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# Some Random Fixed Point Theorems Using Implicit Relation in Polish Spaces

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**Abstract:** In this paper, we establish some random fixed point and common random fixed point theorems for one and two continuous random operators satisfying an implicit relation and defined on a non-empty separable metric space.

**Keywords:** Random operator, random fixed point, Common random fixed point, implicit relation, Polish space.

## 1 Introduction

Fixed point theory has the diverse applications in different branches of mathematics, statistics, engineering and economics in dealing with the problems arising in approximation theory, potential theory, game theory, theory of differential equations, theory of integral equations and others. Development in the investigation on fixed points of nonexpansive mappings, contractive mappings in different spaces like Metric spaces, Banach spaces, Fuzzy metric spaces have almost been saturated.
In 1950's, the Prague school of probabilistic started the study of random fixed point theorems. After that a considerable attention has been given to the study of random fixed point theorems because of its importance in probabilistic functional analysis and probabilistic models with numerous applications. The introduction of randomness however leads to several new questions of measurability of solutions, probabilistic and statistical aspects of random solutions. It is well known that random fixed point theorems are stochastic generalization of classical fixed point theorems what are known as deterministic results. Random fixed point theorems for contraction mappings on separable complete metric spaces were first proved by Spacek [15] and Hans [12]. The survey article by Bharucha-Reid [10] in 1976 attracted the attention of several mathematicians and gave wings to this theory. Itoh [13] extended Spacek's and Hans's theorems to multi-valued contraction mappings. Subsequently Beg and Shahzad [9] obtained the stochastic version of the result of Beg and Azam [5] for random multi-valued operators. Recently, Jhade and Saluja [14] gives the stochastic version of Ciric's [11] fixed point theorems for a pair of multi-valued and single-valued nonexpansive type mappings.
In this work, we establish some fixed and common fixed point theorems satisfying an implicit relation for one and two random operators defined on a separable metric space. Our results extend and unify some well-known results existing in the literature.

## 2 Preliminaries

**Definition 2.1.** Let $\phi$ be the class of real valued continuous functions $\phi : (\Re^+)^3 \to \Re^+$ non-decreasing in the second argument and satisfying the following conditions:

$$x \leq \phi(y, x+y, x)$$

$$or\ x \leq \phi(y, x+y, \frac{1}{2}(x+y))$$

$$or\ x \leq \phi(y, x+y, x+y)$$

Then there exists a real number $0 < k < 1$ such that$x \leq ky$ for all$x, y \geq 0$.

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**Condition (A).** A random mapping $T : \Omega \times X \to CB(X)$, where $X$ is nonempty separable metric space is said to satisfy Condition(A), if

$$H\,(T(w,x),T(w,y)) \le \phi(d(x,y), [d(x,T(w,x))+d(y,T(w,y))], \frac{d(y,T(w,y))[1+d(x,T(w,x))]}{1+d(x,y)})$$

for all $x, y \in X$ and for each $w \in \Omega$.
Here $H$ denotes the Hausdroff metric $CB(X)$ induced by metric $d$.

**Condition (B).**Two random mappings $S, T : \Omega \times X \to CB(X)$, where X is nonempty separable metric space is said to satisfy condition(B), if

$$H\,(S(w,x),T(w,y)) \le \phi(d(x,y), [d(x,S(w,x))+d(y,T(w,y))], \frac{1}{2}d(y,T(w,y))+d(y,S(w,x))])$$

for all$x, y \in X$ and for each $w \in \Omega$.
Here $H$ denotes the Hausdroff metric $CB(X)$ induced by metric $d$.

## 3 Main Results

Now we give our main results of this section.

**Theorem 3.1.** Let $X$ be a Polish space. Let$T : \Omega \times X \to CB(X)$ be a continuous random multi-valued operator such that, for all $x, y \in X, w \in \Omega$, $T$ satisfy condition (A). Then there exists a random fixed point of $T$ in $X$.

**Proof.** Let $\xi_0 : \Omega \to X$ be a arbitrary measurable mapping and choose a measurable mapping$\xi_1 : \Omega \to X$such that $\xi_1(w) \in T(w,\xi_0(w))$for each $w \in \Omega$. Then for each $w \in \Omega$, from condition (A), we have

$$H(T(w,\xi_0(w)),T(w,\xi_1(w)))$$
$$\le \phi(d(\xi_0(w),\xi_1(w)), [d(\xi_0(w),T(w,\xi_0(w)))+d(\xi_1(w),T(w,\xi_1(w)))], \frac{d(\xi_1(w),T(w,\xi_1(w)))[1+d(\xi_0(w),T(w,\xi_0(w)))]}{1+d(\xi_0(w),\xi_1(w))}$$

It further implies that there exists a measurable mapping $\xi_2 : \Omega \to X$ such that $\xi_2(w) \in T(w,\xi_1(w))$ for each $w \in \Omega$. Then for each $w \in \Omega$, from condition (A), we have

$$d(\xi_1(w),\xi_2(w)) = H(T(w,\xi_0(w)),T(w,\xi_1(w)))$$
$$\le \phi(d(\xi_0(w),\xi_1(w)), [d(\xi_0(w),T(w,\xi_0(w))) + d(\xi_1(w),T(w,\xi_1(w)))], \frac{d(\xi_1(w),T(w,\xi_1(w)))\,[1+d(\xi_0(w),T(w,\xi_0(w)))]}{1+d(\xi_0(w),\xi_1(w))}\Bigg)$$
$$\le \phi\,(d(\ \xi_0(w),\xi_1(w)), [d(\xi_0(w),\xi_1(w))+d(\xi_1(w),\xi_2(w))], \frac{d(\xi_1(w),\xi_1(w))\,[1+d(\xi_0(w),\xi_1(w))]}{1+d(\xi_0(w),\xi_1(w))}\Bigg)$$
$$\le \phi(d(\xi_0(w),\xi_1(w)), [d(\xi_0(w),\xi_1(w))+d(\xi_1(w),\xi_2(w))], d(\xi_1(w),\xi_2(w)))$$

which implies that, in view of Definition 2.1,

$$d(\xi_1(w),\xi_2(w)) \le kd(\xi_0(w),\xi_1(w))$$

Again there exists a measurable mapping $\xi_3 : \Omega \to X$ such that $\xi_3(w) \in T(w,\xi_2(w))$ for each $w \in \Omega$. Then for each $w \in \Omega$, from condition (A), we have

$$d(\xi_2(w),\xi_3(w)) = H(T(w,\xi_1(w)),T(w,\xi_2(w)))$$
$$\le \phi\,(d(\ \xi_1(w),\xi_2(w)), [d(\xi_1(w),T(w,\xi_1(w)))+d(\xi_2(w),T(w,\xi_2(w)))], \frac{d(\xi_2(w),T(w,\xi_2(w)))\,[1+d(\xi_1(w),T(w,\xi_1(w)))]}{1+d(\xi_1(w),\xi_2(w))}\Bigg)$$
$$\le \phi\,(d(\ \xi_1(w),\xi_2(w)), [d(\xi_1(w),\xi_2(w))+d(\xi_2(w),\xi_3(w))], \frac{d(\xi_2(w),\xi_3(w))\,[1+d(\xi_1(w),\xi_2(w))]}{1+d(\xi_1(w),\xi_2(w))}\Bigg)$$
$$\le \phi\,(d(\ \xi_1(w),\xi_2(w)), [d(\xi_2(w),\xi_3(w))+d(\xi_1(w),\xi_2(w))], d(\xi_2(w),\xi_3(w)))$$

which implies, in view of Definition 2.1, that

$$d(\xi_2(w),\xi_3(w)) \le kd(\xi_1(w),\xi_2(w)) \le k^2 d(\xi_0(w),\xi_1(w))$$

Proceeding in the same way, by induction, we produce a sequence of measurable mappings $\xi_n : \Omega \to X$ such that for each $w \in \Omega$,$\xi_{n+1}(w) \in T(w,\xi_n(w))$, where $n = 0,1,2\cdots$ and

$$\begin{aligned} d(\xi_n(w),\xi_{n+1}(w)) &\le kd(\xi_{n-1}(w),\xi_n(w)) \\ &\le \\ &\vdots \\ &\le k^n d(\xi_0(w),\xi_1(w)) \end{aligned}$$

Now, we shall prove that, for each $w \in \Omega$,$\{\xi_n(w)\}$ is a Cauchy sequence in X.

Now for $n > m$, we have

$$\begin{aligned} d(\xi_n(w), \xi_m(w)) &\leq d(\xi_n(w), \xi_{n+1}(w)) \\ &\quad + d(\xi_{n+1}(w), \xi_{n+2}(w)) \\ &\quad + \cdots + d(\xi_{m-1}(w), \xi_m(w)) \\ &\leq (k^n + k^{n+1} + \\ &\quad \cdots + k^{m-1}) d(\xi_0(w), \xi_1(w)) \\ &\leq (\frac{k^n}{1-k}) d(\xi_0(w), \xi_1(w)) \end{aligned}$$

Taking the limit as $n, m \rightarrow \infty$, gives $d(\xi_n(w), \xi_m(w)) \rightarrow 0$. It follows that $\{\xi_n(w)\}$ is a Cauchy sequence, for each $w \in \Omega$, and there exists a measurable mapping $\xi : \Omega \rightarrow X$ such that $\xi_n(w) \rightarrow \xi(w)$.

**Existence of random fixed point.** For each $w \in \Omega$,

$$\begin{aligned} d(\xi(w), T(w, \xi(w))) &\leq d(\xi(w), \xi_{n+1}(w)) \\ &\quad + d(\xi_{n+1}(w)), T(w, \xi(w))) \end{aligned}$$

$$\begin{aligned} &= d(\xi(w), \xi_{n+1}(w)) + H(T(w, \xi_n(w)), T(w, \xi(w))) \\ &\leq d(\xi(w), \xi_{n+1}(w)) + \phi(d(\xi_n(w), \xi(w)) \\ &\quad , [d(\xi_n(w), T(w, \xi_n(w))) + d(\xi(w), T(w, \xi(w))) \\ &\quad , \frac{d(\xi(w), T(w, \xi(w)))[1 + d(\xi_n(w), T(w, \xi_n(w)))]}{1 + d(\xi_n(w), \xi(w))}) \\ &= d(\xi(w), \xi_{n+1}(w)) + \phi(d(\xi_n(w), \xi(w)) \\ &\quad , [d(\xi_n(w), \xi_{n+1}(w)) + d(\xi(w), T(w, \xi(w))) \\ &\quad , \frac{d(\xi(w), T(w, \xi(w)))[1 + d(\xi_n(w), \xi_{n+1}(w))]}{1 + d(\xi_n(w), \xi(w))}) \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, we get

$$\begin{aligned} d(\xi(w), T(w, \xi(w))) \leq \phi(0, 0 + d(\xi(w) \\ , T(w, \xi(w))), d(\xi(w), T(w, \xi(w)))) \end{aligned}$$

Therefore, in view of Definition 2.1, we get $d(w, T(w, \xi(w))) \leq 0$, a contradiction. Hence $\xi(w) \in T(w, \xi(w))$.This completes the proof of the theorem.

Next, we give a common random fixed point theorem for two continuous operators.

**Theorem 3.2.**Let $X$ be a Polish space. Let $S, T : \Omega \times X \rightarrow CB(X)$ be two continuous random multi-valued operator such that, for all $x, y \in X, w \in \Omega$, $S$ and $T$ satisfy condition (B). Then $S$ and $T$ have a common random fixed point in $X$.

**Proof.** Let $\xi_0 : \Omega \rightarrow X$ be a arbitrary measurable mapping and choose a measurable mapping $\xi_1 : \Omega \rightarrow X$ such that $\xi_1(w) \in S(w, \xi_0(w))$ for each $w \in \Omega$. Then by condition (B), we have

$$\begin{aligned} H(S(w, \xi_0(w)), T(w, \xi_1(w))) \leq \phi(d(\xi_0(w), \xi_1(w)) \\ , [d(\xi_0(w), S(w, \xi_0(w))) + d(\xi_1(w), T(w, \xi_1(w)))] \\ , \frac{1}{2}[d(\xi_0(w), T(w, \xi_1(w))) + d(\xi_1(w), S(w, \xi_0(w)))]) \end{aligned}$$

It further implies that there exists a measurable mapping $\xi_2 : \Omega \rightarrow X$ such that $\xi_2(w) \in T(w, \xi_1(w))$ for each $w \in \Omega$ and by condition (B), we have

$$\begin{aligned} d(\xi_1(w), \xi_2(w)) &= H(S(w, \xi_0(w)), T(w, \xi_1(w))) \\ &\leq \phi(d(\xi_0(w), \xi_1(w)) \\ &\quad , [d(\xi_0(w), S(w, \xi_0(w))) + d(\xi_1(w), T(w, \xi_1(w)))] \\ &\quad , \frac{1}{2}[d(\xi_0(w), T(w, \xi_1(w))) + d(\xi_1(w), S(w, \xi_0(w)))]) \\ &\leq \phi(d(\xi_0(w), \xi_1(w)) \\ &\quad , [d(\xi_0(w), \xi_1(w)) + d(\xi_1(w), \xi_2(w))] \\ &\quad , \frac{1}{2}[d(\xi_0(w), \xi_2(w)) + d(\xi_1(w), \xi_1(w))]) \\ &\leq \phi(d(\xi_0(w), \xi_1(w)) \\ &\quad , [d(\xi_1(w), \xi_2(w)) + d(\xi_0(w), \xi_1(w))] \\ &\quad , \frac{1}{2}[d(\xi_0(w), \xi_1(w)) + d(\xi_1(w), \xi_2(w))]) \end{aligned}$$

Therefore, in view of Definition 2.1, we get

$$d(\xi_1(w), \xi_2(w)) \leq k d(\xi_0(w), \xi_1(w))$$

In the same manner, there exists a measurable mapping $\xi_3 : \Omega \rightarrow X$ such that $\xi_3(w) \in T(w, \xi_2(w))$ for each $w \in \Omega$. Then by condition (B), we get

$$\begin{aligned} d(\xi_2(w), \xi_3(w)) &= H(S(w, \xi_2(w)), T(w, \xi_1(w))) \\ &\leq \phi(d(\xi_2(w), \xi_1(w)) \\ &\quad , [d(\xi_2(w), S(w, \xi_2(w))) + d(\xi_1(w), T(w, \xi_1(w)))] \\ &\quad , \frac{1}{2}[d(\xi_2(w), T(w, \xi_1(w))) + d(\xi_1(w), S(w, \xi_2(w)))]) \\ &\leq \phi(d(\xi_1(w), \xi_2(w)) \\ &\quad , [d(\xi_2(w), \xi_3(w)) + d(\xi_1(w), \xi_2(w))] \\ &\quad , \frac{1}{2}[d(\xi_2(w), \xi_2(w)) + d(\xi_1(w), \xi_3(w))]) \\ &\leq \phi(d(\xi_2(w), \xi_1(w)) \\ &\quad , [d(\xi_2(w), \xi_3(w)) + d(\xi_1(w), \xi_2(w))] \\ &\quad , \frac{1}{2}[d(\xi_1(w), \xi_2(w)) + d(\xi_2(w), \xi_3(w))]) \end{aligned}$$

Therefore from Definition 2.1, we get

$$\begin{aligned} d(\xi_2(w), \xi_3(w)) &\leq k d(\xi_1(w), \xi_2(w)) \\ &\leq k^2 d(\xi_0(w), \xi_1(w)) \end{aligned}$$

Proceeding in the same way, by induction, we produce a sequence of measurable mappings $\xi_n : \Omega \rightarrow X$ such that for each $w \in \Omega$, and $\gamma > 0$,$\xi_{2\gamma+1}(w) \in S(w,\xi_{2\gamma}(w))$ and $\xi_{2\gamma+2}(w) \in T(w,\xi_{2\gamma+1}(w))$ and

$$\begin{aligned} d(\xi_n(w),\xi_{n+1}(w)) &\leq kd(\xi_{n-1}(w),\xi_n(w)) \\ &\leq \\ &\vdots \\ &\leq k^n d(\xi_0(w),\xi_1(w)) \end{aligned}$$

Now, we shall prove that, for each $w \in \Omega$,$\{\xi_n(w)\}$ is a Cauchy sequence in X.
Now for $n > m$, we have

$$\begin{aligned} d(\xi_n(w),\xi_m(w)) &\leq d(\xi_n(w),\xi_{n+1}(w)) \\ &\quad + d(\xi_{n+1}(w),\xi_{n+2}(w)) \\ &\quad + \cdots + d(\xi_{m-1}(w),\xi_m(w)) \\ &\leq (k^n + k^{n+1} + \\ &\quad \cdots + k^{m-1}) d(\xi_0(w),\xi_1(w)) \\ &\leq (\frac{k^n}{1-k}) d(\xi_0(w),\xi_1(w)) \end{aligned}$$

Taking the limit as $n,m \to \infty$, gives $d(\xi_n(w),\xi_m(w)) \to 0$. It follows that $\{\xi_n(w)\}$ is a Cauchy sequence, for each $w \in \Omega$, and there exists a measurable mapping $\xi : \Omega \to X$ such that $\xi_n(w) \to \xi(w)$ for each $w \in \Omega$.

It further implies that

$$\xi_{2\gamma+1}(w) \to \xi(w) \text{ and } \xi_{2\gamma+2}(w) \to \xi(w).$$

To prove to existence of common random fixed point, for each $w \in \Omega$, we have

$$\begin{aligned} d(\xi(w),S(w,\xi(w)) &\leq d(\xi(w),\xi_{2\gamma+2}(w)) \\ &\quad + d(\xi_{2\gamma+2}(w),S(w,\xi(w))) \\ &= d(\xi(w),\xi_{2\gamma+2}(w)) \\ &\quad + H(S(w,\xi(w)),T(w,\xi_{2\gamma+2}(w))) \\ &\leq d(\xi(w),\xi_{2\gamma+2}(w)) \\ &\quad + \varphi(d(\xi(w),\xi_{2\gamma+1}(w)) \\ &\quad ,[d(\xi(w),S(w,\xi(w))) \\ &\quad + d(\xi_{2\gamma+1}(w),T(w,\xi_{2\gamma+1}(w)))] \\ &\quad ,\frac{1}{2}[d(\xi(w),T(w,\xi_{2\gamma+1}(w))) \\ &\quad + d(\xi_{2\gamma+1}(w),S(w,\xi(w)))]) \\ &= d(\xi(w),\xi_{2\gamma+2}(w)) \\ &\quad + \varphi(d(\xi(w),\xi_{2\gamma+1}(w)) \\ &\quad ,[d(\xi(w),S(w,\xi(w))) \\ &\quad + d(\xi_{2\gamma+1}(w),\xi_{2\gamma+2}(w))] \\ &\quad ,\frac{1}{2}[d(\xi(w),\xi_{2\gamma+2}(w)) \\ &\quad + d(\xi_{2\gamma+1}(w),S(w,\xi(w)))]) \end{aligned}$$

Since $\{\xi_{2\gamma+1}(w)\}$ and $\{\xi_{2\gamma+1}(w)\}$ are subsequence of $\{\xi_{2\gamma+1}(w)\}$, as $\gamma \to \infty$

$$\xi_{2\gamma+1}(w) \to \xi(w) \text{ and } \xi_{2\gamma+2}(w) \to \xi(w)$$

Therefore as $\gamma \to \infty$, we have

$$\begin{aligned} d(\xi(w),S(w,\xi(w)) &\leq d(\xi(w),\xi(w)) \\ &\quad + \varphi(d(\xi(w),\xi(w)) \\ &\quad ,[d(\xi(w),S(w,\xi(w))) \\ &\quad + d(\xi(w),\xi(w))] \\ &\quad ,\frac{1}{2}[d(\xi(w),\xi(w)) \\ &\quad + d(\xi(w),S(w,\xi(w)))]) \\ &\leq \varphi(0,d(\xi(w),S(w,\xi(w))) + 0 \\ &\quad ,\frac{1}{2}[0 + d(\xi(w,S(w,\xi(w)))) \end{aligned}$$

Which implies, in view of Definition 2.1, that $d(w,S(w,\xi(w))) \leq 0$, a contradiction. Hence $\xi(w) \in S(w,\xi(w))$ for each $w \in \Omega$.
Similarly we can prove that $\xi(w) \in T(w,\xi(w))$ for each $w \in \Omega$.
This complete as the proof of the theorem.

## 4 Conclusion

Random fixed point theorems for random contraction mappings on separable complete metric spaces were first proved by Spacek [15] and Hans [12]. In this paper, we introduced a new contractive type implicit relation and proved some random fixed point and common fixed point theorems for one and two continuous random operators satisfying implicit relation in a non-empty separable metric space. Our results may be the motivation to other authors for extending and improving these results to be suitable tools for their applications.

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