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Numerical Scheme for Solving the Space-Time Variable Order Nonlinear Fractional Wave Equation

Nasser Hassan Sweilam[1,*] *and Taghreed Abdulrahman Assiri*[2,*]

[1] Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt
[2] Department of Mathematics, Faculty of Science, Um-Alqura University, Saudi Arabia.

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Abstract: In this paper, the space-time variable order fractional wave equation with a nonlinear source term is considered. The derivative is defined in the Caputo sense. The non-standard finite difference method is proposed for solving the variable order fractional wave equation. Special attention is given to study the stability analysis and the truncation error of the method. Some numerical test examples are presented, and the results demonstrate the effectiveness of the method. The obtained results are compared with exact solutions and the standard finite difference solutions.

Keywords: Non-standard finite difference scheme, Caputo's derivatives, variable order wave equation, stability analysis.

1 Introduction

Fractional calculus has been considered as one of the best mathematical tools to characterize the memory property of complex systems and certain materials [16], it can be considered as an extension of the usual calculus ([1],[2],[10],[13]-[19],[28],[31]). The variable order calculus is the generalization of classical calculus and fractional calculus, which were invented by Newton and Leibnitz hundreds of years ago. Now the study on it becomes a hotpot in recent years ([12],[20],[29],[30]). The variable order fractional derivative is a good tool in depicting the memory property which changes with time or spatial location. So, the physical models could be depicted more accurately by employing the variable order fractional calculus [16]. Samko and Ross [30], proposed the concept of variable order operator and investigated the properties of variable order integration and differentiation of Riemann-Liouville type. Most of the definitions of the variable order differential operators are extensions to the fractional calculus definitions such as Riemann-Liouville, Grünwald, Caputo, Riesz and some not as Coimbra definition ([3]-[6],[9],[29]). Some systems in fluid dynamics and electromagnetics are introduced using the variable order derivatives (for more details see [5]-[7] and the references sited therein).

The wave equation is an important second-order partial differential equation for the description of waves as they occur in physics such as sound waves, light waves and water waves. Variable order wave equation arises in fields like acoustics, electromagnetics, and fluid dynamics([4],[20]).

Difference methods and, in particular, explicit finite difference methods, are simple an important class of numerical methods for solving fractional differential equations. The usefulness of the explicit method and popularity is based on their particularly attractive features. The most attractive feature is that no need to solve resultant system of equations, especially for large scale problems. The main disadvantage of these methods is that the stability condition which can be in general proved in a small interval of space and time.

The genesis of nonstandard finite difference (NSFD) modeling procedures began with the 1989 publication of Mickens [23]. Extensions and a summary of the known results up to 1994 are given in Mickens [26], either for ordinary differential equations (ODEs) or partial differential equations (PDEs) ([23]- [27]). Their use have been investigated in several fields including control, mechanical systems, chaos synchronization and others ([8], [32] and the references cited therein). NSFD scheme is used with arbitrarily large time step sizes, saving computational cost when integrating over long time periods. Also, it is important due to the fact that variables representing subpopulations must never take negative values.

* Corresponding author e-mail: nsweilam@sci.cu.edu.eg, rieda2008@gmail.com

In this work, we shall see that the non-standard discretization is another numerical way to solve the fractional differential equations while preserving their crucial non-local property.

In the following, we present the basic definition for the variable order fractional derivatives and the main rules of the nonstandard discretization methods, which we will use in this paper.

Definition 1.1. [11] The variable order Caputo derivative is defined as follows:

$$D_x^{\alpha(x,t)}u(x,t)=\frac{1}{\Gamma(n-\alpha(x,t))}\int_0^x\frac{1}{(x-\xi)^{\alpha(x,t)-n+1}}\frac{\partial^n u(\xi,t)}{\partial\xi^n}d\xi, \tag{1}$$

where $n-1<\alpha(x,t)<n,\ n\in Z^+$.

NSFD Rules

In this part, we would like to introduce several comments related to NSFD schemes were firstly proposed by Mickens [23]. This class of schemes and their formulations center on two issues. First, how should discrete representations for derivatives be determined, and second, what are the proper forms to be used for nonlinear terms.

In the forward Euler method the derivative term $\frac{dy}{dt}$ is replaced by $\frac{y(t+h)-y(t)}{h}$, where h is the step size. However, in the NSFD schemes this term is replaced by $\frac{y(t+h)-y(t)}{\phi(h)}$,where $\phi(h)$ is a continuous function of step size h, and the function $\phi(h)$ satisfies the following conditions:

$$\phi(h)=h+O(h^2),\quad 0<\phi(h)<1,\quad h\to 0.$$

Examples of functions $\phi(h)$ that satisfy these conditions are [25]:

$\phi(h)=h,\ \sinh h,\ e^h-1,\ \frac{1-e^{-\lambda h}}{\lambda}$, etc., ...

Note that in taking the lim $h\to 0$ to obtain the derivative, the use of any of these $\phi(h)$ will lead to the usual result for the first derivative

$$\frac{dy}{dt}=\lim_{h\to 0}\frac{y[t+\phi_1(h)]-y(t)}{\phi_2(h)}=\lim_{h\to 0}\frac{y(t+h)-y(t)}{h}.$$

A scheme is called nonstandard if at least one of the following conditions is satisfied:

1- Nonlocal approximation is used.

2- Discretization of derivative is not traditional and use a nonnegative function.

One can say that there is no appropriate general method to choose the function $\phi(h)$ or to choose which nonlinear terms are to be replaced ([21], [25], [26]).

The main aim of this work is to use the nonstandard finite difference method (NSFD) to study numerically the following nonlinear space-time variable order wave equation (see for example [20], and the references sited therein):

$$D_t^{\beta(x,t)}u(x,t)=B(x,t)D_x^{\alpha(x,t)}u(x,t)+f(u,x,t),\quad 1<\alpha(x,t),\beta(x,t)\le 2, \tag{2}$$

with the initial conditions

$$u(x,0)=\varphi_1(x)\quad ,u_t(x,0)=\varphi_2(x),\quad 0\le x\le a, \tag{3}$$

and the boundary conditions

$$u(0,t)=\Psi_1(t)\quad ,u(a,t)=\Psi_2(t),\quad 0\le t\le T, \tag{4}$$

where $B(x,t)>0$ is a constant, $\varphi_1(x)$, $\varphi_2(x)$, $\Psi_1(t)$, *and* $\Psi_2(t)$ are smooth functions and $f(u,x,t)$ is a nonlinear scour term satisfies the Lipschitz condition, i.e.,

$$|f(u_1,x,t)-f(u_2,x,t)|\le L|u_1-u_2|, \tag{5}$$

where $L>0$ is called a Lipschitz constant for f.

This paper is organization as follows. In Section 2, we apply the Mickens non-standard discretization scheme to the fractional order wave equation described in Caputo. In Section 3, we study the stability and the truncation error of the method. Numerical test examples are presented to show the efficiency of the method in Section 4. Finally, in Section 5 we give some conclusions.

2 Discretization for NSFD Method

Let us consider the discrete form of the Caputo derivative:

$$D_x^{\alpha(x,t)}u(x,t)=\frac{1}{\Gamma(2-\alpha(x,t))}\int_0^x\frac{1}{(x-\xi)^{\alpha(x,t)-2+1}}\frac{\partial^2 u(\xi,t)}{\partial\xi^2}d\xi$$

$$=\frac{1}{\Gamma(2-\alpha(x,t))}\sum_{k=0}^{i-1}\int_{kh}^{(k+1)h}z^{1-\alpha(x,t)}\frac{\partial^2 u(x-z,t)}{\partial z^2}dz.$$

Let $z=x-\xi$, then

$$D_x^{\alpha(x,t)}u(x,t)\simeq\frac{1}{\Gamma(2-\alpha(x,t))}\sum_{k=0}^{i-1}\frac{u(x-(k-1)h,t)-2u(x-kh,t)+u(x-(k+1)h,t)}{h^2}\times(\int_{kh}^{(k+1)h}z^{1-\alpha(x,t)}dz), \tag{6}$$

then,

$$D_x^{\alpha(x,t)}u(x,t)\simeq\frac{h^{2-\alpha(x,t)}}{\Gamma(3-\alpha(x,t))}\sum_{k=0}^{i-1}\frac{u(x-(k-1)h,t)-2u(x-kh,t)+u(x-(k+1)h,t)}{h^2}\times\left((k+1)^{2-\alpha(x,t)}-k^{2-\alpha(x,t)}\right). \tag{7}$$

In the following, the NSFD notions is introduced. Let N and M be two positive integers, $h=\frac{a}{M}$ and $\tau=\frac{T}{N}$, where h and τ are the step size of space and time respectively. Also we introduce the following notations:

$$x_i=ih,\ for\ i=1,2,\ldots,N,\ t_j=j\tau,\ for\ j=1,\ldots,M, \tag{8}$$

$\alpha_i^j=\alpha(x_i,t_j),\ \beta_i^j=\beta(x_i,t_j),\ u_i^j=u(x_i,t_j),\ B_i^j=B(x_i,t_j)\ \ and\ f_i^j=f\left(u_i^j,x_i,t_j\right).$
Then

$$D_x^{\alpha(x_i,t_j)}u(x_i,t_j)=\frac{h^{-\alpha_i^j}}{\Gamma\left(3-\alpha_i^j\right)}\sum_{k=0}^{i-1}(u_{i-k+1}^j-2u_{i-k}^j+u_{i-k-1}^j)((k+1)^{2-\alpha_i^j}-k^{2-\alpha_i^j}). \tag{9}$$

By the same way, we have:

$$D_t^{\beta(x_i,t_j)}u(x_i,t_j)=\frac{\tau^{-\beta_i^j}}{\Gamma\left(3-\beta_i^j\right)}\sum_{k=0}^{j-1}(u_i^{j-k+1}-2u_i^{j-k}+u_i^{j-k-1})((k+1)^{2-\beta_i^j}-k^{2-\beta_i^j}). \tag{10}$$

Now, using the NSFD discretization scheme to (9) and (10) by replacing the step size h by a function of h, $\phi(h)$ and the step size τ by a function of τ, $\psi(\tau)$.

$$D_x^{\alpha(x_i,t_j)}u(x_i,t_j)=\frac{(\phi(h))^{-\alpha_i^j}}{\Gamma\left(3-\alpha_i^j\right)}\sum_{k=0}^{i-1}(u_{i-k+1}^j-2u_{i-k}^j+u_{i-k-1}^j)((k+1)^{2-\alpha_i^j}-k^{2-\alpha_i^j}).$$

By the same way, we have:

$$D_t^{\beta(x_i,t_j)}u(x_i,t_j)=\frac{(\psi(\tau))^{-\beta_i^j}}{\Gamma\left(3-\beta_i^j\right)}\sum_{k=0}^{j-1}(u_i^{j-k+1}-2u_i^{j-k}+u_i^{j-k-1})((k+1)^{2-\beta_i^j}-k^{2-\beta_i^j}),$$

where $\phi(h)$ *and* $\psi(\tau)$ have the properties:

$$\psi(\tau)=\tau+O(\tau^2)\ \ and\ \ \phi(h)=h+O(h^2).$$

For simplicity let us define:

$$R_i^j=\frac{B_i^j(\phi(h))^{-\alpha_i^j}}{\Gamma\left(3-\alpha_i^j\right)},\ \ Q_i^j=\frac{\Gamma\left(3-\beta_i^j\right)}{(\psi(\tau))^{-\beta_i^j}}, \tag{11}$$

$$G_k^j = \left((k+1)^{2-\alpha_i^j} - k^{2-\alpha_i^j}\right) \text{ and } H_i^k = \left((k+1)^{2-\beta_i^j} - k^{2-\beta_i^j}\right),$$

then, we can rewrite equation (2) in the following form

$$\sum_{k=0}^{j-1} (u_i^{j-k+1} - 2u_i^{j-k} + u_i^{j-k-1})H_i^k \approx Q_i^j R_i^j \sum_{k=0}^{i-1} (u_{i-k+1}^j - 2u_{i-k}^j + u_{i-k-1}^j)G_k^j + Q_i^j f_i^j, \tag{12}$$

that is,

$$u_i^{j+1} = 2u_i^j - u_i^{j-1} - \sum_{k=1}^{j-1} (u_i^{j-k+1} - 2u_i^{j-k} + u_i^{j-k-1})H_i^k +$$

$$Q_i^j R_i^j \sum_{k=0}^{i-1} (u_{i-k+1}^j - 2u_{i-k}^j + u_{i-k-1}^j)G_k^j + Q_i^j f_i^j, \tag{13}$$

$$u_i^{j+1} = (2 - H_i^1)\, u_i^j - \sum_{k=2}^{M-2} \left(H_i^{k-2} - 2H_i^{k-1} + H_i^k\right) u_i^{j-k+1} - \left(H_i^{j-2} - 2H_i^{j-1}\right) u_i^1 - H_i^{j-1} u_i^0 +$$

$$Q_i^j R_i^j \sum_{k=0}^{i-1} (u_{i-k+1}^j - 2u_{i-k}^j + u_{i-k-1}^j)G_k^j + Q_i^j f_i^j.$$

The previous equation can be expressed in the following matrix form:

$$U_i^0 = \emptyset_1,\ U_i^1 = U_i^0 + \psi(\tau)\emptyset_2, \tag{14}$$

and for $j \geq 2$

$$U_i^{j+1} = A^j U_i^j - \sum_{k=2}^{M-2} \left(H_i^{k-2} - 2H_i^{k-1} + H_i^k\right) U_i^{j-k+1} - \left(H_i^{j-2} - 2H_i^{j-1}\right) U_i^1 - H_i^{j-1} U_i^0 + F^j, \tag{15}$$

where $F^j = \left(Q_i^j f\left(u_{m-1}^j, x_{m-1}, t_j\right), \ldots, Q_i^j f\left(u_1^j, x_1, t_j\right)\right)^T$, $U^j = \left(u_{M-1}^j,\ u_{M-2}^j, \ldots, u_1^j\right)^T$,

$$\emptyset_1 = (\varphi_1(x_1)\ , \varphi_1(x_2), \ldots, \varphi_1(x_N))^T,\ \ \emptyset_2 = (\varphi_2(x_1)\ , \varphi_2(x_2), \ldots, \varphi_2(x_N))^T, \tag{16}$$

and $A^j = (a_{nm}^j)$, where

$$a_{nm}^j = \begin{cases} Q_n^j R_n^j G_{n-1}^j, & m = 1, \\ Q_n^j R_n^j \left(G_{n-m}^j - 2G_{n-m+1}^j + \theta G_{n-m+2}^j\right), & m \leq n, \\ 2 - H_n^j + Q_n^j R_n^j (\theta G_1^j - 2G_0^j), & m = n+1, \\ Q_n^j R_n^j G_0^j, & m = n+2, \\ 0\,, & m > n+2, \end{cases}$$

and

$$\theta = \begin{cases} 0\,, & when \quad m = 2, \\ 1\,, & otherwise, \end{cases} \tag{17}$$

for $n = 1, 2, \ldots, K-1$, and $m = 1, 2, \ldots, K-1$. Also, we note that

$$\|A\|_\infty = \max_{1 \leq n \leq K} \sum_{m=1}^{K} |a_{nm}| \ = \max_{1 \leq n \leq K} \{2 - H_i^n\} = 2 - H_i^0, \tag{18}$$

then $\|A\|_\infty = 1$.

Lemma 1. The coefficients G_k^j and H_i^k satisfy the following conditions

1. $G_0^j = 1, and\ H_i^0 = 1.$
2. $G_k^j > G_{k+1}^j$, $and\ H_i^k > H_i^{k+1}$, for $k = 0, 1, \dots$

3 Stability Analysis and Truncation Error

Let us consider W^{j+1} and U^{j+1} be two different numerical solutions of (15) with initial values given by W^0 and U^0, respectively.

Theorem 3.1. NSFD method defined by (15) to equation (2) is unconditionally stable, i.e.,

$$\left|W^{j+1} - U^{j+1}\right| \leq C\left|W^0 - U^0\right|, \ for\ any\ j. \tag{19}$$

Proof. Defined $W^{j+1} - U^{j+1} = \varepsilon^{j+1}$.
From (15) we have

$$\varepsilon_i^{j+1} = A^j \varepsilon_i^j - \sum_{k=2}^{M-2} \left(H_i^{k-2} - 2H_i^{k-1} + H_i^k\right) \varepsilon_i^{j-k+1} - \left(H_i^{j-2} - 2H_i^{j-1}\right) \varepsilon_i^1 - H_i^{j-1} \varepsilon_i^0 \quad + F_\varepsilon^j, \tag{20}$$

where

$$F_\varepsilon^j = \left(Q_{m-1}^j f\left(u_{m-1}^j, x_{m-1}, t_j\right) - Q_{m-1}^j f\left(w_{m-1}^j, x_{m-1}, t_j\right), \dots, Q_1^j f\left(u_1^j, x_1, t_j\right) - Q_1^j f\left(w_1^j, x_1, t_j\right)\right)^T$$

$$\leq \left(Q_{m-1}^j L_{m-1}^j \varepsilon_{m-1}^j, \dots, Q_1^j L_1^j \varepsilon_1^j\right)^T = \triangle F^j \varepsilon^j, \tag{21}$$

and $\triangle F^j = diag\left(Q_{m-1}^j L_{m-1}^j, \dots, Q_1^j L_1^j\right)^T$.
Noting that $\left|L_i^j\right| \leq L$, for any i, j. Let $\overline{Q} = \max\left\{Q_{m-1}^j, \dots, Q_1^j\right\}$.
From (20), we have $\left\|A^j + \triangle F^j\right\|_m \leq \left(2 + \overline{Q}L\right)$, when

$$\psi(\tau)^{\beta_i^j} > \frac{-2}{\Gamma(3 - \beta_i^j)L}, \ where\ L > 0,$$

then

$$\left\|\varepsilon_i^{j+1}\right\|_\infty \leq \left\|A^j + \triangle F^j\right\|_\infty \left\|\varepsilon_i^j\right\|_\infty + \sum_{k=2}^{M-2} \left(H_i^{k-2} - 2H_i^{k-1} + H_i^k\right) \left\|\varepsilon_i^{j-k+1}\right\|_\infty +$$
$$\left(H_i^{j-2} - 2H_i^{j-1}\right) \left\|\varepsilon_i^1\right\|_\infty + H_i^{j-1} \left\|\varepsilon_i^0\right\|_\infty. \tag{22}$$

Now, we analyze the stability via mathematical induction ([11], [22], [31]), from (14) we have $\left\|\varepsilon_i^1\right\|_\infty \leq C\left\|\varepsilon_i^0\right\|_\infty$, where C is a constant.
Now, assume that $\left\|\varepsilon_i^j\right\|_\infty \leq C\left\|\varepsilon_i^0\right\|_\infty$, then from (22) , we have

$$\left\|\varepsilon_i^{j+1}\right\|_\infty \leq C_1 \left(2 + \overline{Q}L\right) \left\|\varepsilon_i^0\right\|_\infty + \sum_{k=2}^{M-2} \left(H_i^{k-2} - 2H_i^{k-1} + H_i^k\right) C_2 \left\|\varepsilon_i^0\right\|_\infty +$$
$$C_3 \left(H_i^{j-2} - 2H_i^{j-1}\right) \left\|\varepsilon_i^0\right\|_\infty + H_i^{j-1} \left\|\varepsilon_i^0\right\|_\infty \leq C\left\|\varepsilon_i^0\right\|_\infty. \tag{23}$$

Then the theorem holds.
Lemma 2: Let

$$\overline{D}_x^{\alpha(x_i,t_j)} u(x_i,t_j) = \frac{(\phi(h))^{-\alpha(x_i,t_j)}}{\Gamma(3 - \alpha(x_i,t_j))} \sum_{k=0}^{j-1} G_i^j (u_{i-k+1}^j - 2u_{i-k}^j + u_{i-k-1}^j), \tag{24}$$

be a smooth function, then

$$\left|\overline{D}_x^{\alpha(x_i,t_j)} u(x_i,t_j) - D_x^{\alpha(x_i,t_j)} u(x_i,t_j)\right| = O(\phi(h)). \tag{25}$$

Proof. In term of standard centered difference formula, we have

$$\begin{aligned}
\overline{D}_x^{\alpha(x_i,t_j)} u(x_i,t_j) &= \tfrac{(\phi(h))^{2-\alpha(x_i,t_j)}}{\Gamma(3-\alpha(x_i,t_j))} \textstyle\sum_{j=0}^{k-1} G_i^j [\tfrac{\partial^2 u(x-jh,t)}{\partial z^2} + O(\phi(h)^2)] \\
&= \tfrac{(\phi(h))^{2-\alpha(x_i,t_j)}}{\Gamma(3-\alpha(x_i,t_j))} \textstyle\sum_{j=0}^{k-1} G_i^j \tfrac{\partial^2 u(x-jh,t)}{\partial z^2} + \tfrac{(\phi(h))^{2-\alpha(x_i,t_j)} k^{2-\alpha(x_i,t_j)}}{\Gamma(3-\alpha(x_i,t_j))} O(\phi(h)^2) \\
&= \tfrac{(\phi(h))^{2-\alpha(x_i,t_j)}}{\Gamma(3-\alpha(x_i,t_j))} \textstyle\sum_{j=0}^{k-1} G_i^j \tfrac{\partial^2 u(x-jh,t)}{\partial z^2} + \tfrac{x^{2-\alpha(x_i,t_j)}}{\Gamma(3-\alpha(x_i,t_j))} O(\phi(h)^2) \\
&= \tfrac{(\phi(h))^{2-\alpha(x_i,t_j)}}{\Gamma(3-\alpha(x_i,t_j))} \textstyle\sum_{j=0}^{k-1} G_i^j \tfrac{\partial^2 u(x-jh,t)}{\partial z^2} + O(\phi(h)^2).
\end{aligned} \tag{26}$$

By the integral mean value theorem, we have

$$D_x^{\alpha(x_i,t_j)} u(x_i,t_j) = \frac{1}{\Gamma(2-\alpha(x_i,t_j))} \sum_{j=0}^{k-1} \int_{jh}^{(j+1)h} z^{1-\alpha(x_i,t_j)} \frac{\partial^2 u(x-z,t)}{\partial z^2} dz = \frac{(\phi(h))^{2-\alpha(x_i,t_j)}}{\Gamma(3-\alpha(x_i,t_j))}, \tag{27}$$

where $\zeta_j \in [jh,(j+1)h]$. Combining the above two formulae, we have

$$\begin{aligned}
&\left|\overline{D}_x^{\alpha(x_i,t_j)} u(x_i,t_j) - D_x^{\alpha(x_i,t_j)} u(x_i,t_j)\right| = \left|\tfrac{(\phi(h))^{2-\alpha(x_i,t_j)}}{\Gamma(3-\alpha(x_i,t_j))} \textstyle\sum_{j=0}^{k-1} G_i^j [\tfrac{\partial^2 u(x-jh,t)}{\partial z^2} - \tfrac{\partial^2 u(x-\zeta_j,t)}{\partial z^2}] + O(\phi(h)^2)\right| \\
&= \left|\tfrac{\phi(h)^{2-\alpha(x_i,t_j)}}{\Gamma(3-\alpha(x_i,t_j))} \textstyle\sum_{j=0}^{k-1} G_i^j . O(\phi(h)) + O(\phi(h)^2)\right| \\
&= \left|\tfrac{h^{2-\alpha(x_i,t_j)} k^{2-\alpha(x_i,t_j)}}{\Gamma(3-\alpha(x_i,t_j))} . O(\phi(h)) + O(\phi(h)^2)\right| \\
&= O(\phi(h)) + O(\phi(h)^2) \\
&= O(\phi(h)).
\end{aligned} \tag{28}$$

Using lemma 2, the truncation error of NSFD scheme (14) can be derive. It has a local truncation error of $O(\psi(\tau))$(from the left side) and $O(\phi(h))$ (from the right side).

Remark
NSFD method was shown to be stable and with a local truncation error, which is $O(\psi(\tau)) + O(\phi(h))$. Therefore, according to the Lax's Equivalence Theorem [28], it converges at this rate.

4 Numerical Examples

Example 4.1. Consider the nonlinear variable order fractional wave equation [20]:

$$D_t^{\beta(x,t)} u(x,t) = -0.5\cos(\alpha(x,t)\pi/2) D_x^{\alpha(x,t)} u(x,t) + f(u,x,t), \tag{29}$$

with $\alpha(x,t) = 1.5 + 0.5\, e^{-(xt)^2-1}$, $\beta(x,t) = 1.5 + 0.25\, cos(x)\, sin(2t)$, and

$$f(u,x,t) = \frac{2u}{t^2+1} - (t^2+1)\left(\frac{16x^{2-\alpha(x,t)}}{\Gamma(3-\alpha(x,t))} + \frac{6x^{3-\alpha(x,t)}}{\Gamma(4-\alpha(x,t))}\right), \tag{30}$$

the initial and boundary conditions are:

$$u(x,0) = x^2(8-x), \quad u_t(x,0) = 0, \ and \ u(0,t) = u(8,t) = 0, \tag{31}$$

where $0 \le x \le 8$ *and* $T = 1$. Let

$$\psi(\tau) = \tanh(\tau) \ and \ \phi(h) = sinh(h).$$

The exact solution is: $u(x,t) = x^2(8-x)(t^2+1)$, when $\alpha = \beta = 2$.

A comparison between the numerical and the exact solutions when $N = 1000$ and $M = 125$ is presented in figure 1. In figures 2, 3 and 4, respectively we report the approximate solutions in three dimensions, where the axis's are (t,x,u), (α,x,u) and (β,x,u), respectively. Figure 2, shows the numerical solution at all values of the time and figures 3 and 4, show the numerical solutions change with respect to α and β, at $T = 1$. In Table 1, we calculate the absolute errors between the exact solution u_{ex} and the approximate solution u_{approx} when $N = 1000$ and $M = 125$. In Table 2, a comparison between the NSFD and the standard finite difference (SFD) solutions, where the accuracy of the NSFD is better than the SFD. From the results displayed in the table 2 and in all the figures, it is obvious that the proposed method is an efficient and able to give numerical solutions coincide closely with the exact solutions.

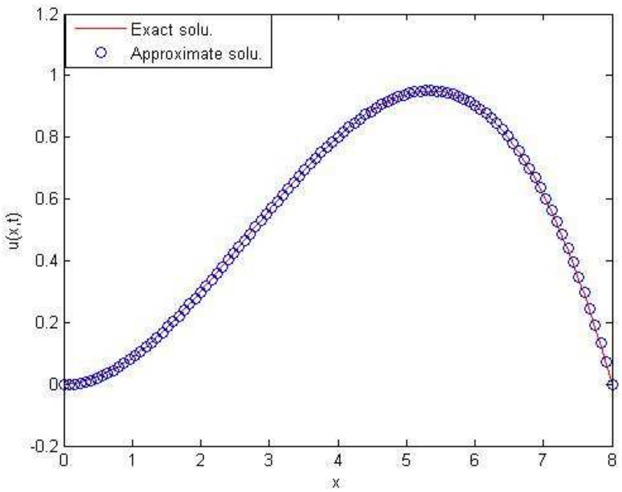

Fig. 1: Comparison between the analytical and the numerical solutions.

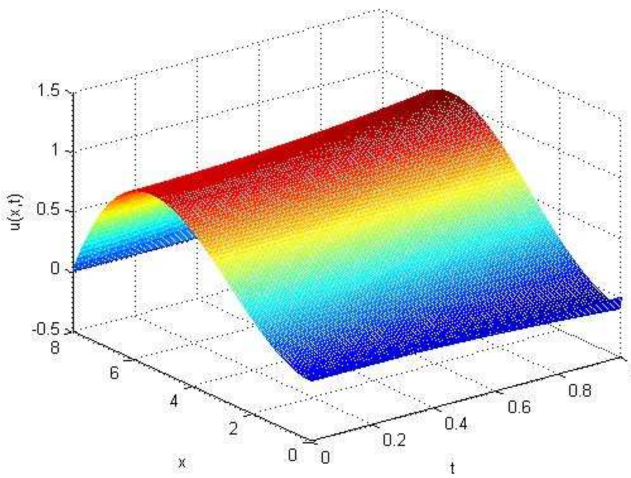

Fig. 2: The numerical solutions where the axis is (t,x,u).

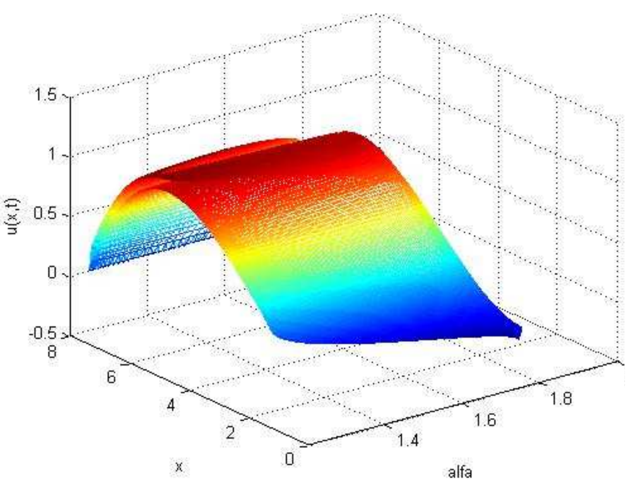

Fig. 3: The numerical solutions where the axis is (α,x,u).

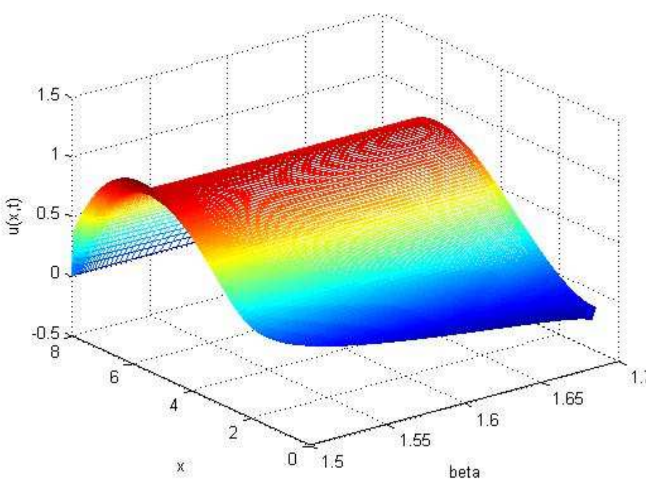

Fig. 4: The numerical solutions where the axis is (β,x,u).

Table 1: The absolute error between the exact solution and the approximate solution when $N = 1000$ and $M = 125$.

x	$\lvert u_{ex} - u_{approx} \rvert$
0.0000	0.00000000
0.8000	0.00270905
1.6000	0.00247795
2.4000	0.00212119
3.2000	0.00162219
4.0000	0.00091329
4.8000	0.00009988
5.6000	0.00152948
6.4000	0.00350169
7.2000	0.00615396
8.0000	0.00000000

Table 2: The maximum error of the NSFD and the SFD methods

T	maximum error of NSFD	maximum error of SFD
T=1	$6.1539e^{-3}$	0.0102
T=4	$3.4818e^{-3}$	0.6469
T=8	$9.0641e^{-5}$	1.0926

Example 4.2. Consider the following variable-order nonlinear fractional wave equation:

$$D_t^{\beta(x,t)} u(x,t) = 2\,cos(t)\; D_x^{\alpha(x,t)} u(x,t) + f(u,x,t), \tag{32}$$

with $\alpha(x,t) = 2 - cos^2(x)\, sin^2(t), \quad \beta(x,t) = 1.8 + 0.5\, e^{-(xt)^2 - 1}$,
and $u(x,0) = 1 + \sin(x)$, $u_t(x,0) = 0,\; and\; u(0,t) = cos(t),\; u(10,t) = 0.174 + cos(t)$,

where $0 \le x \le 10$, $T = 1$, and $f(u,x,t) = u - sin(x) - cos(t) + 2sin(x)cos(t) - cos(t)$.
Let

$$\psi(\tau) = e^{\tau^2} - 1 \;\; and \;\; \phi(h) = sinh(h).$$

The exact solution is:

$$u(x,t) = sin\,x + cos\,t,\;\; when\;\; \alpha = \beta = 2. \tag{33}$$

Figure 5, shows the behavior of the exact solutions and the numerical solutions of the proposed method with $N = 500$ and $M = 125$. In figure 6, a comparison between the SFD and the exact solutions when $N = 500$ and $M = 125$. Figures 5 and 6, show that the accuracy of the NSFD is better than the SFD. Figure 7, shows the approximate solution change with respect to β at $T = 1$, where the axis's are (β, x, u).

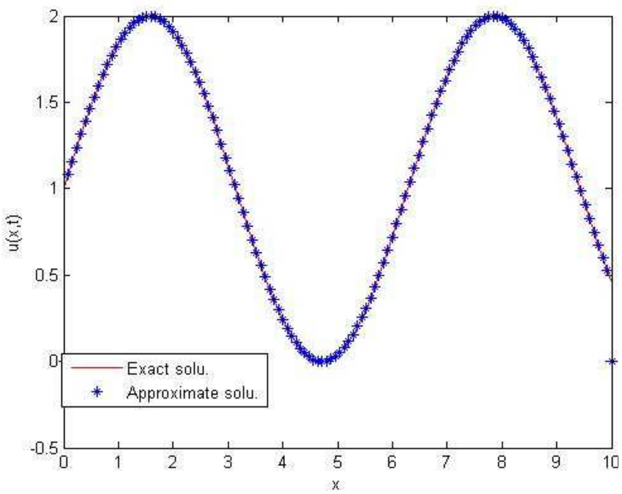

Fig. 5: Comparison between the exact and the NSFD.

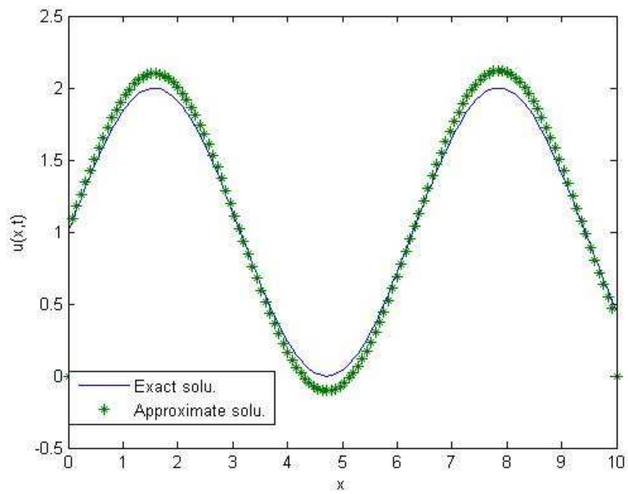

Fig. 6: Comparison between the exact and the SFD.

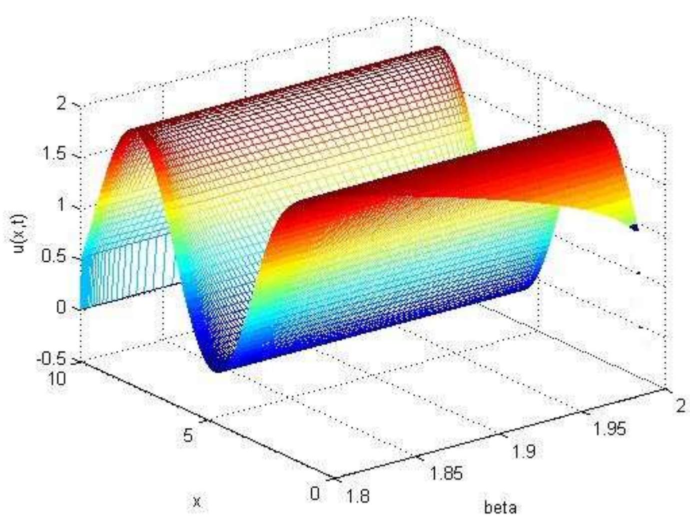

Fig. 7: The numerical solutions where the axis is (β, x, u).

Example 4.3. Consider the following variable-order nonlinear fractional wave equation:

$$D_t^{\beta(x,t)} u(x,t) = D_x^{\alpha(x,t)} u(x,t) + f(u,x,t), \quad 0 < x < 2 \ \ and \ \ 0 < t < 1, \tag{34}$$

with $\alpha(x,t) = 2 - cos^2(x)\, sin^2(t), \quad \beta(x,t) = 1.8 + 0.5\, e^{-(xt)^2 - 1}$,

where $f(u,x,t) = -u - 2sin(t)$,

the initial and boundary conditions are:

$$u(x,0) = 0, \ \ \frac{\partial u}{\partial t}(x,0) = x^2, \ \ u(0,t) = 0, \ \ and \ \ u(2,t) = 4\,sin(t).$$

Let

$$\psi(\tau) = \sinh(\tau^2) \;\; and \;\; \phi(h) = 1 - e^{-h^2}.$$

When $\alpha = 2$, the exact solution is:

$$u(x,t) = x^2 sin(t).$$

In Table 3, the absolute error between the exact solution u_{ex} and the NSFD solution u_{approx} are given where the maximum error is $4.1179e^{-4}$, with $N = 200$ and $M = 100$. In order to test the numerical scheme, we describe in figure 8 the analytical and the approximate solutions at $N = 200$ and $M = 100$. To study the behaviour of the solutions figure 9, shows the 3D solutions. Table 4, shows the absolute error between the exact solution u_{ex} and the SFD solution u_{approx} where the maximum error is $1.7304e^{-2}$, with $N = 200$ and $M = 100$. From the results displayed in Tables 3 and 4, it is obvious that the accuracy of the NSFD is better than the SFD. So, the proposed method is an efficient and able to give numerical solutions coincide closely with the exact solutions.

Table 3: The absolute error between the exact solution u_{ex} and the NSFD solution u_{approx}.

x_i	u_{ex}	u_{approx}	$\lvert u_{ex} - u_{approx} \rvert$
0.0000	0.00000000	0.00000000	0.00000000
0.4000	0.01578229	0.01580835	0.00002605
0.8000	0.06312918	0.06323240	0.00010322
1.2000	0.14204065	0.14227245	0.00023180
1.6000	0.25251671	0.25292850	0.00041179
2.0000	0.39455736	0.39455736	0.00000000

Table 4: The absolute error between the exact solution u_{ex} and the SFD solution u_{approx}.

x_i	u_{ex}	u_{approx}	$\lvert u_{ex} - u_{approx} \rvert$
0.0000	0.00000000	0.00000000	0.00000000
0.4000	0.01578229	0.02754002	0.01175772
0.8000	0.06312918	0.07684702	0.01371785
1.2000	0.14204065	0.15763362	0.01559297
1.6000	0.25251671	0.26982165	0.01730494
2.0000	0.39455736	0.39455736	0.00000000

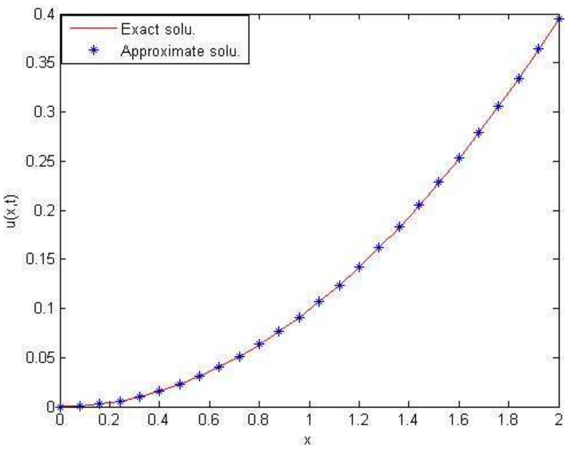

Fig. 8: Comparison between the analytical and the NSFD solutions with $\tau = 0.005$.

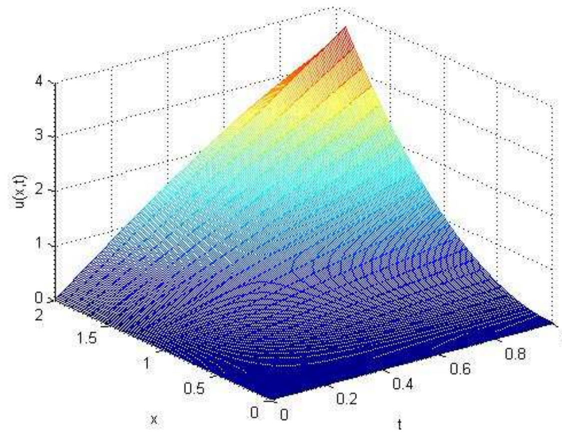

Fig. 9: 3D- solutions with $\tau = 0.005$.

Conclusions

In this paper, the NSFD method is applied for solving the space-time variable order fractional wave equation, where the variable order derivative is defined in the sense of Caputo. Special attentions are given to study the stability analysis and the truncation error of the method. Numerical experiments are done to test the method. The obtained results are compared with the SFD results. Moreover, NSFD gives good results than SFD. From these results, we observed that the NSFD method is more efficient for solving the variable order fractional wave equation than the SFD method. All results are obtained by using MATLAB (R2013b).

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