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# Interval Estimation for Burr Type-X Distribution under Type-I Hybrid Progressive Censoring Scheme

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**Abstract:** In this research, point and interval estimation for unknown parameters of Burr Type-X (Burr-X) distribution based on the Type-I hybrid progressive censoring are obtained. For the estimation process, the maximum likelihood estimations (MLEs) and Bayesian methods (BM) with Markov Chain Monte Carlo (MCMC) are used. Finally, some computations and comparisons between the different methods based on simulation data are obtained.

**Keywords:** Type-I Hybrid progressive censored scheme; Bayesian and non-Bayesin estimation; Burr-X distribution; Markov chain Monte Carlo technique.

## 1 Introduction

The censoring schemes are very important in lifetimes. There are many situations where the experimenter may not be able to obtain full information about the failure times for all experimental elements. There are also situations in which the pre-failure elements are planned in order to reduce the cost and time associated with the test. Traditional censoring, including Type-I and Type-II schemes, does not enjoy the flexibility of removing elements except at the endpoint of the experiment. The hybrid censoring scheme is defined as a mixture between the Type-I and Type-II schemes. Epstein [1,2] introduced the hybrid censoring scheme and now became most popular in life test experiments. Some papers have tackled in this point of research including Amein [3,4], Amein et al. [5], Balakrishnan [6], Balakrishnan and Cramer [7], Kundu [8] and Mohie El-Din et al. [9].

The paper is organized as follows: The description of the model and the fixed of necessary assumptions are presented in Section 2. In section 3, we calculate the MLEs. Then follow the fisher information matrix and construction of confidence intervals are calculated in Section 4. By using the squared error loss function the Bayes estimation is presented in Section 5. Bayesian estimation using MCMC approach is obtained in Section 6. The numerical results and the analysis of obtained data are presented in Section 7. Finally, real data set is presented as application, some remarks and conclusion will be given in Section 8.

## 2 Description of Model

It is known that the Type-I hybrid Progressive censoring scheme (TIHPCS) is a mixture from the Type-II progressive and hybrid censoring scheme. It considers $n$ identical items setting on a life-test and their lifetime distributions denoted by $X_1,....,X_n$. Suppose $m$ and $T$ are the number of failures and the time, respectively which is predetermined beforehand such that $m<n$. Also, suppose the integer numbers $R_1,R_2,\dots,R_m$ represent the number of failure and units which are dropped from the experiment satisfying the equation $\sum_{i=1}^{m} R_i+m=n$. When the first failure occurs $X_{(1:m:n)}$, the $R_1$ of the remaining units are randomly removed from the experiment and so on for reach to the $m^{th}$ failure or the predetermined time $T$. Life

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test is terminated at a random time $T_1 = min(X_{(m:m:n)}, T)$. If the test quits at time $T$ it will satisfy $X_{(j:m:n)} < T < X_{(j+1:m:n)}$ and $R_j^* = n - R_1 - \ldots - R_j - j$.

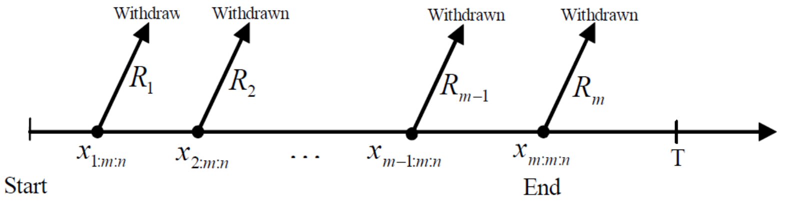


**Fig. 1** experiment terminate at $m^{th}$ failure i.e. $x_m < T$

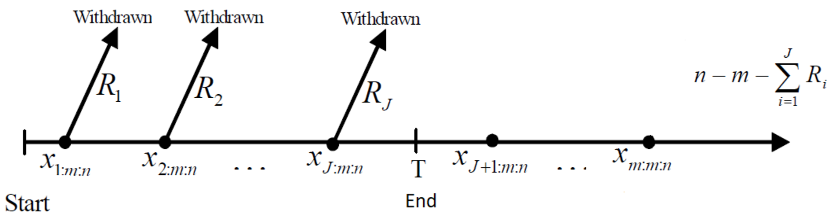


**Fig. 2** experiment terminate at time $T$, i.e. $T < x_m$

The observed sample might be one of the follows two cases:

$$CaseI: \{X_{(1:m:n)}, X_{(2:m:n)}, \ldots, X_{(m:m:n)}\}, \quad if \quad X_{(m:m:n)} < T(see fig.1), \tag{1}$$

$$CaseII: \{X_{(1:m:n)}, X_{(2:m:n)}, \ldots, X_{(j:m:n)}\}, \quad if \quad X_{(j:m:n)} < T < X_{(j+1:m:n)}, (see fig.2). \tag{2}$$

Burr [10] has derived a family of Burr distributions. One of this family is Burr Type-X and also is called the generalization of the Rayleigh distribution. It is known that the Burr distributions played an important practical role in reliability study, phenomena modeling, health, agriculture and biology.

The cumulative distribution function and probability density function with shape parameter is $\alpha$ and scale parameter is $\beta$ are given by:

$$F(x;\alpha,\beta) = (1 - e^{-(\beta x)^2})^{\alpha} \quad, x > 0\,, \alpha, \beta > 0. \tag{3}$$

$$f(x;\alpha,\beta) = 2\alpha\beta x e^{-(\beta x)^2}(1 - e^{-(\beta x)^2})^{(\alpha-1)} \quad, x > 0, \alpha, \beta > 0. \tag{4}$$

The reliability function and hazard rate function are respectively:

$$S(x,\alpha,\beta) = 1 - (1 - e^{-(\beta x)^2})^{\alpha} \tag{5}$$

and

$$H(x,\alpha,\beta) = \frac{2\,\alpha\,\beta\,x e^{-(\beta x_i^2)}(1 - e^{-(\beta x_i^2)})^{(\alpha-1)}}{1 - (1 - e^{-(\beta x_i^2)})^{\alpha}} \tag{6}$$

## 3 Maximum Likelihood Estimation (MLE)

Here, we use the MLE technique to make a confidence interval for the unknown parameters $\alpha$ and $\beta$ of Burr-X. The likelihood function can written as:

$$Case \quad I: \quad L(\alpha,\beta) \propto \prod_{i=1}^{m} f(x_i)\,[1 - F(x_i)]^{R_i}, \tag{7}$$

$$Case \quad II: \quad L(\alpha,\beta) \propto \prod_{i=1}^{m} f(x_i)\,[1 - F(x_i)]^{R_i}\,[1 - F(T)]^{R_j^*}, \tag{8}$$

Substituting from Eqs. (3 - 4) in Eqs. (7 - 8); yields:

$$Case \quad I: \quad L(\alpha,\beta|x) \propto \prod_{i=1}^{m} 2\alpha\beta x_i \; e^{-(\beta x_i^2)} \left(1-e^{-(\beta x_i^2)}\right)^{\alpha-1} [1-(1-e^{-(\beta x_i^2)})^{\alpha}]^{R_j}, \tag{9}$$

$$Case \quad II: \quad L(\alpha,\beta|x) \propto \prod_{i=1}^{j} 2\alpha\beta x_i \; e^{-(\beta x_i^2)} \left(1-e^{-(\beta x_i^2)}\right)^{\alpha-1} [1-(1-e^{-(\beta x_i^2)})^{\alpha}]^{R_i}[1-(1-e^{-\beta T^2})^{\alpha}]^{R_j^*}, \tag{10}$$

where $R_j^* = n - R_1 - \ldots - R_j - j$.
The likelihood in the two case can be written as:

$$L(\alpha,\beta|x) \propto \prod_{i=1}^{d} 2\alpha\beta x_i \; e^{-(\beta x_i^2)} \left(1-e^{-(\beta x_i^2)}\right)^{\alpha-1} [1-(1-e^{-(\beta x_i^2)})^{\alpha}]^{R_i}[1-(1-e^{-\beta T^2})^{\alpha}]^{R_j^*}, \tag{11}$$

From Eq. (11), the associated log-likelihood function can be written as:

$$\begin{aligned}\log L(\alpha,\beta|x) \propto d\log\alpha + d\log\beta + \sum_{i=1}^{d}\log x_i - \beta\sum_{i=1}^{d} x_i^2 + (\alpha-1)\sum_{i=1}^{d}\log(1-e^{-(\beta x_i^2)}) \\ + \sum_{i=1}^{d} R_j \log(1-(1-e^{-(\beta x_i^2)})^{\alpha}) + \sum_{i=1}^{d} R_j^* \log(1-(1-e^{-(\beta x_i^2)})^{\alpha}),\end{aligned} \tag{12}$$

differentiating Eq. (12) by $\alpha$and$\beta$, we find

$$\frac{\partial \log L}{\partial \alpha} = \frac{d}{\alpha} + \sum_{i=1}^{d}\log\left(1-e^{-(\beta x_i^2)}\right) - \sum_{i=1}^{d} R_j \frac{(1-e^{-(\beta x_i^2)})^{\alpha}}{1-(1-e^{-(\beta x_i^2)})^{\alpha}} \log(1-e^{-(\beta x_i^2)}) = 0, \tag{13}$$

$$\frac{\partial \log L}{\partial \beta} = \frac{d}{\beta} - \sum_{i=1}^{d} x_i^2 + (\alpha-1)\sum_{i=1}^{d}\frac{x_i^2 e^{-(\beta x_i^2)}}{(1-e^{-(\beta x_i^2)})} - \sum_{i=1}^{d} R_j \frac{\alpha(1-e^{-(\beta x_i^2)})^{\alpha-1} e^{-(\beta x_i^2)} x_i^2}{1-(1-e^{-(\beta x_i^2)})^{\alpha}} = 0, \tag{14}$$

$$\begin{aligned}= \frac{d}{\beta} - \sum_{i=1}^{d} x_i^2 + (\alpha-1)\sum_{i=1}^{d}\frac{x_i^2 e^{-(\beta x_i^2)}}{(1-e^{-(\beta x_i^2)})} - \sum_{i=1}^{d} R_j \frac{\alpha(1-e^{-(\beta x_i^2)})^{\alpha-1} e^{-(\beta x_i^2)} x_i^2}{1-(1-e^{-(\beta x_i^2)})^{\alpha}} \\ - \sum_{i=1}^{d} R_j^* \frac{\alpha T^2(1-e^{-\beta T^2})^{\alpha-1} e^{-\beta T^2}}{1-(1-e^{-\beta T^2})^{\alpha}} = 0.\end{aligned}$$

Estimators of $\alpha$and$\beta$ can be obtained by solving the non-linear Eqs. (14 - 15) by numerical technique.
In addition, the MLE of $\hat{S}(x,\alpha,\beta)$ and $\hat{H}(x,\alpha,\beta)$ take the following form:

$$\hat{H}(x,\alpha,\beta) = \frac{2\,\hat{\alpha}\,\hat{\beta}\, x e^{(-\hat{\beta}x_i^2)}(1-e^{(-\hat{\beta}x_i^2)})^{(\hat{\alpha}-1)}}{1-(1-e^{(-\hat{\beta}x_i^2)})^{\hat{\alpha}}}, \tag{15}$$

and

$$\hat{S}(x,\alpha,\beta) = 1-(1-e^{(-\hat{\beta}x_i^2)})^{\hat{\alpha}}. \tag{16}$$

## 4 Asymptotic Confidence Intervals

Finding confidence interval estimation for unknown parameters, associated hazard rate and survival function is the main aim of this section. For this purpose, the Fisher information matrix $I = [I_{i,j}]$ , $i, j = 1,2$ is given by:

$$I = -\begin{bmatrix} \frac{\partial^2 \log L}{\partial \alpha^2} & \frac{\partial^2 \log L}{\partial \alpha \partial \beta} \\ \frac{\partial^2 \log L}{\partial \beta \partial \alpha} & \frac{\partial^2 \log L}{\partial \beta^2} \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix}. \tag{17}$$

From the above equations, the variance-covariance matrix is approximated as $[V_{ij}] = [I]^{-1}$.
The $(1-\eta)\%$ approximate confidence intervals for $\alpha, \beta$ respectively are:

$$\hat{\alpha} \pm Z_{(\eta/2)}\sqrt{\hat{V}_{11}} \quad \text{and} \quad \hat{\beta} \pm Z_{(\eta/2)}\sqrt{\hat{V}_{22}},$$

where $\hat{V}_{11}$and $\hat{V}_{22}$ are the elements on the main diagonal of the covariance matrix $I^{-1}(\hat{\alpha}, \hat{\beta})$ and $z_{\eta/2}$ is the percentile of the standard normal distribution with right-tail probability $\eta/2$.

## 5 Bayes Method

Now, we will estimate the unknown parameters of Burr-X distribution by using Bayesian approach. Suppose that $X_1, X_2, ..., X_n$ denotes a Type-I progressive hybrid censored sample drawn from a Burr $X(\alpha, \beta)$ distribution. It is assumed that $\alpha$ and $\beta$ are a prior distributed as independent Gamma$(a,b)$ and Gamma$(p,q)$ distributions respectively.

$$\Pi(\alpha) \propto \alpha^{a-1} e^{-b\alpha}, \ \alpha > 0, a > 0, b > 0 \tag{18}$$

and

$$\Pi(\beta) \propto \beta^{p-1} e^{-q\beta}, \ \beta > 0, p > 0, q > 0. \tag{19}$$

Therefore the joint prior distribution of $\alpha$ and $\beta$ takes the following form,

$$g(\alpha, \beta) = L(\alpha, \beta|x)\pi(\alpha)\pi(\beta). \tag{20}$$

From the previous equation Eq. (20) the posterior density function for the parameters $\alpha$ and $\beta$ for given the data, denoted by $\pi^*(\alpha, \beta|x)$, becomes

$$\pi^*(\alpha, \beta|x) = \frac{L(\alpha, \beta|x)\pi(\alpha)\pi(\beta)}{\int_0^\infty \int_0^\infty L(\alpha, \beta|x)\pi(\alpha)\pi(\beta) d\alpha d\beta}. \tag{21}$$

For any function under squared error loss function, the Bayes estimator of is the posterior mean which takes the following form:

$$\hat{g}(\alpha, \beta|x) = E(g(\alpha, \beta|x)) = \frac{\int_0^\infty \int_0^\infty g(\alpha, \beta) L(\alpha, \beta|x)\pi(\alpha)\pi(\beta)}{\int_0^\infty \int_0^\infty L(\alpha, \beta|x)\pi(\alpha)\pi(\beta) d\alpha d\beta}. \tag{22}$$

In general, the Eq. (22) is complicated and difficult to solve. Therefore, the MCMC approach is used to solve the double integration and may be a suitable method in this case. The mechanism of MCMC is generating samples from the posterior distributions and then computing the Bayes estimator of $\hat{g}(\alpha, \beta|x)$.

## 6 Markov Chain Monte-Carlo (MCMC) Approach

The Metropolis-Hastings method is widely used in case of Eq. (23). It simulates samples from a prescribed posterior distribution. It was developed by Metropolis et al. [11] and extended later by Hastings [12] and now it is the important method in case of Eq. (23) compared with the Lindly approximation, Importance sampling and others. The posterior density function becomes:

$$g(\alpha, \beta|x) \propto \alpha^{d+a-1} e^{-b\alpha} \beta^{d+p-1} e^{-q\beta} x_i \ e^{-(\beta x_i^2)} \left(1 - e^{-(\beta x_i^2)}\right)^{\alpha-1} [1-(1-e^{-(\beta x_i^2)})^\alpha]^{R_i} [1-(1-e^{-\beta T^2})^\alpha]^{R_j^*}. \tag{23}$$

From Eq. (23), the prior density function of $\alpha$ for given $\beta$ and $\beta$ for given $\alpha$ becomes:

$$g_1(\alpha|\beta,x) \propto \alpha^{d+a-1}\left(1-e^{-(\beta x_i^2)}\right)^{\alpha-1}[1-(1-e^{-(\beta x_i^2)})^{\alpha}]^{R_i}[1-(1-e^{-\beta T^2})^{\alpha}]^{R_j^*}, \tag{24}$$

and

$$g_2(\beta|\alpha,x) \propto \beta^{d+p-1}e^{-q\beta}e^{-(\beta x_i^2)}\left(1-e^{-(\beta x_i^2)}\right)^{\alpha-1}[1-(1-e^{-(\beta x_i^2)})^{\alpha}]^{R_i}[1-(1-e^{-\beta T^2})^{\alpha}]^{R_j^*}. \tag{25}$$

The prior density function of $\alpha$ and $\beta$ given cannot be reduced analytically to well known distributions. So, the normal distribution will be used as proposal distribution for generating $\alpha$ and $\beta$ from the posterior density functions and in turn obtain the Bayes estimates and the corresponding suitable intervals take the following form:

1.Start with $\beta^{(0)},\alpha^{(0)}$
2.Set $\nu=1$.
3.Generate $\alpha^{\nu}$ from $g_1$ with the normal $N(\alpha^{\nu}-1,V)$.
4.Generate $\beta^{\nu}$ from $g_2$ with the normal $N(\beta^{\nu-1},V_{22})$.
5.Calculate $\alpha^{\nu}$ and $\beta^{\nu}$.
6.Set $\nu=\nu+1$.
7.Repeat 3-6 steps $N$ times.
8.Rearrange the values $\alpha_i$ and $\beta_i, i=M+1,M+2,\cdots,N$.
9.Calculate the Bayes estimates of $\alpha$ and $\beta$ as:
$E(\alpha|data)=\frac{1}{N-M}\sum_{i=M+1}^{N}\alpha_i$ and $E(\beta|data)=\frac{1}{N-M}\sum_{i=M+1}^{N}\beta_i$ , where $M$ is burn-in.
10.$(1-\eta)\%$ suitable intervals of $\alpha$ and $\beta$ can be caculated as: $\left(\alpha_{\{\frac{\eta}{2}(N-M)\}},\alpha_{\{\frac{1-\eta}{2}(N-M)\}}\right)$ and $\left(\beta_{\{\frac{\eta}{2}(N-M)\}},\beta_{\{\frac{1-\eta}{2}(N-M)\}}\right)$.

## 7 Simulation Study

For achieving the simulation studies, we used (Mathematica ver. 8.0) for illustrating the theoretical results of estimation problem. Confidence interval for $\alpha$, $\beta$, $S$ and $H$ using ML and BE include MCMC approach. Non-informative denoted by (MCMC0) and informative priors denoted by (MCMC1) are calculated numerically. (500) samples have been generated from Type-I hybrid progressive samples from Burr-X distribution with different censoring schemes $R$ that contain $N=11000$ values with discarding the first $M=1000$ values in the case of MCMC as burn-in. The performances of ML and Bayesian (with joint gamma priors) methods are compared via mean squared errors technique. In Tables (2 - 7) below, average point estimation (mean), mean squared error (mse), interval estimation lower limit (LL), interval estimation upper limit (UL), interval length (IL) and coverage probability (cov) for $\alpha$, $\beta$, $S$ and $H$ are computed. All results are obtained at $\alpha=2;\beta=3;T=0.8;t=0.5;a=2;b=1;p=3;q=1;N=11000;M=1000;\eta=0.05$ and different censoring schemes (see Table 1).

**Table 1.** Censoring schemes with different values for $n$ and $m$ , where $0^r$ means that 0 repeated $r$ times.

| Censoring scheme | $n$ | $m$ | $R$ |
|---|---|---|---|
| $CS_1$ | 30 | 10 | $R_1=\{2,1,2,3,1,3,2,2,3,1\}$ |
| $CS_2$ | 30 | 15 | $R_2=\{2,1,2,0^3,2,2,0,1,1,0,2,1,1\}$ |
| $CS_3$ | 40 | 10 | $R_3=\{2,3,2,6,4,3,2,4,3,1\}$ |
| $CS_4$ | 40 | 15 | $R_4=\{2,1,2,0,2,1,2,1,4,1,1,4,2,1,1\}$ |
| $CS_5$ | 40 | 20 | $R_5=\{2,1,0^2,1,1,2,2,1,0,2,1,2,0^2,1,0,2,1,1\}$ |
| $CS_6$ | 50 | 10 | $R_6=\{2,3,2,3,4,6,4,5,5,6\}$ |
| $CS_7$ | 50 | 20 | $R_7=\{2,1,2,1,1,1,2,2,1,0,2,1,2,2,3,1,2,2,1,1\}$ |
| $CS_8$ | 50 | 30 | $R_8=\{2,1,0^4,2,0^2,2,0,2,0^3,2,2,1,0^3,2,2,0^5,1,1\}$ |
| $CS_9$ | 60 | 30 | $R_9=\{2,1,3,0^3,2,0,3,2,0,2,0^3,2,2,1,0^3,2,2,0^2,3,0,1,1,1\}$ |

Censoring schemes with different values for $n$ and $m$ , where $0^r$ means that 0 repeated $r$ times.

**Table 2.** Mean, MSE, LL, UL, IL and cov for $\alpha$ and $\beta$ using maximum likelihood method.

| $CS$ | $n$ | $m$ | $mean$ | $MSE$ | $LL$ | $UL$ | $IL$ | $Cov$ |
|---|---|---|---|---|---|---|---|---|
| | | | | $\alpha$ | | | | |
| $CS_1$ | 30 | 10 | 2.0318 | 0.061 | 1.3203 | 2.7432 | 1.4229 | 0.974 |
| $CS_2$ | 30 | 15 | 1.9816 | 0.1163 | 1.4348 | 2.5284 | 1.0936 | 0.87 |
| $CS_3$ | 40 | 10 | 2.0332 | 0.0483 | 1.3129 | 2.7535 | 1.4406 | 0.992 |
| $CS_4$ | 40 | 15 | 2.0589 | 0.0451 | 1.482 | 2.6359 | 1.1539 | 0.988 |
| $CS_5$ | 40 | 20 | 2.02 | 0.094 | 1.5408 | 2.4993 | 0.9585 | 0.906 |
| $CS_6$ | 50 | 10 | 2 | 0.052 | 1.2391 | 2.7609 | 1.5218 | 1 |
| $CS_7$ | 50 | 20 | 2.0412 | 0.039 | 1.5503 | 2.5321 | 0.9818 | 0.984 |
| $CS_8$ | 50 | 30 | 1.9734 | 0.1893 | 1.5917 | 2.3551 | 0.7634 | 0.778 |
| $CS_9$ | 60 | 30 | 2.0727 | 0.0737 | 1.6783 | 2.4671 | 0.7887 | 0.938 |
| | | | | $\beta$ | | | | |
| $CS_1$ | 30 | 10 | 2.9611 | 0.3072 | 0.508 | 5.4142 | 4.9062 | 1 |
| $CS_2$ | 30 | 15 | 2.9977 | 0.3373 | 0.833 | 5.1625 | 4.3294 | 1 |
| $CS_3$ | 40 | 10 | 3.0018 | 0.2871 | 0.6404 | 5.3632 | 4.7227 | 1 |
| $CS_4$ | 40 | 15 | 3.0524 | 0.274 | 0.933 | 5.1718 | 4.2388 | 1 |
| $CS_5$ | 40 | 20 | 3.0341 | 0.3051 | 1.1394 | 4.9289 | 3.7896 | 1 |
| $CS_6$ | 50 | 10 | 2.9825 | 0.2867 | 0.6079 | 5.3572 | 4.7493 | 1 |
| $CS_7$ | 50 | 20 | 3.0379 | 0.2844 | 1.2001 | 4.8757 | 3.6756 | 1 |
| $CS_8$ | 50 | 30 | 3.0516 | 0.3802 | 1.4525 | 4.6507 | 3.1982 | 1 |
| $CS_9$ | 60 | 30 | 3.121 | 0.285 | 1.5255 | 4.7166 | 3.1911 | 1 |

Mean, MSE, LL, UL, IL and cov for $\alpha$ and $\beta$ using maximum likelihood method.

**Table 3.** Mean, MSE, LL, UL, IL and cov for $S$ and $H$ using maximum likelihood method.

| $CS$ | $n$ | $m$ | $mean$ | $MSE$ | $LL$ | $UL$ | $IL$ | $Cov$ |
|---|---|---|---|---|---|---|---|---|
| | | | | **S** | | | | |
| $CS_1$ | 30 | 10 | 0.7166 | 0.0855 | 0.5619 | 0.8714 | 0.3096 | 0.952 |
| $CS_2$ | 30 | 15 | 0.7362 | 0.0947 | 0.6048 | 0.8676 | 0.2629 | 0.856 |
| $CS_3$ | 40 | 10 | 0.7227 | 0.0779 | 0.5737 | 0.8718 | 0.2981 | 0.962 |
| $CS_4$ | 40 | 15 | 0.718 | 0.0705 | 0.589 | 0.8469 | 0.2579 | 0.964 |
| $CS_5$ | 40 | 20 | 0.7269 | 0.0879 | 0.6106 | 0.8431 | 0.2326 | 0.868 |
| $CS_6$ | 50 | 10 | 0.7354 | 0.0755 | 0.5935 | 0.8772 | 0.2837 | 0.954 |
| $CS_7$ | 50 | 20 | 0.7245 | 0.0617 | 0.6124 | 0.8365 | 0.2241 | 0.948 |
| $CS_8$ | 50 | 30 | 0.739 | 0.1065 | 0.6444 | 0.8337 | 0.1893 | 0.72 |
| $CS_9$ | 60 | 30 | 0.7171 | 0.0762 | 0.619 | 0.8151 | 0.196 | 0.87 |
| | | | | **H** | | | | |
| $CS$ | $n$ | $m$ | $mean$ | $MSE$ | $LL$ | $UL$ | $IL$ | $Cov$ |
| $CS_1$ | 30 | 10 | 2.6338 | 0.849 | 0.7772 | 4.4905 | 3.7133 | 0.968 |
| $CS_2$ | 30 | 15 | 2.4867 | 0.9384 | 1.1219 | 3.8515 | 2.7296 | 0.87 |
| $CS_3$ | 40 | 10 | 2.5914 | 0.7894 | 0.7054 | 4.4775 | 3.7721 | 0.976 |
| $CS_4$ | 40 | 15 | 2.6583 | 0.726 | 1.1562 | 4.1604 | 3.0042 | 0.976 |
| $CS_5$ | 40 | 20 | 2.5838 | 0.8624 | 1.3534 | 3.8143 | 2.4609 | 0.904 |
| $CS_6$ | 50 | 10 | 2.4711 | 0.7968 | 0.5622 | 4.38 | 3.8177 | 0.972 |
| $CS_7$ | 50 | 20 | 2.5834 | 0.6105 | 1.3375 | 3.8294 | 2.4919 | 0.982 |
| $CS_8$ | 50 | 30 | 2.5254 | 1.1112 | 1.5534 | 3.4975 | 1.9441 | 0.76 |
| $CS_9$ | 60 | 30 | 2.7114 | 0.7877 | 1.6607 | 3.7622 | 2.1015 | 0.924 |

Mean, MSE, LL, UL, IL and cov for $S$ and $H$ using maximum likelihood method.

**Table 4.** Mean, MSE, LL, UL, IL and cov for $\alpha$ and $\beta$ using Bayesian estimation (non-informative prior).

| $CS$ | $n$ | $m$ | $mean$ | $MSE$ | $LL$ | $UL$ | $IL$ | $Cov$ |
|---|---|---|---|---|---|---|---|---|
| | | | | $\alpha$ | | | | |
| $CS_1$ | 30 | 10 | 1.9553 | 0.0534 | 1.2235 | 2.6089 | 1.3854 | 0.988 |
| $CS_2$ | 30 | 15 | 2.068 | 0.0441 | 1.5167 | 2.5663 | 1.0496 | 0.978 |
| $CS_3$ | 40 | 10 | 1.9448 | 0.0564 | 1.2069 | 2.6027 | 1.3957 | 0.986 |
| $CS_4$ | 40 | 15 | 2.0014 | 0.0431 | 1.4201 | 2.5306 | 1.1105 | 0.988 |
| $CS_5$ | 40 | 20 | 2.0773 | 0.0348 | 1.5851 | 2.5198 | 0.9347 | 0.98 |
| $CS_6$ | 50 | 10 | 1.8772 | 0.0749 | 1.1046 | 2.5531 | 1.4486 | 0.966 |
| $CS_7$ | 50 | 20 | 2.0122 | 0.0318 | 1.5245 | 2.4631 | 0.9386 | 0.986 |
| $CS_8$ | 50 | 30 | 2.2042 | 0.0671 | 1.8114 | 2.5603 | 0.7489 | 0.866 |
| $CS_9$ | 60 | 30 | 2.1046 | 0.0336 | 1.7119 | 2.4691 | 0.7573 | 0.966 |
| | | | | $\beta$ | | | | |
| $CS_1$ | 30 | 10 | 2.8991 | 1.9721 | 1.2 | 5.8658 | 4.6657 | 0.99 |
| $CS_2$ | 30 | 15 | 3.1852 | 2.636 | 1.6014 | 6.1831 | 4.5817 | 0.95 |
| $CS_3$ | 40 | 10 | 2.9252 | 0.8207 | 1.2603 | 5.7399 | 4.4796 | 0.986 |
| $CS_4$ | 40 | 15 | 2.9854 | 0.9989 | 1.4558 | 5.4615 | 4.0057 | 0.976 |
| $CS_5$ | 40 | 20 | 3.2131 | 1.6063 | 1.7012 | 5.6087 | 3.9075 | 0.964 |
| $CS_6$ | 50 | 10 | 2.8672 | 0.3975 | 1.2031 | 5.4253 | 4.2221 | 0.988 |
| $CS_7$ | 50 | 20 | 3.0039 | 0.7613 | 1.6348 | 5.1135 | 3.4787 | 0.964 |
| $CS_8$ | 50 | 30 | 3.5935 | 3.1328 | 2.2216 | 5.9699 | 3.7483 | 0.85 |
| $CS_9$ | 60 | 30 | 3.23 | 0.8096 | 1.9326 | 5.1126 | 3.18 | 0.952 |

Mean, MSE, LL, UL, IL and cov for $\alpha$ and $\beta$ using Bayesian estimation (non-informative prior).

**Table 5.** Mean, MSE, LL, UL, IL and cov for $S$ and $H$ using Bayesian estimation (non-informative prior).

| $CS$ | $n$ | $m$ | $mean$ | $MSE$ | $LL$ | $UL$ | $IL$ | $Cov$ |
|---|---|---|---|---|---|---|---|---|
| | | | | **S** | | | | |
| $CS_1$ | 30 | 10 | 0.7475 | 0.0704 | 0.599 | 0.8959 | 0.2969 | 0.952 |
| $CS_2$ | 30 | 15 | 0.7435 | 0.0698 | 0.6124 | 0.8746 | 0.2622 | 0.904 |
| $CS3$ | 40 | 10 | 0.7557 | 0.0693 | 0.6105 | 0.9009 | 0.2904 | 0.934 |
| $CS_4$ | 40 | 15 | 0.7392 | 0.0598 | 0.6144 | 0.864 | 0.2496 | 0.958 |
| $CS_5$ | 40 | 20 | 0.733 | 0.0667 | 0.6171 | 0.8489 | 0.2318 | 0.9 |
| $CS_6$ | 50 | 10 | 0.771 | 0.0714 | 0.6277 | 0.9143 | 0.2866 | 0.91 |
| $CS_7$ | 50 | 20 | 0.739 | 0.0527 | 0.6296 | 0.8484 | 0.2188 | 0.966 |
| $CS_8$ | 50 | 30 | 0.7234 | 0.0658 | 0.6244 | 0.8224 | 0.198 | 0.876 |
| $CS_9$ | 60 | 30 | 0.7219 | 0.059 | 0.6242 | 0.8195 | 0.1954 | 0.904 |
| | | | | **H** | | | | |
| $CS_1$ | 30 | 10 | 2.3075 | 0.6873 | 0.3932 | 4.2217 | 3.8285 | 0.952 |
| $CS_2$ | 30 | 15 | 2.4793 | 0.5944 | 1.0689 | 3.8897 | 2.8208 | 0.968 |
| $CS_3$ | 40 | 10 | 2.2541 | 0.7194 | 0.2003 | 4.3079 | 4.1076 | 0.958 |
| $CS_4$ | 40 | 15 | 2.4282 | 0.6108 | 0.9312 | 3.9252 | 2.994 | 0.964 |
| $CS_5$ | 40 | 20 | 2.5649 | 0.5807 | 1.3137 | 3.8161 | 2.5023 | 0.972 |
| $CS_6$ | 50 | 10 | 2.0801 | 0.7577 | −0.2033 | 4.3635 | 4.5668 | 0.92 |
| $CS_7$ | 50 | 20 | 2.4368 | 0.5038 | 1.2002 | 3.6733 | 2.4732 | 0.978 |
| $CS_8$ | 50 | 30 | 2.8336 | 0.6401 | 1.7715 | 3.8957 | 2.1241 | 0.982 |
| $CS_9$ | 60 | 30 | 2.6861 | 0.5728 | 1.6288 | 3.7434 | 2.1146 | 0.982 |

Mean, MSE, LL, UL, IL and cov for $S$ and $H$ using Bayesian estimation (non-informative prior).

**Table 6.** Mean, MSE, LL, UL, IL and cov for $\alpha$ and $\beta$ using Bayesian estimation (informative prior).

| $CS$ | $n$ | $m$ | $mean$ | $MSE$ | $LL$ | $UL$ | $IL$ | $Cov$ |
|---|---|---|---|---|---|---|---|---|
| | | | | $\alpha$ | | | | |
| $CS_1$ | 30 | 10 | 1.9727 | 0.0332 | 1.3531 | 2.5401 | 1.187 | 0.996 |
| $CS_2$ | 30 | 15 | 2.048 | 0.025 | 1.5553 | 2.5075 | 0.9521 | 1 |
| $CS_3$ | 40 | 10 | 1.9652 | 0.0342 | 1.331 | 2.5425 | 1.2115 | 0.996 |
| $CS_4$ | 40 | 15 | 2.0113 | 0.0274 | 1.4923 | 2.4951 | 1.0027 | 1 |
| $CS_5$ | 40 | 20 | 2.0709 | 0.0247 | 1.6315 | 2.4776 | 0.8462 | 0.99 |
| $CS_6$ | 50 | 10 | 1.9156 | 0.0459 | 1.2717 | 2.5041 | 1.2324 | 0.99 |
| $CS_7$ | 50 | 20 | 2.0116 | 0.0212 | 1.5658 | 2.4322 | 0.8664 | 0.996 |
| $CS_8$ | 50 | 30 | 2.1783 | 0.0465 | 1.8198 | 2.5127 | 0.6929 | 0.918 |
| $CS_9$ | 60 | 30 | 2.103 | 0.0278 | 1.7324 | 2.4493 | 0.7169 | 0.98 |
| | | | | $\beta$ | | | | |
| $CS_1$ | 30 | 10 | 2.8991 | 0.3001 | 1.3895 | 5.1293 | 3.7398 | 0.996 |
| $CS_2$ | 30 | 15 | 3.1852 | 0.5579 | 1.6605 | 5.3313 | 3.6707 | 0.988 |
| $CS_3$ | 40 | 10 | 2.9252 | 0.2254 | 1.4388 | 5.0768 | 3.638 | 0.994 |
| $CS_4$ | 40 | 15 | 2.9854 | 0.2282 | 1.5794 | 4.9781 | 3.3987 | 0.996 |
| $CS_5$ | 40 | 20 | 3.2131 | 0.704 | 1.805 | 5.0963 | 3.2913 | 0.972 |
| $CS_6$ | 50 | 10 | 2.8672 | 0.1863 | 1.4366 | 4.9557 | 3.519 | 1 |
| $CS_7$ | 50 | 20 | 3.0039 | 0.2509 | 1.7081 | 4.7997 | 3.0916 | 0.994 |
| $CS_8$ | 50 | 30 | 3.5935 | 1.2608 | 2.2019 | 5.3229 | 3.1209 | 0.884 |
| $CS_9$ | 60 | 30 | 3.23 | 0.4817 | 1.9831 | 4.8437 | 2.8606 | 0.97 |

Mean, MSE, LL, UL, IL and cov for $\alpha$ and $\beta$ using Bayesian estimation (informative prior).

**Table 7.** Mean, MSE, LL, UL, IL and cov for $S$ and $H$ using Bayesian estimation under (informative prior).

| $CS$ | $n$ | $m$ | $mean$ | $MSE$ | $LL$ | $UL$ | $IL$ | $Cov$ |
|---|---|---|---|---|---|---|---|---|
| | | | | **S** | | | | |
| $CS_1$ | 30 | 10 | 0.7386 | 0.0707 | 0.5884 | 0.8888 | 0.3004 | 0.96 |
| $CS_2$ | 30 | 15 | 0.7359 | 0.0658 | 0.6031 | 0.8687 | 0.2656 | 0.944 |
| $CS_3$ | 40 | 10 | 0.7462 | 0.0673 | 0.6012 | 0.8912 | 0.29 | 0.954 |
| $CS_4$ | 40 | 15 | 0.7336 | 0.0596 | 0.6077 | 0.8596 | 0.2518 | 0.974 |
| $CS_5$ | 40 | 20 | 0.7281 | 0.0657 | 0.6113 | 0.845 | 0.2337 | 0.902 |
| $CS_6$ | 50 | 10 | 0.7609 | 0.0697 | 0.6207 | 0.9011 | 0.2804 | 0.938 |
| $CS_7$ | 50 | 20 | 0.7361 | 0.0524 | 0.6261 | 0.846 | 0.22 | 0.96 |
| $CS_8$ | 50 | 30 | 0.7173 | 0.0644 | 0.6175 | 0.8171 | 0.1996 | 0.91 |
| $CS_9$ | 60 | 30 | 0.7189 | 0.0595 | 0.6208 | 0.8171 | 0.1963 | 0.9 |
| | | | | **H** | | | | |
| $CS_1$ | 30 | 10 | 2.3856 | 0.6576 | 0.5414 | 4.2297 | 3.6884 | 0.958 |
| $CS_2$ | 30 | 15 | 2.5065 | 0.5765 | 1.1178 | 3.8952 | 2.7775 | 0.968 |
| $CS_3$ | 40 | 10 | 2.3346 | 0.6491 | 0.4274 | 4.2418 | 3.8144 | 0.966 |
| $CS_4$ | 40 | 15 | 2.4748 | 0.5809 | 0.9953 | 3.9543 | 2.959 | 0.972 |
| $CS_5$ | 40 | 20 | 2.5863 | 0.5641 | 1.3432 | 3.8295 | 2.4863 | 0.978 |
| $CS_6$ | 50 | 10 | 2.1832 | 0.7011 | 0.1699 | 4.1965 | 4.0266 | 0.964 |
| $CS_7$ | 50 | 20 | 2.4537 | 0.4926 | 1.2238 | 3.6835 | 2.4597 | 0.98 |
| $CS_8$ | 50 | 30 | 2.8357 | 0.6383 | 1.7852 | 3.8861 | 2.1009 | 0.986 |
| $CS_9$ | 60 | 30 | 2.703 | 0.5772 | 1.6475 | 3.7584 | 2.1109 | 0.988 |

Mean, MSE, LL, UL, IL and cov for $S$ and $H$ using Bayesian estimation under (informative prior).

## 8 Remarks and Conclusion

In the above sections, the ML and BE methods are used to fix the interval estimation for the unknown parameters of Burr-X distribution in case of TIHPCS. MCMC approximation is used in Bayesian procedure to solve the hard integrations. Some numerical computations and comparisons are presented to illustrate the methods of inference developed here. The simulation approach was due to examine and compare the fulfillment of the proposed methods for meany values of the sample sizes, different censoring schemes. From the results, we observe the following.

1. For fixed values of the sample size, by increasing the failure times, the MSEs and Cov for both hazard rate function and reliability function of the considered parameters are decreased.
2. For fixed values of the sample size the MSEs, for $\alpha$ less than the MSEs for $\beta$ in all methods studied here.
3. The results show that, the Bayesian estimation with informative prior is better comparing with the other methods.
4. We observed that, in most cases, mean squared errors and interval lengths calculated for Bayesian under MCMC approximation procedure are smaller than calculated for maximum likelihood, so Bayesian estimation under MCMC approximation procedure is better than the maximum likelihood as expected. Coverage probabilities in the two methods are nearly so close.

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