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Chaos synchronization of complex Rössler system

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Abstract: This paper presents chaos synchronization between two identical chaotic systems via nonlinear control technique. The used system is complex Rössler system. The maximal Lyapunov exponent and the sensitivity on initial conditions are calculated of such system and they emphasis its chaotic behavior. The nonlinear control method has been applied successfully to synchronize the proposed system. Based on Lyaponov function the control input vectors are selected and activated to achieve synchronization. Finally, the numerical simulation are used to show the robustness and effectiveness of proposed method.

Keywords: Complex, Rössler system, Chaos, Synchronization, Error.

1. Introduction

Since the original papers by Pecora and Carrol's of chaotic synchronization [1,2], this topic has attracted the interest of many researchers and is still an important problem in the modern theory of nonlinear oscillator. This interest is related to the fact that the phenomenon of chaos synchronization is a key of secret communication [3,4,5,6] and also for its application of numerous problems in biology , chemistry, physics, electrical and automation engineering [7,8,9,10,11,12,13], etc. Generally the synchronized systems usually consists of two parts: the master (drive) system and slave (response) system. The idea of synchronization is to use the signal generated by the master system as an input in the slave system, so that the trajectory of the slave system asymptotically approaches that of the master system and the error signal is zero [14].

Some techniques have been made recently to solve the problem of chaos synchronization, such as active control, generalized active control, backstepping design, nonlinear control [15,16,17,18,19] and so on. The nonlinear control technique is the one of these important methods, where it is effective method for making two identical chaotic systems or two different chaotic systems be synchronized. However, this method usually assumed that the Lyapunov function of error dynamic of synchronization is formed as: $v(e) = \frac{1}{2}\exp^T e$ [15].

Nonlinear dynamical systems of complex nonlinear oscillators constitute some of the most fascinating development in physics and mathematics. A natural way to obtain complicated dynamical system is to couple several identical simple oscillators is rather rich and can display properties which are not observed in the behavior of the individual oscillator. The complex variable are a more convenient way to study these coupled oscillators. The advantage of introducing complex variables is that the reduction of the dimensions of phase aces to the half and equation of motions become easier to study.

Several researchers have used the chaotic and hyperchaotic Rössler system as a standard examples of chaos synchronization and control strategy [17]. If we replace the variables (x, y, z) of Rössler system by complex variables $x_1 + ix_2, x_3 + ix_4, x_5 + ix_6$ respectively, we will obtain the new system called the complex Rössler system which exhibit chaotic behavior. Here, we will apply the nonlinear control technique to synchronize two identical complex Rössler chaotic system and determine the controller based on Lyapunov function.

The rest of this paper is organized as follows: In section 2, complex Rössler system is introduced. The maximal Lyapunov exponent and the sensitivity of initial conditions of the system are calculated and plotted to show the chaotic behavior of proposed system. In section 3, theory of nonlinear control is presented. In section 4, the theory of nonlinear control is applied to study the chaos synchronization of two identical complex Rössler system. Also, we calculate the Lyapunov function to drive

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the expression of the control input vectors. The numerical simulations are presented in section 5 to show the robustness and effectiveness of the method. Finally, the present work is concluded in section 6.

2. System description

In 1976, Rössler [20] first introduced Rössler system which has a standard chaotic system to verify the effectiveness of the chaos control strategy. The so called Rössler system is arose from work in chemical kinetics. The system is described with 3 coupled nonlinear differential equations.

$$\begin{aligned}\dot{x} &= -y-z,\\ \dot{y} &= x+a\,y,\\ \dot{z} &= b+z(x-y).\end{aligned} \tag{1}$$

For the parameter values (a = b = 0.2 and 5.7) system (1) exhibit the well known chaotic attractor. Here, if we replace the real variables (x,y,z) of system by complex variables $x = x_1 + i\,x_2,\ y = x_3 + i\,x_4$ *and* $z = x_5 + i\,x_6$ we will get the complex Rössler system of the form:

$$\begin{aligned}\dot{x_1} &= -x_3 - x_5,\\ \dot{x_2} &= -x_4 - x_6,\\ \dot{x_3} &= x_1 + a\,x_3,\\ \dot{x_4} &= x_2 + a\,x_4,\\ \dot{x_5} &= b - c\,x_5 + x_1\,x_5 - x_2\,x_6,\\ \dot{x_6} &= -c\,x_6 + x_2\,x_5 + x_1\,x_6.\end{aligned} \tag{2}$$

It easy to check that the system (2) has two nontrivial fixed points at

$$\begin{aligned}\bar{x_1} &= \{\frac{c+s}{2},\ 0,\ \frac{-c-s}{2a},\ 0,\ \frac{c+s}{2a},\ 0\},\\ \bar{x_2} &= \{\frac{c-s}{2},\ 0,\ \frac{-c-s}{2a},\ 0,\ \frac{c+s}{2a},\ 0\},\end{aligned} \tag{3}$$

where $s = \sqrt{-4ab+c^2}$, Also the Jacobian of system (2) is

$$A = \begin{pmatrix} 0 & 0 & -1 & 0 & -1 & 0\\ 0 & -1 & 0 & -1 & 0 & 0\\ 1 & 0 & a & 0 & 0 & 0\\ 0 & 1 & 0 & a & 0 & 0\\ x_5 & -x_6 & 0 & 0 & x_1-c & -x_2\\ x_6 & x_5 & 0 & 0 & x_2 & x_1-c \end{pmatrix}$$

For the parameter values $(a\ = b\ = 0.2$ *and* $5.7)$ the fixed points becomes:

$$\begin{aligned}\bar{x_1} &= \{0.0070262,\ 0,\ -0.035131,\ 0, 0.035131,\ 0\}\\ \bar{x_2} &= \{5.69297,\ 0,\ -28.4694,\ 0, 28.4694,\ 0\}\end{aligned}$$

The value of $\bar{x_2}$ makes the $Trac\,A <\ 0$, then system (2) is dissipative and exhibit chaotic attractor as shown in Figure 1(a-c) with: $x(0) = (0,1,0,0,1,0,0,1,0)^T$.

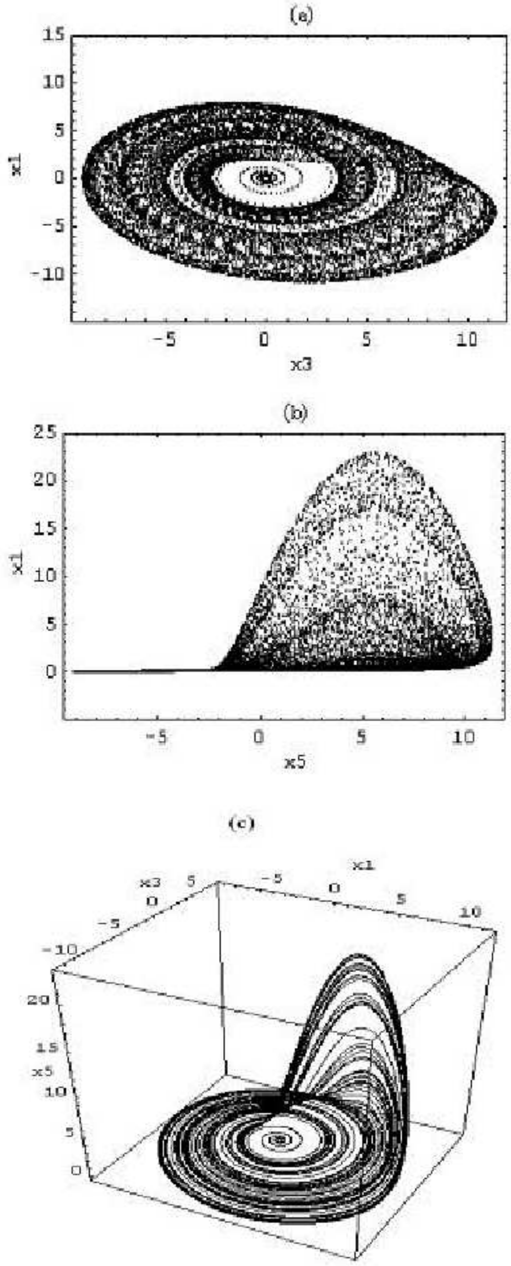

Figure 1: Chaotic behavior of system (2.2) for For the parameter values $(a\ = b\ = 0.2$ *and* $5.7)$ and $x(0) = (0.1,0,0.1,0,0.1,0)^T$.

The maximal Lyapunov exponent of system (2) with the same parameter values and initial conditions of Figure 1 is calculated and it is positive as shown in Figure (2a). Also Figure (2b) displays two solutions of system (2) with two close initial conditions: $x_1(0) = (0,1,0,0,1,0,0,1,0)^T$

$x_2(0) = (0.1+0.001, 0+0.001, 0.1, 0, 0.1, 0)^T$
Despite the slight change in the terms of the primary, but the solutions separated after a short time, as shown in Fig.2.

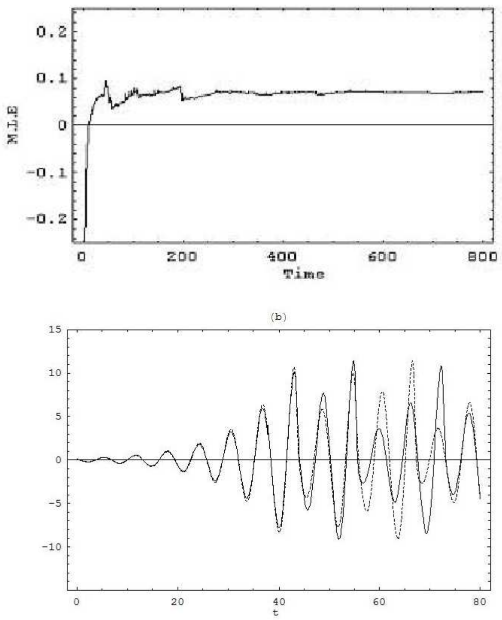

Figure 2: (a) Maximal Lyapunov exponent of system (2) with the same parameter values and initial conditions of Figure 1. (b) Two closed solution of system (2) with the same parameter values of Figure 1 and the initial conditions are: solid line:$x(0) = (0.1, 0, 0.1, 0, 0.1, 0)^T$ and dashed line: $x(0) = (0.1+0.001, 0, 0.1, 0, 0.1, 0)^T$.

3. Design of nonlinear control method

Consider a chaotic system described by the following relation:

$$\dot{x}_1 = A_1\, x_1 + h_1(x_1) \qquad (4)$$

Where $x_1(t)$ is the n-dimensional state vector of the system,is the matrix of the system, $A_1 \in \mathbb{R}^{n\times n}$ parameters, and $h_1(x_1) : \mathbb{R}^n \to \mathbb{R}^n$ is the nonlinear function of the system.

The equation (4) is considered as the master system and the slave system is obtained by adding the control input vector $u \in \mathbb{R}^n$ to the master system (4), so the slave system is described as follows:

$$\dot{x_2} = A_2\, x_2 + h_2(x_2) + u \qquad (5)$$

Where $x_2(t) \in \mathbb{R}^n$ is the state vector of the system, $A_2 \in \mathbb{R}^{n\times n}$ is the matrix of the slave system parameters, and $h_2(x_2) : \mathbb{R}^n \to \mathbb{R}^n$ is the nonlinear part of the slave system. $A_1 = A_2$ and $h_1(x_1) = h_2(x_2)$ for two identical chaotic systems, $A_1 \neq A_2$ and $h_1(x_1) \neq h_2(x_2)$ for two different chaotic systems.

The considered synchronization problem is to design an appropriate controller such that the trajectory of the slave system asymptotically approaches the one of the master system. We subtract equation (4) from equation (5) to get the dynamics of synchronization errors as follows:

$$\dot{e} = A_2\, x_2 + h_2\,(x_2) + u - A_1\, x_1 + h_1(x_1) \qquad (6)$$

Where $e = x_2 - x_1$ is the function of the error vector. The aim of synchronization is to make the error vector e converges to zero as time goes to infinity.
We suppose the following Lyapunov error function:

$$v(e) \;=\; \frac{1}{2}\exp^T\, e \qquad (7)$$

Where $v(e)$ is a positive definite function. Assuming that the parameters of the master and slave systems are known and the states of both system are measurable. The synchronization is achieved by selecting the controller u to make the first derivative of (7) is negative i.e. $\dot{v}(e) < 0$. Then the master system and the slave system are synchronized with each other.

4. Synchronization of two identical complex Rössler system

In this section we will use the nonlinear control method to synchronize two identical complex Rössler chaotic system. We consider the equation (2) is a master system and the slave system is given by:

$$\begin{aligned}
\dot{y_1} &= -y_3 - y_5 + u_1,\\
\dot{y_2} &= -y_4 - y_6 + u_2,\\
\dot{y_3} &= y_1 + a\, y_3 + u_3,\\
\dot{y_4} &= y_2 + a\, y_4 + u_4,\\
\dot{y_5} &= b - c\, y_5 + y_1\, y_5 - y_2\, y_6 + u_5,\\
\dot{y_6} &= -c\, y_6 + y_2\, y_5 + y_1\, y_6 + u_6.
\end{aligned} \qquad (8)$$

Where $u = [u_1, u_2, u_3, u_4, u_5, u_6]^T$ is the control input vector to be determined and T is the transpose. We wish to estimate an appropriate nonlinear controller $u_i, i = 1, ..., 6$ such that the trajectory of the slave system asymptotically approaches the trajectory of the master system. Hereby, the two systems are synchronized. To do that let us consider the error equation is : $e_i = y_i - x_i, i\; = 1, ..., 6$.
According to the nonlinear control method we subtract system (8) from system (2) to give:

$$\begin{aligned}
\dot{e_1} &= -e_3 - e_5 + u_1,\\
\dot{e_2} &= -e_4 - e_6 + u_2,\\
\dot{e_3} &= e_1 + a\, e_3 + u_3,\\
\dot{e_4} &= e_2 + a\, e_4 + u_4,\\
\dot{e_5} &= -c\, e_5 + (y_1\, y_5 - x_1\, x_5) - (y_2\, y_6 - x_2\, x_6) + u_5,\\
\dot{e_6} &= -c\, e_6 + (y_2\, y_5 - x_2\, x_5) + (y_1\, y_6 - x_1\, x_6) + u_6.
\end{aligned} \qquad (9)$$

System (9) can be considered as a control problem with control input vector v_i which is a function of the error vector e_i. Consider a Lyapunov function candidate of the form:

$$v(e) = \frac{1}{2}e^T e. \quad (10)$$

In order to make the time derivative of is negative, we must select the controller u_i as follows:

$$\begin{aligned} u_1 &= e_3 + e_5 - e_1, \\ u_2 &= e_4 + e_6 - e_2, \\ u_3 &= -e_1 - 2a\, e_3, \\ u_4 &= -e_2 - 2a\, e_4, \\ u_5 &= u_{5\,a} + u_{5\,b}, \\ u_6 &= u_{6\,a} + u_{6\,b}. \end{aligned} \quad (11)$$

Where :

$$\begin{aligned} u_{5\,a} &= (y_2\, y_6 - x_2\, x_6) - (y_1\, y_5 - x_1\, x_5), \\ u_{5\,b} &= (c-1)\, e_5, \\ u_{6\,a} &= -(y_2\, y_5 - x_2\, x_5) - (y_1\, y_6 - x_1\, x_6), \\ u_{6\,b} &= (c-1)\, e_6. \end{aligned} \quad (12)$$

With this choice of the controller u_i the time derivative of (10) becomes:

$$\dot{v}(e) = -e_1^2 - e_2^2 - a\, e_3^2 - a\, e_4^2 - e_5^2 - e_6^2 < 0. \, if a > 0 \quad (13)$$

Since $\dot{v}(e)$ is a negative-definite function, the error states $\lim_{t\to\infty} \|e(t)\| = 0$, implying synchronization of master-slave systems.By using the equations (11) and (12) the final form of the slave system is :

$$\begin{aligned} \dot{y_1} &= x_1 - x_3 - x_5 - y_1, \\ \dot{y_2} &= x_2 - x_4 - x_6 - y_2, \\ \dot{y_3} &= x_1 + 2a\, x_3 - ay_3, \\ \dot{y_4} &= x_2 + 2a\, x_4 - ay_4, \\ \dot{y_5} &= b - y_5 + x_1\, x_5 - x_2\, x_6 + (1-c)x_5, \\ \dot{y_6} &= -y_6 + x_2\, x_5 + x_1\, x_6 + (1-c)x_6. \end{aligned} \quad (14)$$

5. Simulation results

The two systems (1) and (13) are solved numerically by using the software Mathematica 5.1.The parameters are selected as $a = b2$ and c = 5.7 the initial conditions are:

$x(0 = (0,1,0,0,1,0,0,1,0)^T$

$y(0) = (-3.9,-4,-1.9,-2,-1.9,-1.4)^T$

The results are illustrated in Figures 3 and 4 that the trajectories of the slave system $y = (y_1, y_2, y_3, y_4, y_5, y_6)^T$ asymptotically approach the ones of the master system $y = (x_1, x_2, x_3, x_4, x_5, x_6)^T$ as shown in Figure 3(a-c). Figures 3(a-c) show the time series of signals x_i and $y_i (i = 1,3,5)$ respectively. While Figure 4(a-c) displays the synchronization error $e_i (i = 1,3,5)$ versus time. From these Figures, we can see that the synchronization error e_i eventually converges to zero after short time and the two identical complex Rössler chaotic systems are synchronized.

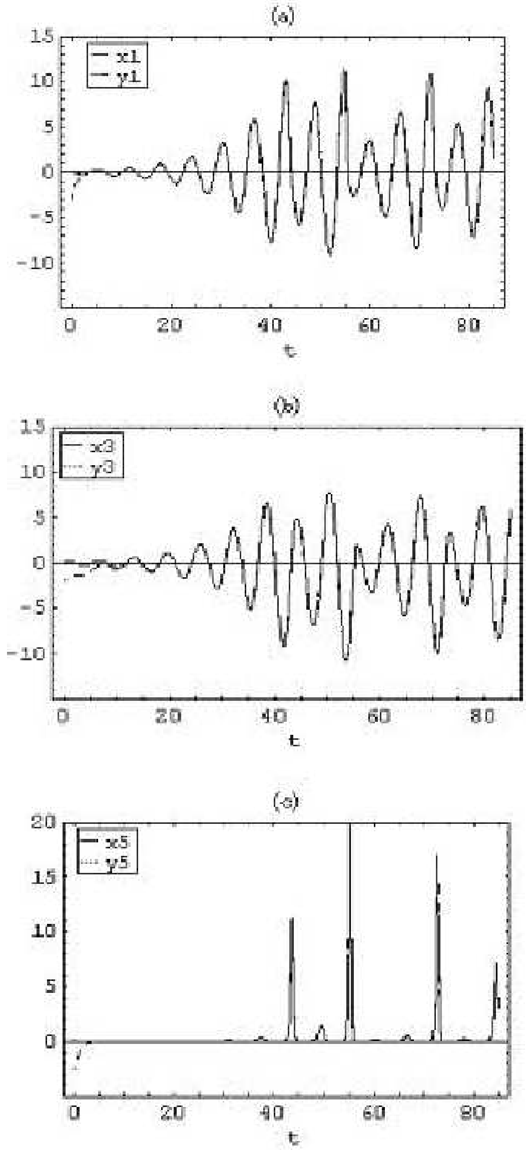

Figure 3: Synchronization of two identical complex Rössler systems using nonlinear control method (a) Signals x_1 and y_1 versus t (b)x_3 and y_3 versus t (c)x_5 and y_5 versus t .

6. Conclusion

In this paper, nonlinear control technique has been used to synchronize two identical complex Rössler systems. The maximal Lyapunov exponent and the sensitivity on initial conditions are calculated to emphasis the chaotic behavior of that system. The proposed technique is effective and convenient for synchronization where it has been successfully applied to synchronize various systems like: Lorenz, Rössler, Lu and four-scrol attractor. The

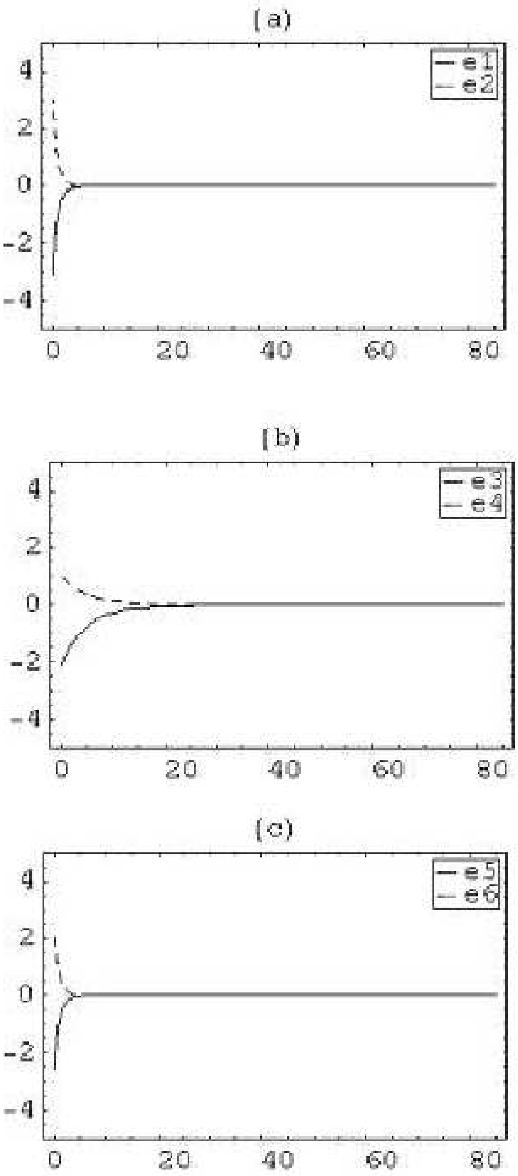

Figure 4: Dynamics of synchronization error for two identical complex Rössler chaotic systems (a)e_1 and e_2 versus t (b)e_3 and e_4 versus t (c) e_5 and e_6 versus t .

numerical results are presented to show the effectiveness of this approach. A good agreement is found between the analytical and numerical calculation.

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