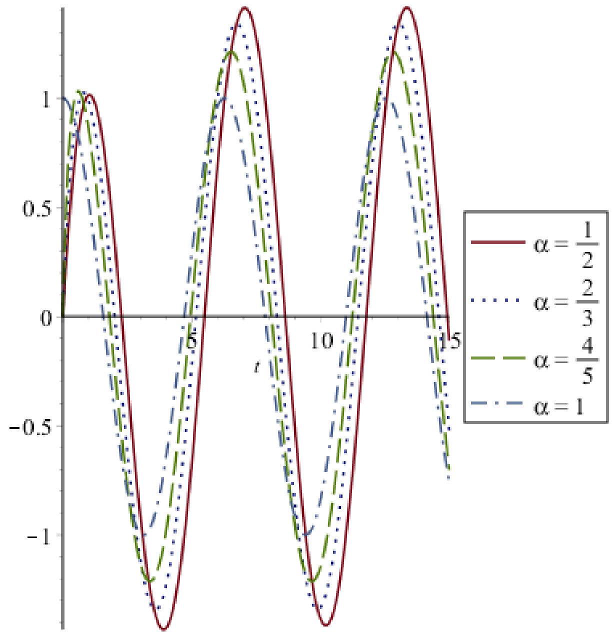

Fig. 2: ${}^{CF}D^{\alpha}\sin t$ for some values of α.

$$f_2(t) = \begin{cases} t & 0 \le t \le 1 \\ t^2 & 1 < t, \end{cases} \tag{30}$$

and

$$f_3(t) = \begin{cases} 2t-1 & 0 \le t \le 1 \\ t^2 & 1 < t. \end{cases} \tag{31}$$

Note that f_i agree in a neighbourhood of $t = 2$ for $i = 1,2,3$, being f_1 and f_3 differentiable functions and $f_2 \in H^1(0,b)$ for every $b > 0$ with a jump for its derivative at $t = 1$.

Let us see that ${}_0^{CF}D^{\alpha} f_1(2) \neq {}_0^{CF}D^{\alpha} f_2(2) \neq {}_0^{CF}D^{\alpha} f_3(2)$ for every $\alpha \in (0,1)$, while for the local classical derivative we have that $f_1'(2) = f_2'(2) = f_3'(2)$.
From Corollary 1, we have that

$${}_0^{CF}D^{\alpha} f_1(2) = \frac{4}{\alpha} - \frac{2(1-\alpha)}{\alpha^2} + \frac{2(1-\alpha)}{\alpha^2} e^{-\frac{2\alpha}{1-\alpha}}. \tag{32}$$

Now, for every $t > 1$

$$\begin{aligned} {}_0^{CF}D^{\alpha} f_2(t) &= \frac{1}{1-\alpha} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f_2'(\tau) \mathrm{d}\tau. \\ &= \frac{1}{1-\alpha} \int_0^1 e^{-\frac{\alpha}{1-\alpha}(t-\tau)} \mathrm{d}\tau + \frac{1}{1-\alpha} \int_1^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} 2\tau \mathrm{d}\tau. \end{aligned} \tag{33}$$

Integrating (33) and then evaluating at $t = 2$ we have

$${}_0^{CF}D^{\alpha} f_2(2) = \frac{4}{\alpha} - \frac{2(1-\alpha)}{\alpha^2} - \frac{e^{-\frac{2\alpha}{1-\alpha}}}{\alpha} + \frac{2-3\alpha}{\alpha} e^{-\frac{\alpha}{1-\alpha}}. \tag{34}$$

Let h be the function defined as $h(\alpha) = {}_0^{CF}D^{\alpha} f_1(2) - {}_0^{CF}D^{\alpha} f_2(2)$ for every α in $(0,1)$. Then

$$h(\alpha) = \frac{2-\alpha}{\alpha^2} e^{-\frac{2\alpha}{1-\alpha}} - \frac{2-3\alpha}{\alpha} e^{-\frac{\alpha}{1-\alpha}} \neq 0 \quad \text{for every } \alpha \in (0,1). \tag{35}$$

Note that $h(\alpha) \to 0$ when $\alpha \nearrow 1$.
Analogously, for every $t > 1$, we have

$$\begin{aligned} {}_0^{CF}D^{\alpha} f_3(t) &= \frac{1}{1-\alpha}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f_3'(\tau)\mathrm{d}\tau. \\ &= \frac{1}{1-\alpha}\int_0^1 e^{-\frac{\alpha}{1-\alpha}(t-\tau)}\cdot 2\mathrm{d}\tau + \frac{1}{1-\alpha}\int_1^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} 2\tau\mathrm{d}\tau. \end{aligned} \quad (36)$$

Substituting by $t = 2$ it yields that

$$ {}_0^{CF}D^{\alpha} f_3(2) = \frac{4}{\alpha} - \frac{2(1-\alpha)}{\alpha^2} - \frac{2}{\alpha}e^{-\frac{2\alpha}{1-\alpha}} + \frac{2(1-\alpha)}{\alpha^2}e^{-\frac{\alpha}{1-\alpha}}. \quad (37)$$

Again, let g be the function defined as $g(\alpha) = {}_0^{CF}D^{\alpha} f_1(2) - {}_0^{CF}D^{\alpha} f_3(2)$ for every α in $(0,1)$ be. Then

$$ g(\alpha) = \frac{2}{\alpha^2}e^{-\frac{2\alpha}{1-\alpha}} - \frac{2(1-\alpha)}{\alpha^2}e^{-\frac{\alpha}{1-\alpha}} \neq 0 \quad \text{for every } \alpha \in (0,1), \quad (38)$$

while $g(\alpha) \to 0$ when $\alpha \nearrow 1$.

3 Global solution to a nonlinear fractional differential equation

The following Theorem is similar (not equal and the differences will be specified later) to Theorem 1 in the work of Losada and Nieto [16] as well as its proof.

Theorem 1. *Let $\varphi : [a,\infty) \times \mathbb{R} \to \mathbb{R}$ be a Lipschitz function respect on the second variable with constant L, i.e.*

$$ |\varphi(t,s_1) - \varphi(t,s_2)| \leq L|s_1 - s_2| \quad \forall s_1, s_2 \in \mathbb{R}, $$

and let $\alpha \in (0,1)$ be such that $L < \frac{1}{1-\alpha}$. Then if $\varphi(a,a_0) = 0$, problem

$$ \begin{cases} {}_a^{CF}D^{\alpha} f(t) = \varphi(t, f(t)), & t > a, \\ f(a) = a_0 \end{cases} \quad (39)$$

has a unique solution $f \in \mathscr{C}[a,T]$, for every $T \in \left(a, a + \frac{1-(1-\alpha)L}{\alpha L}\right)$.

The differences from Theorem 1 and [16, Theorem 1] are:

i) The definitions of the Caputo–Fabrizio derivative (3) and integral (17) are different from those considered in [16].
ii) The initial time is general (not necessary given by 0).
iii) The order of differentiation α depends on the Lipschitz constant L.
iv) The assumption $\varphi(a,a_0) = 0$ is imposed by Proposition 2-1 (and it is also necessary in the performance of the proof).

Theorem 2. *Let $\varphi : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a Lipschitz function respect on the second variable with constant L, and let be $\alpha \in (0,1)$ such that $L < \frac{1}{1-\alpha}$. Then, the problem*

$$ \begin{cases} {}^{CF}D^{\alpha} f(t) = \varphi(t, f(t)), & t > 0, \\ f(0) = a_0 \end{cases} \quad (40)$$

has a unique solution $f \in \mathscr{C}[0,T]$, for every finite time $T \in \mathbb{R}^+$, that is, globally in time.

Proof. Let the pair $\{T_1, f_1\}$ given by Theorem 1 which solves the problem

$$ \begin{cases} {}^{CF}D^{\alpha} f(t) = \varphi(t, f(t)), & t > 0, \\ f(0) = a_0 \end{cases} \quad (41)$$

in the interval $[0,T_1]$. Consider next the problem

$$\begin{cases} {}^{CF}D^{\alpha}f(t) = \varphi(t,f(t)), & t > T_1 \\ f(t) = f_1(t) & \forall t \in [0,T_1]. \end{cases} \tag{42}$$

By using the translation formula given in Proposition 2-2, it holds that problem (42) is equivalent to

$$\begin{cases} {}^{CF}_{T_1}D^{\alpha}f(t) = \varphi(t,f(t)) - e^{-\frac{\alpha(t-T_1)}{1-\alpha}}\,{}^{CF}D^{\alpha}f(T_1), & t > T_1, \\ f(t) = f_1(t), & \forall t \in [0,T_1]. \end{cases} \tag{43}$$

Also, being f_1 the solution to problem (41), problem (43) is equivalent to

$$\begin{cases} {}^{CF}_{T_1}D^{\alpha}f(t) = \varphi(t,f(t)) - e^{-\frac{\alpha(t-T_1)}{1-\alpha}}\varphi(T_1,f_1(T_1)), & t > T_1 \\ f(t) = f_1(t) & \forall t \in [0,T_1]. \end{cases} \tag{44}$$

Let us focus now in the sub-problem related to (44) given by

$$\begin{cases} {}^{CF}_{T_1}D^{\alpha}f(t) = \Phi(t,f(t)), & t > T_1, \\ f(T_1) = f_1(T_1) \end{cases} \tag{45}$$

where $\Phi(t,x) = \varphi(t,x) - e^{-\frac{\alpha(t-T_1)}{1-\alpha}}\varphi(T_1,f_1(T_1))$.

Being φ a Lipschitz function respect on the second variable with constant L and $e^{-\frac{\alpha(t-T_1)}{1-\alpha}} \le 1$ for every $t \ge T_1$, easily follows that Φ is a Lipschitz function respect on the second variable with constant L. By hypothesis $L < \frac{1}{1-\alpha}$, then we can apply Theorem 1 to (45), and there exists a pair $\{T_2, f_2\}$ such that f_2 is the unique solution to problem (45) in the interval $[T_1,T_2]$, where

$$T_2 - T_1 < \frac{1-(1-\alpha)L}{\alpha L}. \tag{46}$$

Note that the same argument can be used to obtain a solution to problem

$$\begin{cases} {}^{CF}_{T_2}D^{\alpha}f(t) = \varphi(t,f(t)) - e^{-\frac{\alpha(t-T_2)}{1-\alpha}}\,{}^{CF}D^{\alpha}f(T_2), & t > T_2 \\ f(t) = \begin{cases} f_2(t) & t \in (T_1,T_2], \\ f_1(t) & t \in [0,T_1] \end{cases} \end{cases}. \tag{47}$$

Also recalling that f_1 is a solution to problem (42) and f_2 is a solution to problem (45), we have

$$\begin{aligned} {}^{CF}D^{\alpha}f(T_2) =& \frac{1}{1-\alpha}\int_0^{T_2} f'(\tau)e^{-\frac{\alpha(T_2-\tau)}{1-\alpha}}\mathrm{d}\tau \\ &= \frac{1}{1-\alpha}e^{-\frac{\alpha(T_2-T_1)}{1-\alpha}}\int_0^{T_1} f_1'(\tau)e^{-\frac{\alpha(T_1-\tau)}{1-\alpha}}\mathrm{d}\tau + \frac{1}{1-\alpha}\int_{T_1}^{T_2} f_2'(\tau)e^{-\frac{\alpha(T_2-\tau)}{1-\alpha}}\mathrm{d}\tau \\ &= e^{-\frac{\alpha(T_2-T_1)}{1-\alpha}}\varphi(T_1,f(T_1)) + {}^{CF}_{T_1}D^{\alpha}f(T_2) \\ &= e^{-\frac{\alpha(T_2-T_1)}{1-\alpha}}\varphi(T_1,f(T_1)) + \varphi(T_2,f(T_2)) - e^{-\frac{\alpha(T_2-T_1)}{1-\alpha}}\,{}^{CF}D^{\alpha}f(T_1), \\ &= \varphi(T_2,f(T_2)) \end{aligned}$$

and then, problem (47) can be written as

$$\begin{cases} {}^{CF}_{T_2}D^{\alpha}f(t) = \Phi_2(t,f(t)), & t > T_2, \\ f(t) = \begin{cases} f_2(t) & t \in (T_1,T_2] \\ f_1(t) & t \in [0,T_1], \end{cases} \end{cases} \tag{48}$$

where $\Phi_2(t,x) = \varphi(t,x) - e^{-\frac{\alpha(t-T_2)}{1-\alpha}}\varphi(T_2, f(T_2))$. Once again, by Theorem 1 there exists a unique solution to the sub-problem

$$\begin{cases} {}^{CF}_{T_2}D^\alpha f(t) = \Phi(t, f(t)), & t > T_2, \\ f(T_2) = f_2(T_2) \end{cases} \tag{49}$$

for every T_3 such that

$$T_2 < T_3 < T_2 + \frac{1-(1-\alpha)L}{\alpha L}. \tag{50}$$

Calling ΔT to some positive constant such that $0 < \Delta T < \frac{1-(1-\alpha)L}{\alpha L}$, successively applying the former procedure, there exists a continuous function f_N which is the unique solution to

$$\begin{cases} {}^{CF}D^\alpha f(t) = \varphi(t, f(t)), & 0 < t < N\Delta T, \\ f(0) = a_0 \end{cases} \tag{51}$$

for every $N \in \mathbb{N}$. Being N an arbitrary natural, the solution of problem (41) is globally defined in time.

*Remark.*The effects of memory in the Caputo-Fabrizio derivative are shown in the need of considering the sub-problem (44) with an "initial condition" which must be known all over the interval $[0, T_1]$, in contrast to the local property of the classical derivative which requires only the initial condition at the time T_1.

4 Conclusion

We have analyzed and proved some useful properties related to the fractional Caputo–Fabrizio derivative such as translation property, convergence to integer order derivatives and inverse operator. Also a computation of this fractional derivative to power functions, sin and cosine functions, and exponential function were given, attempting to provide, in each case, expressions as simple as possible. Note that the terms that converges to zero when $\alpha \nearrow 1$ were visually separated than the terms that converges to the classical derivatives. Finally, an existence and uniqueness of a global solution to a nonlinear fractional differential equation was proved.

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