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Characterization of Exponential Distribution through Equidistribution Conditions for Consecutive Maxima

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Abstract: A characterization of the exponential distribution based on equidistribution conditions for maxima of random samples with consecutive sizes $n-1$ and n for an arbitrary and fixed $n \geq 3$ is proved. This solves an open problem stated recently in Arnold and Villasenor [3].

Keywords: characterizations, exponential distribution, order statistics, maxima

1 Introduction

Characterizations of the exponential distribution are abundant. Comprehensive surveys can be found in Ahsanullah and Hamedani [1], Arnold and Huang [2], and Johnson, Kotz, and Balakrishnan [5]. Recently, Arnold and Villasenor [3] obtained a series of characterizations based on random sample of size two. They also identified a list of conjectures for possible extensions of their results to larger samples. In this work we confirm that one of these conjectures is true for a sample of any fixed size $n \geq 2$. Note that in Yanev and Chakraborty [8] the case of random sample of size three was considered.

Let $X_1, X_2, \ldots X_n$, $n \geq 2$ be a random sample from an exponentially distributed parent X. It is known that

$$\max\{X_1, X_2, \ldots X_{n-1}\} + \frac{1}{n} X_n \stackrel{d}{=} \max\{X_1, X_2, \ldots X_n\}, \quad (1)$$

where $\stackrel{d}{=}$ denotes equality in distribution. We write $X \sim \exp(\lambda)$ if the probability density function (pdf) of X equals $f_X(x) = \lambda e^{-\lambda x} I(x > 0)$. Our goal is to prove that (1), under analyticity assumptions on the cumulative distribution function (cdf) F of X, is a sufficient condition for X to be exponential.

Theorem Let X be a non-negative continuous random variable with pdf f. If f is analytic in a neighborhood of zero and (1) holds true, then $X \sim \exp(\lambda)$ with some $\lambda > 0$.

Wesołowski and Ahsanullah [7] and more recently Castaño-Martinez et al. [4] proved characterizations of probability distributions in the context of random translations. The characterization (1) above can be deduced from their results (see Corollary 1 in Wesołowski and Ahsanullah [7] and Corollary 3 in Castaño-Martinez et al. [4]). However, our proof is different from theirs in not referring to uniqueness results for integral equations. The direct approach we follow may also be used in obtaining some more general results, a possibility which we will explore in the future.

2 Preliminaries

Define for all non-negative integers n, i, and any real number x

$$H_{n,i}(x) := \sum_{j=0}^{n} (-1)^j \binom{n}{j} (x-j)^i.$$

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It is known, (e.g., Ruiz [6]) that for all integers $n \geq 0$ and all real x

$$H_{n,i}(x) = \begin{cases} n! \text{ if } & i = n; \\ 0 \text{ if } & 0 \leq i \leq n-1. \end{cases} \tag{2}$$

Define $G_m(x) := F^m(x) f(x)$ for $m \geq 1$ and denote by $g^{(i)}(x)$ for $i \geq 1$ the ith derivative of a function $g(x)$; $g^{(0)}(x) := g(x)$.

Lemma 1 Let X be a continuous random variable with cdf F satisfying $F(0) = 0$. If for $0 \leq r \leq m-1$

$$f^{(r)}(0) = \left[\frac{f'(0)}{f(0)}\right]^{r-1} f'(0), \tag{3}$$

then for $0 \leq i \leq 2m$

$$G_m^{(i)}(0) = \left[\frac{f'(0)}{f(0)}\right]^{i-m} f^{m+1}(0) H_{m,i}(m+1). \tag{4}$$

Proof. **Case** $\mathbf{0 \leq i \leq m-1}$**.** In this case (2) implies $H_{m,i}(m+1) = 0$. On the other hand, in the left-hand side of (4), we have $G_m^{(i)}(0) = 0$ because each term in the expansion of $G_m^{(i)}(0)$ has a factor $F(0) = 0$.

Case $\mathbf{i = m}$**.** From (2) it follows that (4) is equivalent to

$$G_m^{(m)}(0) = m! f^{m+1}(0). \tag{5}$$

We shall prove (5) by induction. If $m = 1$, then (5) follows from the definition of $G(x)$ and the assumption $F(0) = 0$. Assuming that (5) is true for $m = k$, we will prove it for $m = k+1$. Since $G_{k+1}(x) = F(x)G_k(x)$ and $F(0) = 0$, we have

$$\begin{aligned} G_{k+1}^{(k+1)}(0) &= \sum_{j=0}^{k+1} \binom{k+1}{j} F^{(j)}(0) G_k^{(k+1-j)}(0) \\ &= F(0)G_k^{(k+1)}(0) + (k+1)F^{(1)}(0)G_k^{(k)}(0) \\ &= (k+1) f(0) k! f^{k+1}(0) \\ &= (k+1)! f^{k+2}(0), \end{aligned}$$

where we have used that $G_k^{(r)}(0) = 0$ for $0 \leq r \leq k-1$ and the induction assumption $G_k^{(k)}(0) = k! f^{k+1}(0)$.

Case $\mathbf{m < i \leq 2m}$**.** Suppose we have proved (4) for $m = 1, 2, \ldots k$. We want to prove it for $m = k+1$. Observe that

$$G_{k+1}^{(i)}(0) = \sum_{j=0}^{i} \binom{i}{j} F^{(j)}(0) G_k^{(i-j)}(0).$$

Since $G_k^{(r)}(0) = 0$ for $0 \leq r \leq k-1$, making use of (3) and the induction assumption, we obtain

$$\begin{aligned} G_{k+1}^{(i)}(0) &= \sum_{j=1}^{k} \binom{i}{j} f^{(j-1)}(0) G_k^{(i-j)}(0) + \sum_{j=k+1}^{i} \binom{i}{j} f^{(j-1)}(0) G_k^{(i-j)}(0) \\ &= \sum_{j=1}^{k} \binom{i}{j} \left[\frac{f'(0)}{f(0)}\right]^{j-2} f'(0) \left[\frac{f'(0)}{f(0)}\right]^{i-j-k} f^{k+1}(0) H_{k,i-j}(k+1) \\ &= \left[\frac{f'(0)}{f(0)}\right]^{i-k-1} f^{k+2}(0) \sum_{j=1}^{i} \binom{i}{j} H_{k,i-j}(k+1), \end{aligned} \tag{6}$$

where in the last equality we used that (2) implies $H_{k,i-j}(k+1)=0$ for $j=i+1,\dots,k$. Further, we have

$$\sum_{j=1}^{i}\binom{i}{j}H_{k,i-j}(k+1)=\sum_{r=0}^{k}(-1)^r\binom{k}{r}\sum_{j=1}^{i}\binom{i}{j}(k+1-r)^{i-j}$$
$$=\sum_{r=0}^{k}(-1)^r\binom{k}{r}\left[(k+2-r)^i-(k+1-r)^i\right]$$
$$=(k+2)^i-\left[(k+1)^i+\binom{k}{1}(k+1)^i\right]+\left[\binom{k}{1}k^i+\binom{k}{2}k^i\right]+\dots+(-1)^k\left[\binom{k}{k-1}2^i+2^i\right]+(-1)^{k+1}$$
$$=(k+1)^i-\binom{k+1}{1}(k+1)^i+\dots+(-1)^k\binom{k+1}{k}2^i+(-1)^{k+1}$$
$$=\sum_{j=0}^{k+1}(-1)^j\binom{k+1}{j}(k+2-j)^i=H_{k+1,i}(k+2).$$

The lemma's claim follows by induction, taking into account (6).

The identity below may be of independent interest.

Lemma 2 For any integers $m\geq 0$ and $k\geq 0$

$$\sum_{j=0}^{m}(k+2)^{m-j}H_{k,j}(k+1)=\sum_{j=0}^{m}\binom{m+1}{j+1}H_{k,j}(k+1). \tag{7}$$

Proof. The left-hand side of (7) equals

$$\sum_{j=0}^{m}(k+2)^{m-j}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k+1-i)^j=\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k+2)^m\sum_{j=0}^{m}\left(\frac{k+1-i}{k+2}\right)^j \tag{8}$$
$$=\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{1}{i+1}\left[(k+2)^{m+1}-(k+1-i)^{m+1}\right]$$
$$=\sum_{i=0}^{k}(-1)^i\binom{k+1}{i+1}\frac{1}{k+1}\left[(k+2)^{m+1}-(k+1-i)^{m+1}\right]$$
$$=-\frac{(k+2)^{m+1}}{k+1}\sum_{r=1}^{k+1}(-1)^r\binom{k+1}{r}+\frac{1}{k+1}\sum_{r=1}^{k+1}(-1)^r\binom{k+1}{r}(k+2-r)^{m+1}$$
$$=-\frac{(k+2)^{m+1}}{k+1}\left[\sum_{r=0}^{k+1}(-1)^r\binom{k+1}{r}-1\right]+\frac{1}{k+1}\left[\sum_{r=0}^{k+1}(-1)^r\binom{k+1}{r}(k+2-r)^{m+1}-(k+2)^{m+1}\right]$$
$$=\frac{1}{k+1}\sum_{r=0}^{k+1}(-1)^r\binom{k+1}{r}(k+2-r)^{m+1}.$$

For the right-hand side of (7) we obtain

$$\sum_{j=0}^{m}\binom{m+1}{j+1}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k+1-i)^j=\sum_{i=0}^{k}(-1)^i\binom{k}{i}\sum_{j=0}^{m}\binom{m+1}{j+1}(k+1-i)^j$$
$$=\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{1}{k+1-i}\sum_{j=0}^{m}\binom{m+1}{j+1}(k+1-i)^{j+1}$$
$$=\frac{1}{k+1}\sum_{i=0}^{k}(-1)^i\binom{k+1}{i}\sum_{r=1}^{m+1}\binom{m+1}{r}(k+1-i)^r$$
$$=\frac{1}{k+1}\sum_{i=0}^{k}(-1)^i\binom{k+1}{i}\left[\sum_{r=0}^{m+1}\binom{m+1}{r}(k+1-i)^r-1\right]$$
$$=\frac{1}{k+1}\sum_{i=0}^{k}(-1)^i\binom{k+1}{i}(k+2-i)^{m+1}-\frac{1}{k+1}\left[\sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}-(-1)^{k+1}\right]$$
$$=\frac{1}{k+1}\sum_{r=0}^{k+1}(-1)^r\binom{k+1}{r}(k+2-r)^{m+1},$$

which equals (8). The proof of the lemma is complete.

Next lemma (see also Arnold and Villaseñor [3]) will play a crucial role in the proof of the theorem. In private communications, P. Fitzsimmons pointed out to us that the assumption of analyticity of the density function f is missing in [3].

Lemma 3 If $F(0) = 0$, the pdf f is analytic in a neighborhood of 0, and

$$f^{(k)}(0) = \left[\frac{f'(0)}{f(0)}\right]^{k-1} f'(0), \qquad k = 1,2,\ldots, \tag{9}$$

then $X \sim \exp\{\lambda\}$ for some $\lambda > 0$.

Proof. For the Maclaurin series of $f(x)$, we have for $x > 0$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + \sum_{k=1}^{\infty} \left[\frac{f'(0)}{f(0)}\right]^{k-1} f'(0)\frac{x^k}{k!} = f(0)\exp\left\{\frac{f'(0)}{f(0)}x\right\}. \tag{10}$$

Since $f(x)$ is a pdf, we have $f'(0)/f(0) < 0$. Denoting $\lambda = -f'(0)/f(0) > 0$ and setting the integral of (10) from 0 to ∞ to be 1, we obtain $\lambda = f(0)$. Therefore, $f(x) = \lambda e^{-\lambda x} I(x > 0)$, i.e., $X \sim \exp\{\lambda\}$.

3 Proof of the theorem

Equation (1) can be written as

$$\int_0^x f_{X_n/n}(y) f_{\max\{X_1,\ldots,X_{n-1}\}}(x-y)\,dy = n(n-1)f(x)\int_0^x G_{n-2}(y)\,dy.$$

This is equivalent to

$$\int_0^x nf(ny)(n-1)F^{n-2}(x-y)f(x-y)\,dy = n(n-1)f(x)\int_0^x G_{n-2}(y)\,dy,$$

which simplifies to

$$\int_0^x f(ny)G_{n-2}(x-y)\,dy = f(x)\int_0^x G_{n-2}(y)\,dy. \tag{11}$$

Differentiating the left-hand side of (11) with respect to x, we obtain

$$\frac{d}{dx}\int_0^x f(ny)G_{n-2}(x-y)\,dy = f(nx)G_{n-2}(0) + \int_0^x f(ny)G'_{n-2}(x-y)\,dy.$$

Differentiating the last equation $2n-3$ times, we obtain

$$\frac{d^{2n-2}}{dx^{2n-2}}\int_0^x f(ny)G_{n-2}(x-y)\,dy = \sum_{i=0}^{2n-3} n^{2n-3-i} f^{(2n-3-i)}(x)G_{n-2}^{(i)}(0) + \int_0^x f(ny)G_{n-2}^{(2n-2)}(x-y)\,dy. \tag{12}$$

On the other hand, applying to the right-hand side of (11) the Leibnitz product rule of differentiation, we have

$$\frac{d^{2n-2}}{dx^{2n-2}}\left[f(x)\int_0^x G_{n-2}(y)\,dy\right] = \sum_{i=0}^{2n-3}\binom{2n-2}{i+1} f^{(2n-3-i)}(x)G_{n-2}^{(i)}(x) + f^{(2n-2)}(x)\int_0^x G_{n-2}(y)\,dy \tag{13}$$

Therefore, the equation (11), taking into account (12) and (13), becomes

$$\sum_{i=0}^{2n-3} n^{2n-3-i} f^{(2n-3-i)}(x)G_{n-2}^{(i)}(0) + \int_0^x f(ny)G_{n-2}^{(2n-2)}(x-y)\,dy \tag{14}$$
$$= \sum_{i=0}^{2n-3}\binom{2n-2}{i+1} f^{(2n-3-i)}(x)G_{n-2}^{(i)}(x) + f^{(2n-2)}(x)\int_0^x G_{n-2}(y)\,dy.$$

Setting $x = 0$ and taking into account that $G^{(i)}_{n-2}(0) = 0$ for $0 \le i \le n-3$, we obtain that (14) is equivalent to

$$\sum_{i=n-2}^{2n-4} n^{2n-3-i} f^{(2n-3-i)}(0)G^{(i)}_{n-2}(0) = \sum_{i=n-2}^{2n-4} \binom{2n-2}{i+1} f^{(2n-3-i)}(0)G^{(i)}_{n-2}(0).$$

For $i = n-2$, we have $f^{(n-1)}(0)G^{(n-2)}_{n-2}(0) = f^{(n-1)}(0)f^{n-1}(0)(n-2)!$. Thus, the equation above can be written as

$$\left[n^{n-1} - \binom{2n-2}{n-1}\right] f^{(n-1)}(0)f^{n-1}(0)(n-2)! = \sum_{i=n-1}^{2n-4} \left[\binom{2n-2}{i+1} - n^{2n-3-i}\right] f^{(2n-3-i)}(0)G^{(i)}_{n-2}(0). \tag{15}$$

In view of Lemma 3, to complete the proof it suffices to show

$$f^{(r)}(0) = \left[\frac{f'(0)}{f(0)}\right]^{r-1} f'(0), \qquad r = 1,2,\dots \tag{16}$$

Assume (16) for all $1 \le r \le n-2$. We shall prove it for $r = n-1$, i.e.,

$$f^{(n-1)}(0) = \left[\frac{f'(0)}{f(0)}\right]^{n-2} f'(0), \qquad r = 1,2,\dots \tag{17}$$

It follows from Lemma 1 with $m = n-2$ that for $n-1 \le i \le 2n-4$

$$f^{(2n-3-i)}(0)G^{(i)}_{n-2}(0) = \left[\frac{f'(0)}{f(0)}\right]^{i-n+2} f^{n-1}(0)H_{n-2,i}(n-1). \tag{18}$$

Substituting (18) in the right-hand side of (15) we obtain

$$\left[n^{n-1} - \binom{2n-2}{n-1}\right] f^{(n-1)}(0)(n-2)! = \left[\frac{f'(0)}{f(0)}\right]^{n-2} f'(0) \sum_{i=n-1}^{2n-4} \left[\binom{2n-2}{i+1} - n^{2n-3-i}\right] H_{n-2,i}(n-1).$$

To establish (18) we need to prove

$$\left[n^{n-1} - \binom{2n-2}{n-1}\right] = \sum_{i=n-1}^{2n-4} \left[\binom{2n-2}{i+1} - n^{2n-3-i}\right] H_{n-2,i}(n-1)$$

or equivalently

$$\sum_{i=n-2}^{2n-4} n^{2n-3-i} H_{n-2,i}(n-1) = \sum_{i=n-2}^{2n-4} \binom{2n-2}{i+1} H_{n-2,i}(n-1). \tag{19}$$

Since (2) implies $H_{n-2,i}(n-1) = 0$ for $0 \le i \le n-3$ and for $i = 2n-3$ we have $n^{2n-3-i} = \binom{2n-2}{i+1} = 1$, we obtain that (19) is equivalent to

$$\sum_{i=0}^{2n-3} n^{2n-3-i} H_{n-2,i}(n-1) = \sum_{i=0}^{2n-3} \binom{2n-2}{i+1} H_{n-2,i}(n-1),$$

which follows from Lemma 3 with $m = 2n-3$. This completes the induction argument and thus proves (16). Referring to (16) and Lemma 2 we complete the proof of the theorem.

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