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Jensen's Inequality for Convex Functions on N-Coordinates

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Abstract: In recent years, new classes of convex functions have been introduced in order to generalize the results and to obtain new estimations. In this paper, we give generalization of the Jensen's inequality by using definition of convex functions on n–coordinates. Results given in [10] are particular cases of results given here.

Keywords: convex functions, Jensen's inequality, convex functions on n–coordinates.

1 Introduction

Jensen's Inequality, was an inequality discovered by the Danish mathematician and engineer Johan L. W. V. Jensen (1859–1925), see [5], relates the value of a convex function of an integral to the integral of the convex function. The concept of convexity and its various generalizations is very important in various fields of mathematics as well as the area of applied mathematics. The origin of interest in convexity arises from areas of application related to fixed point theory and optimization theory (see [6,9,11]).

A set of inequalities in literature, are due to convex functions see [1,2,3,4,6,7,13]. One of the classical inequalities is presented in the following theorem:

Theorem 1. *Let $(\Omega,\mathscr{A},\mu)$ be a measure space, let $g:\Omega\to I$, $I\subset\mathbb{R}$, be a function from $L^\infty(\mu)$ and $p:\Omega\to\mathbb{R}$ be a nonnegative function from $L^1(\mu)$ such that $\int_\Omega p\,d\mu\neq 0$. Then for any convex function $\varphi: I\to\mathbb{R}$, the inequality*

$$\varphi\left(\frac{1}{\int_\Omega p\,d\mu}\int_\Omega pg\,d\mu\right)\leq\frac{1}{\int_\Omega p\,d\mu}\int_\Omega p\varphi(g)\,d\mu \tag{1}$$

holds. This inequality is a variant of the well-known integral Jensen's inequality (see [12]).

In [14], S. S. Dragomir gave Hadamard's inequality for rectangle in plane by defining convex functions on coordinates. A function $f:[a,b]\times[c,d]\to\mathbb{R}$, $[a,b]\times[c,d]\subset\mathbb{R}^2$ with $a<b$ and $c<d$, is called convex on coordinates if the partial mappings $f_y:[a,b]\to\mathbb{R}$ defined as $f_y(u):=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$ defined as $f_x(v):=f(x,v)$ are convex, where defined for all $y\in[c,d]$ and $x\in[a,b]$.

Jensen-type inequalities for convex functions on the coordinates were investigated in [10]. In the same paper the following two theorems were proved:

Theorem 2. *Suppose that*

(i) $(\Omega_1,\mathscr{A},\mu)$ *and* $(\Omega_2,\mathscr{B},\nu)$*, are measure spaces;*

(ii) $p:\Omega_1\to\mathbb{R}$, $p\in L^1(\mu)$ *and* $w:\Omega_2\to\mathbb{R}$, $w\in L^1(\mu)$ *are nonnegative functions such that* $\int_{\Omega_1}p\,d\mu\neq 0$ *and* $\int_{\Omega_2}w\,d\nu\neq 0$;

(iii) $g:\Omega_1\to I$, $g\in L^\infty(\mu)$ *and* $h:\Omega_2\to I$, $h\in L^\infty(\nu)$, $I,J\subset\mathbb{R}$;

(iv) $\varphi: I\times J\to\mathbb{R}$*, are convex on the coordinates on* $I\times J$.

Then the following inequalities hold:

$$\begin{aligned}\varphi(\overline{g},\overline{h})&\leq\frac{1}{2}\left[\frac{1}{P}\int_{\Omega_1}p\varphi\left(g,\overline{h}\right)d\mu+\frac{1}{W}\int_{\Omega_2}w\varphi\left(\overline{g},h\right)d\nu\right]\\&\leq\frac{1}{PW}\int_{\Omega_1}\int_{\Omega_2}pw\varphi\left(g,h\right)d\nu\,d\mu,\end{aligned} \tag{2}$$

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where

$$P=\int_{\Omega_1} p d\mu, \qquad W=\int_{\Omega_2} w d\nu$$
$$\overline{g}=\frac{1}{P}\int_{\Omega_1} pg d\mu, \qquad \overline{h}=\frac{1}{W}\int_{\Omega_2} wh d\nu.$$

Theorem 3. *Let φ be convex on the coordinates on $I\times J\subset\mathbb{R}^2$. If $\boldsymbol{x}$ an n–tuple in I, $\boldsymbol{y}$ an m–tuple in J, $\boldsymbol{p}$ nonnegative n–tuple, such that $P_n=\sum_{i=1}^n p_i\neq 0$ and $\boldsymbol{w}$ nonnegative m–tuple, such that $W_m=\sum_{j=1}^m w_j\neq 0$, then*

$$\varphi(\overline{x},\overline{y})\leq\frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^n p_i\varphi(x_i,\overline{y})+\frac{1}{W_m}\sum_{i=1}^m w_j\varphi(\overline{x},y_j)\right] \quad (3)$$
$$\leq\frac{1}{P_nW_m}\sum_{i=1}^n\sum_{j=1}^m p_iw_j\varphi(x_i,y_i),$$

where

$$\overline{x}=\frac{1}{P_n}\sum_{i=1}^n p_ix_i, \quad \text{and} \quad \overline{y}=\frac{1}{W_m}\sum_{j=1}^m w_jy_j.$$

In paper [?], Ghulam Farid and Atiq Ur Rehman gave generalization of the work of S. S. Dragomir (see [14]) by defining convex functions on n–coordinates as follows:

Definition 1. *For $n\geq 2$, let a_i,b_i; ($i=1,\ldots,n$) be real numbers such that $a_i<b_i$ for $i=1,\ldots,n$. Consider n–dimentional interval Δ^n defined as $\Delta^n=\prod_{i=1}^n[a_i,b_i]$. A mapping $f:\Delta^n\to\mathbb{R}$ is called convex on n–coordinates if the functions $f^i_{x_n}$, where $f^i_{x_n}(t)=f(x_1,\ldots,x_{i-1},t,x_{i+1},\ldots,x_n)$, are convex on $[a_i,b_i]$ for $i=1,\ldots,n$.*

Recall that a mapping $f:\Delta^n\to R$ is convex in Δ^n if for $\boldsymbol{x}=(x_1,\ldots,x_n),\boldsymbol{y}=(y_1,\ldots,y_n)\in\Delta^n$ and $\alpha\in[0,1]$, the following inequality holds:

$$f(\alpha\boldsymbol{x}+(1-\alpha)\boldsymbol{y})\leq\alpha f(x)+(1-\alpha)f(y).$$

The main objective of this paper is to introduce new inequalities of the type of Jensen for notions of convex functions on n–coordinates and show that results proved in [14] are particular case of results in this paper.

2 Main results

To obtain a generalization of Theorem 2, we introduce some notation. Throughout the rest of the paper we assume that:

(i) $(\Omega_j,\mathscr{A}_j,\mu_j)$, are measure spaces for $j=1,\ldots,n$;

(ii) $p_j:\Omega_j\to\mathbb{R}$, $p_j\in L^1(\mu_j)$, are nonnegative functions such that $\int_{\Omega_j}p_jd\mu_j\neq 0$ for $j=1,\ldots,n$;

(iii) $g_j:\Omega_j\to I_j$, $g_j\in L^\infty(\mu_j)$, $I_j\subset\mathbb{R}$ for $j=1,\ldots,n$;

(iv) $\varphi:\prod_{j=1}^n I_j\to\mathbb{R}$, are convex on the n–coordinates on $\prod_{j=1}^n I_j$.

Theorem 4. *Let φ,g_j and p_j as the above, for $j=1,\ldots,n$. Then the following inequalities hold:*

$$\varphi(\overline{g}_1,\ldots,\overline{g}_n)$$
$$\leq\frac{1}{n}\sum_{j=1}^n\frac{1}{P_j}\int_{\Omega_j}p_j\widetilde{\varphi}_jd\mu_j$$
$$\leq\frac{1}{\prod_{j=1}^n P_j}\int_{\Omega_n}\cdots\int_{\Omega_1}\left(\prod_{j=1}^n p_j\right)\varphi(g_1,\ldots,g_n)d\mu_1\cdots d\mu_n, \quad (4)$$

where

$$P_j=\int_{\Omega_j}p_jd\mu_j,\ \ \overline{g}_j=\frac{1}{P_j}\int_{\Omega_j}p_jg_jd\mu_j$$

and

$$\widetilde{\varphi}_j=\varphi(\overline{g}_1,\ldots,\overline{g}_{j-1},g_j,\overline{g}_{j+1},\ldots,\overline{g}_n)$$

for $j=1,\ldots,n$.

Proof. Applying the one–dimensional Jensen's inequality (1), we get

$$\varphi(\overline{g}_1,g_2,\ldots,g_n)\leq\frac{1}{P_1}\int_{\Omega_1}p_1\varphi(g_1,g_2,\ldots,g_n)d\mu_1.$$

Multiplying the previous inequality by p_3 and $\frac{1}{P_3}$ and integrating over Ω_3. Later applying the one–dimensional Jensen's inequality (1), we obtain

$$\varphi(\overline{g}_1,g_2,\overline{g}_3,\ldots,g_n)$$
$$\leq\frac{1}{P_1P_3}\int_{\Omega_3}\int_{\Omega_1}p_1p_3\varphi(g_1,g_2,\ldots,g_n)d\mu_1d\mu_3.$$

Doing this procedure successively, we have

$$\varphi(\overline{g}_1,g_2,\overline{g}_3,\ldots,\overline{g}_n)$$
$$\leq\frac{1}{P_1P_3\cdots P_n}\int_{\Omega_n}\cdots\int_{\Omega_3}\int_{\Omega_1}(p_1p_3\cdots p_n)\times$$
$$\varphi(g_1,g_2,\ldots,g_n)d\mu_1d\mu_3\cdots\mu_n.$$

Multiplying the above inequality by p_2 and $\frac{1}{P_2}$ and integrating over Ω_2. Later by Jensen's inequality (1), we obtain

$$\varphi(\overline{g}_1,\overline{g}_2,\ldots,\overline{g}_n)$$
$$\leq\frac{1}{P_2}\int_{\Omega_2}p_2\widetilde{\varphi}_2d\mu_2$$
$$\leq\frac{1}{\prod_{j=1}^n P_j}\int_{\Omega_n}\cdots\int_{\Omega_1}\left(\prod_{j=1}^n p_j\right)\varphi(g_1,\ldots,g_n)d\mu_1\cdots d\mu_n. \quad (5)$$

Now applying the one–dimensional Jensen's inequality (1), we get

$$\varphi(g_1, g_2, \ldots, \overline{g}_{n-1}, g_n) \leq \frac{1}{P_{n-1}} \int_{\Omega_{n-1}} p_{n-1} \varphi(g_1, g_2, \ldots, g_n)\, d\mu_{n-1}.$$

Using a procedure similar to that which is done to obtain the inequality (5)

$$\varphi(\overline{g}_1, \ldots, \overline{g}_n) \leq \frac{1}{P_n} \int_{\Omega_n} p_n \widetilde{\varphi}_n d\mu_n \leq \frac{1}{\prod_{j=1}^n P_j} \int_{\Omega_n} \cdots \int_{\Omega_1} \left(\prod_{j=1}^n p_j \right) \varphi(g_1, \ldots, g_n)\, d\mu_1 \cdots d\mu_n. \quad (6)$$

Finally applying again the Jensen's inequality (1), we have

$$\varphi(g_1, \ldots, g_{n-1}, \overline{g}_n) \leq \frac{1}{P_n} \int_{\Omega_n} p_n \varphi(g_1, g_2, \ldots, g_n)\, d\mu_n.$$

By a procedure similar to that which is done to obtain the inequality (5)

$$\varphi(\overline{g}_1, \ldots, \overline{g}_n) \leq \frac{1}{P_1} \int_{\Omega_1} p_1 \widetilde{\varphi}_1 d\mu_1 \leq \frac{1}{\prod_{j=1}^n P_j} \int_{\Omega_n} \cdots \int_{\Omega_1} \left(\prod_{j=1}^n p_j \right) \varphi(g_1, \ldots, g_n)\, d\mu_1 \cdots d\mu_n. \quad (7)$$

Thus from inequalities (5)–(7), we have

$$n\varphi(\overline{g}_1, \ldots, \overline{g}_n) \leq \sum_{j=1}^n \frac{1}{P_j} \int_{\Omega_j} p_j \widetilde{\varphi}_j d\mu_j \leq \frac{n}{\prod_{j=1}^n P_j} \int_{\Omega_n} \cdots \int_{\Omega_1} \left(\prod_{j=1}^n p_j \right) \varphi(g_1, \ldots, g_n)\, d\mu_1 \cdots d\mu_n,$$

Therefore we obtain the desired inequalities.

Corollary 1. *Under the same conditions of Theorem 4 for $n = 2$. Then (2) is valid.*

Theorem 5. *Let φ be convex on the n–coordinates on $\prod_{j=1}^n I_j \subset \mathbb{R}^n$. If $x_j = (x_{j1}, \ldots, x_{jm_j})$ is an m_j–tuple on I_j, p_j a nonnegative m_j–tuple such that $P_{m_j} = \sum_{i=1}^{m_j} p_{ji} \neq 0$, for each $j = 1, \ldots, n$, then the following inequalities hold*

$$\varphi(\overline{x}_1, \ldots, \overline{x}_n) \leq \frac{1}{n} \sum_{j=1}^n \left(\frac{1}{P_{m_j}} \sum_{i=1}^{m_j} p_{ji} \widetilde{\varphi}_{ji} \right) \leq \frac{1}{\prod_{j=1}^n P_{m_j}} \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} \left(\prod_{j=1}^n p_{ji_j} \right) \varphi(x_{1i_1}, \ldots, x_{ni_n}) \quad (8)$$

where

$$\overline{x}_j = \frac{1}{P_{m_j}} \sum_{i=1}^{m_j} p_{ji} x_{ji} \text{ and}$$
$$\widetilde{\varphi}_{ji} = \varphi(\overline{x}_1, \ldots, \overline{x}_{j-1}, x_{ji}, \overline{x}_{j+1}, \ldots, \overline{x}_n)$$

for $j = 1, \ldots, n$ and $i = 1, \ldots, m_j$.

Proof. The demonstration is immediate by theorem 4, choose $\Omega_j = \{1, \ldots, n\}$, $g_j(i) = x_{ji_j}$, $p_j(i) = 1$, $\mu_j(\{i\}) = p_{ji}$, for $j = 1, \ldots, n$ and $i = 1, \ldots, m_j$.

Corollary 2. *Under the same conditions of Theorem 5 for $n = 2$. Then (3) is valid.*

3 Applications

In this section we apply some of the above established inequalities of Jensen type.

Using a function φ defined in the Section 2, we define a new function $H : [0,1]^n \to \mathbb{R}$ as,

$$H(t_1, \ldots, t_n) = \frac{1}{\prod_{j=1}^n P_j} \int_{\Omega_1} \cdots \int_{\Omega_n} \left(\prod_{j=1}^n p_j \right) \times \varphi(t_1 g_1 + (1-t_1)\overline{g}_1, \ldots, t_n g_n + (1-t_n)\overline{g}_n)\, d\mu_n \cdots d\mu_1.$$

Theorem 6. *Let φ_j, p_j and g_j as in the Theorem 4, for $j = 1, \ldots, n$. Then*

(i) The function H defined as above is convex on the n–coordinates on $[0,1]^n$.

(ii) We have the bound:

$$\inf_{(t_1,\ldots,t_n)\in[0,1]^n} H(t_1, \ldots, t_n) = H(0,0,\ldots,0) = \varphi(\overline{g}_1, \ldots, \overline{g}_n),$$

where P_j and $\overline{g}_j$ are defined as in Theorem 4, for $j = 1, \ldots, n$.

Proof. (i)

Let $x,y \in [0,1]$, $\alpha \in [0,1]$ and for each $i = 1,\ldots,n$, we have

$$\begin{aligned}
& H^i_{t_n}(\alpha x+(1-\alpha)y) \\
&= H(t_1,\ldots,t_{i-1},\alpha x+(1-\alpha)y,t_{i+1},\ldots,t_n) \\
&= \frac{1}{\prod_{j=1}^n P_j}\int_{\Omega_1}\cdots\int_{\Omega_n}\left(\prod_{j=1}^n p_j\right)\times \\
&\quad \varphi(t_1g_1+(1-t_1)\overline{g}_1,\ldots,[\alpha x+(1-\alpha)y]g_i+[1-(\alpha x+(1-\alpha)y)]\overline{g}_i, \\
&\quad \ldots,t_ng_n+(1-t_n)\overline{g}_n)\,d\mu_n\cdots d\mu_1 \\
&= \frac{1}{\prod_{j=1}^n P_j}\int_{\Omega_1}\cdots\int_{\Omega_n}\left(\prod_{j=1}^n p_j\right)\times \\
&\quad \varphi(t_1g_1+(1-t_1)\overline{g}_1,\ldots,[\alpha x+(1-\alpha)y]g_i+[1-(\alpha x+(1-\alpha)y)]\overline{g}_i, \\
&\quad \ldots,t_ng_n+(1-t_n)\overline{g}_n)\,d\mu_n\cdots d\mu_1 \\
&= \frac{1}{\prod_{j=1}^n P_j}\int_{\Omega_1}\cdots\int_{\Omega_n}\left(\prod_{j=1}^n p_j\right)\times \\
&\quad \varphi(t_1g_1+(1-t_1)\overline{g}_1,\ldots,\alpha[xg_i+(1-x)\overline{g}_i]+(1-\alpha)[yg_i+(1-y)\overline{g}_i], \\
&\quad \ldots,t_ng_n+(1-t_n)\overline{g}_n)\,d\mu_n\cdots d\mu_1 \\
&\le \frac{\alpha}{\prod_{j=1}^n P_j}\int_{\Omega_1}\cdots\int_{\Omega_n}\left(\prod_{j=1}^n p_j\right)\times \\
&\quad \varphi(t_1g_1+(1-t_1)\overline{g}_1,\ldots,[xg_i+(1-x)\overline{g}_i],\ldots,t_ng_n+(1-t_n)\overline{g}_n)\,d\mu_n\cdots d\mu_1 \\
&\quad +\frac{1-\alpha}{\prod_{j=1}^n P_j}\int_{\Omega_1}\cdots\int_{\Omega_n}\left(\prod_{j=1}^n p_j\right)\times \\
&\quad \varphi(t_1g_1+(1-t_1)\overline{g}_1,\ldots,[yg_i+(1-y)\overline{g}_i],\ldots,t_ng_n+(1-t_n)\overline{g}_n)\,d\mu_n\cdots d\mu_1 \\
&= \alpha H(t_1,\ldots,t_{i-1},x,t_{i+1},\ldots,t_n)+(1-\alpha)H(t_1,\ldots,t_{i-1},y,t_{i+1},\ldots,t_n) \\
&= \alpha H^i_{t_n}(x)+(1-\alpha)H^i_{t_n}(y).
\end{aligned}$$

Thus $H^i_{t_n}$ is convex function on $[0,1]$, for each $i = 1,\ldots,n$. Hence H is convex on n–coordinates.

(ii) Let $(t_1,\ldots,t_n) \in [0,1]^n$. Using the integral Jensen's inequality (1) on the n–coordinates, we get

$$\begin{aligned}
& H(t_1,\ldots,t_n) \\
&= \frac{1}{\prod_{j=1}^n P_j}\int_{\Omega_1}\cdots\int_{\Omega_n}\left(\prod_{j=1}^n p_j\right)\varphi(t_1g_1+(1-t_1)\overline{g}_1,\ldots,t_ng_n+(1-t_n)\overline{g}_n)\times \\
&\quad d\mu_n\cdots d\mu_1 \\
&= \frac{1}{\prod_{j=1}^{n-1} P_j}\int_{\Omega_1}\cdots\int_{\Omega_{n-1}}\left(\prod_{j=1}^{n-1} p_j\right)\times \\
&\quad \left[\frac{1}{P_n}\int_{\Omega_n}\varphi(t_1g_1+(1-t_1)\overline{g}_1,\ldots,t_ng_n+(1-t_n)\overline{g}_n)\,d\mu_n\right]d\mu_{n-1}\cdots d\mu_1 \\
&\ge \frac{1}{\prod_{j=1}^{n-1} P_j}\int_{\Omega_1}\cdots\int_{\Omega_{n-1}}\left(\prod_{j=1}^{n-1} p_j\right)\times \\
&\quad \varphi\left(t_1g_1+(1-t_1)\overline{g}_1,\ldots,\frac{1}{P_n}\int_{\Omega_n}p_n[t_ng_n+(1-t_n)\overline{g}_n]\,d\mu_n\right)d\mu_{n-1}\cdots d\mu_1 \\
&= \frac{1}{\prod_{j=1}^{n-1} P_j}\int_{\Omega_1}\cdots\int_{\Omega_{n-1}}\left(\prod_{j=1}^{n-1} p_j\right)\times \\
&\quad \varphi(t_1g_1+(1-t_1)\overline{g}_1,\ldots,t_{n-1}g_{n-1}+(1-t_{n-1})\overline{g}_{n-1},\overline{g}_n)\,d\mu_{n-1}\cdots d\mu_1 \\
&\ge \varphi(\overline{g}_1,\ldots,\overline{g}_n) = H(0,\ldots,0).
\end{aligned}$$

Thus,

$$\inf_{(t_1,\ldots,t_n)\in[0,1]^n} H(t_1,\ldots,t_n) = H(0,\ldots,0).$$

Theorem 7. *Let the function $\varphi : \prod_{j=1}^n I_J \to \mathbb{R}$, be convex on $\prod_{j=1}^n I_J$ and let functions g_j, p_j be as in the Theorem 4 for each $j = 1,\ldots,n$. Then*

(i) *Function H is convex on $[0,1]^n$.*

(ii) *We define the function $G : [0,1] \to \mathbb{R}$, with $G(t) = H(t,t,\ldots,t)$. Then G is convex and has bound:*

$$\inf_{t\in[0,1]} G(t) = G(0) = \varphi(\overline{g}_1,\ldots,\overline{g}_n).$$

Proof. (i)
Let $(t_1,\ldots,t_n), (t_1,\ldots,t_n) \in [0,1]^n$ and $\alpha \in [0,1]$,

$$\begin{aligned}
& H(\alpha(t_1,\ldots,t_n)+(1-\alpha)(s_1,\ldots,s_n)) \\
&= H(\alpha t_1+(1-\alpha)s_1,\ldots,\alpha t_n+(1-\alpha)s_n) \\
&= \frac{1}{\prod_{j=1}^n P_j}\int_{\Omega_1}\cdots\int_{\Omega_n}\left(\prod_{j=1}^n p_j\right)\times \\
&\quad \varphi([\alpha t_1+(1-\alpha)s_1]g_1+[1-(\alpha t_1+(1-\alpha)s_1)]\overline{g}_1,\ldots, \\
&\quad [\alpha t_n+(1-\alpha)s_n]g_n+[1-(\alpha t_n+(1-\alpha)s_n)]\overline{g}_n)\,d\mu_n\cdots d\mu_1 \\
&= \frac{1}{\prod_{j=1}^n P_j}\int_{\Omega_1}\cdots\int_{\Omega_n}\left(\prod_{j=1}^n p_j\right)\varphi(\alpha(t_1g_1+(1-t_1)\overline{g}_1,\ldots,t_ng_n+(1-t_n)\overline{g}_n) \\
&\quad +(1-\alpha)(s_1g_1+(1-s_1)\overline{g}_1,\ldots,s_ng_n+(1-s_n)\overline{g}_n))\,d\mu_n\cdots d\mu_1 \\
&\le \frac{\alpha}{\prod_{j=1}^n P_j}\int_{\Omega_1}\cdots\int_{\Omega_n}\left(\prod_{j=1}^n p_j\right)\varphi(\alpha(t_1g_1+(1-t_1)\overline{g}_1,\ldots,t_ng_n+(1-t_n)\overline{g}_n))\times \\
&\quad d\mu_n\cdots d\mu_1 \\
&\quad +\frac{1-\alpha}{\prod_{j=1}^n P_j}\int_{\Omega_1}\cdots\int_{\Omega_n}\left(\prod_{j=1}^n p_j\right)\varphi(\alpha(s_1g_1+(1-s_1)\overline{g}_1,\ldots,s_ng_n+(1-s_n)\overline{g}_n))\times \\
&\quad d\mu_n\cdots d\mu_1 \\
&= \alpha H(t_1,\ldots,t_n)+(1-\alpha)H(s_1,\ldots,s_n).
\end{aligned}$$

So H is convex on $[0,1]^n$.

(ii) Let $x,y \in [0,1]$ and $\alpha \in [0,1]$. Using the convexity of H, we obtain

$$\begin{aligned}
G(\alpha x+(1-\alpha)y) &= H(\alpha x+(1-\alpha)y,\ldots,\alpha x+(1-\alpha)y) \\
&= H(\alpha(x,\ldots,x)+(1-\alpha)(y,\ldots,y)) \\
&\le \alpha H(x,\ldots,x)+(1-\alpha)H(y,\ldots,y) \\
&= \alpha G(x)+(1-\alpha)G(y),
\end{aligned}$$

therefore it is shown that G is convex on $[0,1]$.
By Theorem 6, for $t \in [0,1]$, we get

$$G(t) = H(t,t) \ge H(0,0).$$

Thus,

$$\inf_{t\in[0,1]} G(t) = g(0) = \varphi(\overline{g}_1,\ldots,\overline{g}_n).$$

4 Conclusions

The principal contribution of this paper is the study of a new class of function of generalized convexity on coordinates. We have shown that this class contains some previously known classes as special cases as well as Jensen's inequalities type for these functions. We expect that the ideas and techniques used in this paper may inspire interested readers to explore some new applications of these newly introduced functions in various fields of pure and applied sciences.

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