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Stability Analysis on Synchronizing Two Parametrically Excited Chaotic Oscillators by a Single Control Function

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Abstract: In chaos synchronization, a feedback controller is designed in a way that one of the chaotic oscillator completely traces the dynamics of another chaotic (master) oscillator. This paper aims to investigate the synchronization behavior between two identical chaotic gyros and two non-identical chaotic gyro and the Double-Hump Duffing-Van der Pol (DHDVP) oscillators. Based on the Routh-Hurwitz criterion and Lyapunov stability theory and using the active control strategy, a single input control function is designed that establishes the synchronization globally. The linear controller gain coefficients are determined by our own choice that ensures the globally exponential stability of the closed-loop. Effect of the unknown time varying external disturbances is under our discussions. The simulation results are carried out to verify the effectiveness of the proposed active control strategy and possible feasibility in synchronizing two identical and two non-identical chaotic oscillators.

Keywords: Chaos Synchronization, Lyapunov stability theory, Routh-Hurwitz criterion, Active control, Chaotic oscillators,

1 Introduction

Chaotic synchronization has become an important topic in nonlinear science not only for its importance in theory but also for its successful applications in certain fields of engineering and sciences [1,2,3,4]. A variety of control techniques and algorithms are developed to carry out chaotic synchronization. These include, the linear error state feedback control [5], nonlinear control [6], active control [7], the projective lag synchronization [8], adaptive control [9] and sliding mode control [10] (to name but a few). Among the aforesaid techniques and algorithms, active control methods are attractive control strategies both for synchronization of two identical and non-identical chaotic systems due to the powerful applications in different scientific fields such as the Bonhoffer-Van der Pol oscillator [11], Bose-Einstein Condensate [12], HIV/AIDS chaotic systems [13] and Duffing-Van der Pol oscillator [14], (are worth citing here among others). If the nonlinearity of a system is known, an active control technique can be easily designed according to the given conditions of the chaotic system [15] to achieve synchronization. It is not necessary to calculate the Lyapunov exponents to execute the controller. There is no derivative in the controller. These features have initiated research interest in the use of active control strategies for the synchronization of chaotic systems.

A gyro is particularly an interesting form of nonlinear system. Because of the nonlinear terms in the dissipative gyro system, the gyro exhibits both sub-harmonic and chaotic motions with period-doubling route to chaos [16, 17]. Gyro has been used to describe the mode in navigation, aeronautics, space engineering and the control of complex physical systems [18]. The synchronization of a chaotic gyro has received tremendous interest in the concerned literature [17,19]. Ref. [20] investigated the nonlinear motion of a symmetric gyro with nonlinear damping mounts on a vibrating base. A one-way coupling approach is used to synchronize two identical chaotic gyros. Ref. [21] extended the findings of [20] and applied an active control technique [22] to synchronize two identical chaotic gyros with nonlinear damping. In these cases, Globally Exponential Synchronization (GES) condition is achieved based on the Routh-Hurwitz criterion.

However, in the findings of these results [20,21], there are two main limitations. Firstly, the GES behavior of two

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identical chaotic gyros using active control [21] is investigated with the control inputs numerically equal to the number of error states. This property of active control design puts an extra effort into the system being synchronized. Secondly, the synchronization between two identical gyros is achieved without considering the effect of unknown external disturbance. In real-life applications, effect of the external disturbance cannot be ignored. This arises from the fact that even a small bounded disturbance can cause the synchronization error control system to be unstable.

Thus, it is more motivating and significant from theoretical as well as practical point of view to investigate the synchronization problem of a chaotic gyro with less control effort and fast synchronization speed in the presence of unknown time varying external disturbance. Contributing to this line of literature, the ultimate aim is to adopt an active control strategy to investigate further the synchronization problem for a chaotic gyro based on the synchronization cost and controller complexity. In this research effort, Routh-Hurwitz criterion [23], Lyapunov stability theory and active control strategy are applied to synchronize two identical chaotic gyros. Sufficient algebraic conditions are derived to compute the linear controller gains. These gains are then employed to achieve the GES. This study also discusses the synchronization phenomena between two different chaotic oscillators; the gyro [21] and DHDVP [24] for further synchronization investigation. A single control input is designed that guarantees the GES. The designed controller contains only feedback terms and partial nonlinear terms of the systems, and they are easy to implement practically. Furthermore, the stability of the synchronization error system with the effect of unknown external disturbance is investigated. In comparison with some previous results, the external disturbance signals are assumed to be time-varying with unknown bounds. The work in this research paper is an improvement to the existing results [20,21] in terms of synchronization speed, quality and controller cost.

The rest of the paper is organized as follows.

In section 2, some basic preliminaries and problem statement are given. A systematic procedure for the proposed active controller design is given in section 3. In section 4, the description of a chaotic gyro is given and solution is provided for the synchronization problem of two identical chaotic gyros. The synchronization problem between two non-identical chaotic gyro and the Duffing-Van der Pol oscillators is solved in section 5 and the synchronization behavior under the effect of unknown time varying external disturbance is investigated. Finally, the concluding remarks are made in section 6.

2 Proposed active control strategy

The following notations will be used in this paper;

R^n denotes the n-dimensional space; $R^{n\times n}$, the space of all matrices of $(n\times n)$ dimension; A^T denotes the transpose of a matrix A; The vector norm is represented by $\|\ \|$.

2.1 Problem formulation

A certain chaotic system is called the master or drive system and the second system is called slave or response system. Most of the synchronization procedures belong to the master-slave system configuration, in which the slave system is forced to track the master system and the two systems show common behavior for all future states. Consider a master-slave system synchronization scheme that is described by the following differential equations:

$$\begin{cases} \text{Master system: } \dot{X}(t) = (X(t))M_1 + F(X(t)) \\ \text{Slave system: } \dot{Y}(t) = (Y(t))M_2 + G(Y(t)) + \mu(t) \end{cases} \tag{1}$$

where $X(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T \in R^n$ and $Y(t) = [y_1(t), y_2(t), \cdots, y_n(t)]^T \in R^n$ are the state vectors, $F(X(t))$ and $G(Y(t))$ are the nonlinear bounded smooth functions and $M_1, M_2 \in R^{n\times n}$ are the constant system matrices of the corresponding master and slave systems (1) respectively. The controller vector is denoted as $\mu(t)\in R^n$. In these circumstances, our aim is to design a feedback control law that describes the synthesis of a bounded control input $\mu(t) \in R^n$. This control input realizes the GES of the slave system to the master system in scheme (1). Moreover, $\mu(t) \in R^n$ establishes the globally exponential stability of the closed-loop.

Definition 2.1. The synchronization error dynamical system is defined as the difference between the master and slave systems given by:

$$\|e_i(t)\| = \|y_i(t) - x_i(t)\|, \quad e_i(t)\in R^n$$

Thus, the error dynamical system for the synchronization scheme (1) is described as follows:

$$\dot{e}(t) = \left.\begin{matrix} Y(t)M_2 - X(t)M_1 + \\ G(Y(t)) + F(X(t)) + \mu(t) \end{matrix}\right\}$$

$$\Rightarrow \dot{e}(t) = M_3 e(t) + H(X(t), Y(t), e(t)) + \mu(t), \tag{2}$$

where $M_3 = \bar{M}_2 - \bar{M}_1$ is the matrix of the linear part and the function $H(X(t), Y(t), e(t))$ described by,

$$H(X(t), Y(t), e(t)) = \left.\begin{matrix} G(Y(t)) - F(X(t)) + \\ (M_2 - \bar{M}_2)Y(t) - (M_1 - \bar{M}_1)X(t) \end{matrix}\right\}$$

contains the nonlinear functions and un-common parts of the error system (2).

Objective 2.1. The states of the master and slave chaotic systems (1) are globally synchronized exponentially, i.e.

$$\lim_{t\to\infty}\|e_i(t)\| = \lim_{t\to\infty}\|y_i(t) - x_i(t)\| = 0, \ \forall\, e_i(0)\in R^n$$

Objective 2.2. The closed-loop system (4) is globally exponentially stable.

2.2 Controller design

Theorem 2.1. The GES synchronization of master and slave systems (1) is accomplished by using the following active controller:

$$\mu(t) = -H(X(t), Y(t), e(t)) + v(t), \quad (3)$$

where the sub-controller matrix $v(t) \in R^{n\times 1}$ is defined as follows:

$$v_i(t) = -Ke_i(t) = -K(Y_i(t) - X_i(t)), \quad i = 1, 2, \cdot, n,$$

which is the control function of $e(t)$ and $K \in R^{n\times n}$ is a feedback linear control gain matrix that adjusts the strength of the feedback controller into the slave system and is to be determined.

Proof. Let us assume that the parameters of the master and slave systems (1) are available and measurable. Using systems of Eqs. (2) and (3), the closed-loop is given by the following:

$$\begin{aligned} \dot{e}(t) &= M_3 e(t) + v(t) \\ &= M_3 e(t) - Ke(t) \\ &= (M_3 - K) e(t) \\ &= Me(t) \end{aligned} \quad (4)$$

where,

$$M = (M_3 - K) \in R^{n\times n}$$

Corollary 2.1. Objective (2) will be established if the coefficient matrix $M \in R^{n\times n}$ (4) is Hurwitz.

The chaos synchronization problem is the same as stabilization of the closed-loop system (4) at the origin. A linear control gain matrix $K \in R^{n\times n}$ can be chosen in such a way that the real part of the eigenvalues of the system matrix $M \in R^{n\times n}$ is negative. This confirms that the closed-loop (4) is GES stable. Thus, the synchronization scheme (1) achieves GES.

3 Identical synchronization for two coupled chaotic gyros

3.1 Dynamics of a chaotic gyro

A gyro is a device for measuring or maintaining orientation, based on the principles of angular momentum. Mechanical gyros typically comprise a spinning wheel or disc in which the axle is free to assume any orientation. In this paper, a periodically forced, nonlinear and symmetric gyro is considered, as shown in Fig. 1. The equation of motion for a symmetric chaotic gyro in terms of the rotating angle 'θ' is given by [20]:

$$\left.\begin{matrix} \ddot{\theta} + \alpha^2 \frac{(1-\cos\theta)^2}{\sin^3\theta} + \\ c_1\dot{\theta} + c_2\dot{\theta}^3 - \beta\sin\theta \end{matrix}\right] = f\sin\omega t \sin\theta, \quad (5)$$

where, $\alpha^2 \frac{(1-\cos\theta)^2}{\sin^3\theta} - \beta\sin\theta$; is a nonlinear resilience force, $f\sin\omega t$; represents a parametric excitation and the value of f being in the range of, $32 < f < 36$, ω is the frequency of external excitation disturbance, $c_1\dot{\theta}$ and $c_2\dot{\theta}^3$ represent the linear and nonlinear damping terms, respectively. According to [20], the spin Euler's angle ϕ and precession ψ have cyclic motions, and hence their momentum integrals are constant and equal to each other. So the governing equations of motion depend only on the mutational angle θ.

Given the states,

$$x = \theta, y = \dot{\theta} \text{ and } h(\theta) = -\alpha^2 \frac{(1-\cos\theta)^2}{\sin^3\theta},$$

the system of equations (6) can be transformed to the following normalized form:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{matrix} \begin{bmatrix} 0 & 1 \\ 0 & -c_1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \\ \begin{bmatrix} 0 & 0 & 0 \\ h(x(t)) & \sin x(t) & -y^3(t) \end{bmatrix} \begin{bmatrix} 1 \\ \beta + f\sin\omega t \\ c_2 \end{bmatrix} \end{matrix} \quad (6)$$

The gyro exhibits a chaotic attractor [20] for the following parameters:

$$\alpha^2 = 100, \beta = 1, c_1 = 0.5, c_2 = 0.05, \omega = 2, f = 35.5$$

The dynamics of the chaotic gyro system (6) has been extensively studied by Chen [20] and Lei et al [21] for the above chosen parameter values $(\alpha^2, \beta, c_1, c_2, \omega, f)$ and for a space area range of the amplitude of the parameter excitation [21]. The synchronization problem of two identical gyros is usually employed in attitude control of long-duration space crafts, signal processing in optical gyro and secure communications [17].

3.2 Problem statement

In order to observe the synchronization behavior in the chaotic gyro oscillator, the two coupled chaotic gyro oscillators are described in a master-slave system synchronization in (7). In this system, master oscillator is described with two state variables, denoted by subscript 1, which drives the slave oscillator having identical equations, denoted by subscript 2. However, initial conditions of the master gyro oscillator are different from that of the slave gyro oscillator.

$$
\begin{aligned}
&\text{(Master Gyro)}\\
&\begin{cases}
\begin{bmatrix}\dot{x}_1(t)\\ \dot{y}_1(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -c_1\end{bmatrix}\begin{bmatrix}x_1(t)\\ y_1(t)\end{bmatrix} + \\
\begin{bmatrix}0 & 0 & 0\\ h(x_1(t)) & \sin x_1(t) & y_1^3(t)\end{bmatrix}\begin{bmatrix}1\\ \beta + f\sin\omega t\\ -c_2\end{bmatrix}
\end{cases}\\
&\text{(Slave Gyro)} \qquad (7)\\
&\begin{cases}
\begin{bmatrix}\dot{x}_2(t)\\ \dot{y}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -c_1\end{bmatrix}\begin{bmatrix}x_2(t)\\ y_2(t)\end{bmatrix} + \\
\begin{bmatrix}0 & 0 & 0\\ h(x_2(t)) & \sin x_2(t) & y_2^3(t)\end{bmatrix}\begin{bmatrix}1\\ \beta + f\sin\omega t\\ -c_2\end{bmatrix}\\
+\begin{bmatrix}\mu_1(t)\\ \mu_2(t)\end{bmatrix},
\end{cases}
\end{aligned}
$$

where $[x_1(t), y_1(t)]^T \in R^2, [x_2(t), y_2(t)]^T \in R^2$ are the state variables and $\alpha^2, \beta, c_1, c_2, \omega$ and f are the parameters of the master and slave systems respectively. $\mu(t) = [0, \mu_2(t)]^T \in R^2$ is the control input. The controller $\mu(t) \in R^2$ is to be determined for the purpose of synchronizing the two identical chaotic gyro oscillators with the same parameters $\alpha^2, \beta, c_1, c_2, \omega$ and f in spite of the differences in their initial conditions.

Definition 3.1. Subtracting the slave oscillator from the master oscillator, yields the error dynamical system for the synchronization scheme (7) given as follows:

$$
\begin{bmatrix}\dot{e}_1(t)\\ \dot{e}_2(t)\end{bmatrix} = \begin{bmatrix}0 & e_2(t)\\ (\beta + f\sin\omega t)(\sin x_2(t) - \sin x_1(t)) + & \\ h(x_2(t) - x_1(t)) + c_2\left(y_1^3(t) - y_2^3(t)\right) & -c_1 e_2(t)\end{bmatrix} + \begin{bmatrix}0\\ \mu_2(t)\end{bmatrix} \qquad (8)
$$

where $e_1(t) = x_2(t) - x_1(t), e_2(t) = y_2(t) - y_1(t)$

Lemma 3.1. It follows from the differential mean-value theorem that:

$$
\frac{\sin x_2 - \sin x_1}{(x_2 - x_1)} = \cos\phi, \text{ where, } \phi \in [x_1, x_2] \text{ and } (x_1 < x_2)
$$

$$
\Rightarrow \sin x_2 - \sin x_1 = \cos\phi\,(x_2 - x_1) \qquad (9)
$$

Using (9), the error dynamics (8) can be expressed as follows:

$$
\begin{bmatrix}\dot{e}_1(t)\\ \dot{e}_2(t)\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}0 & 1\\ (\beta + f\sin\omega t)\cos\phi & -c_1\end{bmatrix}\begin{bmatrix}e_1(t)\\ e_2(t)\end{bmatrix} + \\ \begin{bmatrix}0 & 0\\ h(x_2(t)) - h(x_1(t)) & \left(y_1^3(t) - y_2^3(t)\right)\end{bmatrix}\begin{bmatrix}1\\ c_2\end{bmatrix}\\ + \begin{bmatrix}0\\ \mu_2(t)\end{bmatrix}\end{bmatrix} \qquad (10)
$$

Objective 3.1. The states of the master and slave gyro oscillators are globally exponentially synchronized in the sense that,

$$
\lim_{t\to\infty}\|e_1(t)\| = \lim_{t\to\infty}\|x_2(t) - x_1(t)\| = 0
$$

$$
\lim_{t\to\infty}\|e_2(t)\| = \lim_{t\to\infty}\|y_2(t) - y_1(t)\| = 0
$$

Objective 3.2. The closed-loop system (10) is globally exponentially stable.

Theorem 3.1. The GES of master and slave oscillators (7) is accomplished by using the following active controller:

$$
\begin{bmatrix}\mu_1(t)\\ \mu_2(t)\end{bmatrix} = \begin{bmatrix}0\\ h(x_1(t)) - h(x_2(t)) + c_2\left(y_2^3(t) - y_1^3(t)\right)\end{bmatrix} + \begin{bmatrix}v_1(t)\\ v_2(t)\end{bmatrix}, \qquad (11)
$$

where, $\begin{bmatrix}v_1(t)\\ v_2(t)\end{bmatrix}$ is the sub-controller matrix and is defined as follows:

$$
\begin{bmatrix}v_1(t)\\ v_2(t)\end{bmatrix} = -K\begin{bmatrix}e_1(t)\\ e_2(t)\end{bmatrix} \text{ and } K = \begin{bmatrix}k_{11} & k_{12}\\ k_{21} & k_{22}\end{bmatrix} \qquad (12)
$$

where, $k_{ij}(i, j = 1, 2)$ are the gain coefficients that adjust the strength of the feedback controller into the slave system and to be determined.

Proof. Using systems of Eqs. (10) and (11), the closed-loop system is given by the following:

$$
\begin{bmatrix}\dot{e}_1(t)\\ \dot{e}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ (\beta + f\sin\omega t)\cos\phi & -c_1\end{bmatrix}\begin{bmatrix}e_1(t)\\ e_2(t)\end{bmatrix} + \begin{bmatrix}v_1(t)\\ v_2(t)\end{bmatrix} \qquad (13)
$$

Using Eq. (12), rewriting system of Eq. (13), we get:

$$
\begin{bmatrix}\dot{e}_1(t)\\ \dot{e}_2(t)\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}0 & 1\\ (\beta + f\sin\omega t)\cos\phi & -c_1\end{bmatrix} - \\ \begin{bmatrix}k_{11} & k_{12}\\ k_{21} & k_{22}\end{bmatrix}\end{bmatrix}\begin{bmatrix}e_1(t)\\ e_2(t)\end{bmatrix}
$$

$$
\begin{bmatrix}\dot{e}_1(t)\\ \dot{e}_2(t)\end{bmatrix} = \begin{bmatrix}0 - k_{11} & 1 - k_{12}\\ (\beta + f\sin\omega t)\cos\phi - k_{21} & -c_1 - k_{22}\end{bmatrix}\begin{bmatrix}e_1(t)\\ e_2(t)\end{bmatrix} \qquad (14)
$$

Since, $\mu_1(t) = 0 \Rightarrow v_1(t) = 0$, therefore considering, $k_{11} = k_{12} = 0$, and rewrite system of equations (14) as follows:

$$
\begin{bmatrix}\dot{e}_1(t)\\ \dot{e}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ (\beta + f\sin\omega t)\cos\phi - k_{21} & -c_1 - k_{22}\end{bmatrix}\begin{bmatrix}e_1(t)\\ e_2(t)\end{bmatrix}
$$

$$
\Rightarrow \dot{e}(t) = Me(t), \qquad (15)
$$

where,

$$
M = \begin{bmatrix}0 & 1\\ (\beta + f\sin\omega t)\cos\phi - k_{21} & -c_1 - k_{22}\end{bmatrix} \qquad (16)
$$

The goal is to stabilize the closed-loop system (15) at the origin. Thus, objective 2 will be accomplished if the real part of all eigenvalues of the coefficient matrix $M \in R^{2\times 2}$ in eq. (16) are negative and hence matrix $M \in R^{2\times 2}$

becomes Hurwitz. Then, the closed-loop system (15) is globally exponentially stable.

Remark 3.1. It is noticed that the controller is associated to the feedback controller gain matrix $K \in R^{2\times 2}$ so that the choice of $K \in R^{2\times 2}$ is to make all eigenvalues of the coefficient matrix $M \in R^{2\times 2}$ in eq. (15) in the open left-half of the complex plane.

For a particular choice, the linear controller gain matrix is $M \in R^{2\times 2}$, selected as follows:

$$K = \begin{bmatrix} 0 & 0 \\ (\beta + f\sin\omega t)\cos\phi + \Delta & -c_1 + \Delta \end{bmatrix}, \qquad (17)$$

where, Δ is a numerical value which is set equals to -2. Therefore, the closed–loop system (14) is globally exponentially stable. Thus, objectives 1 and 2 are established.

3.3 Numerical simulation and discussion

Numerical results are furnished to justify the efficiency of the proposed approach. Parameters for the chaotic gyro are selected as in [21], i.e.,

$$\alpha^2 = 100, \beta = 1, c_1 = 0.5, c_2 = 0.05, \omega = 2, f = 35.5$$

with initial conditions $[x_1(0), y_1(0)]^T = [0.5, 1]^T$ and $[x_2(0), y_2(0)]^T = [1, 2]^T$, respectively. All simulation routines are coded and executed using the software, Mathematica^{TM} 10, to solve the systems of differential equations (7) and (10) using the same parameters and different initial conditions. The simulations results are illustrated in Figs. 2 to 5.

For the two identical chaotic gyros, the time histories of the state vectors of synchronized master and slave oscillators and the state trajectories of unsynchronized master and slave oscillators are shown in Figs 2 to 3. These figures demonstrate that the states of the slave oscillator converge to that of the master oscillator under the synthesized control action (11), while the uncontrolled state trajectories are completely different from the master state trajectories when the controller is deactivated.

According to theorem 3.1 and linear controller gain matrix (17), the two eigenvalues of the closed-loop system (15) are $(-0.75 \pm 1.19896i)$, which confirms that the closed-loop system (15) is globally exponentially stable.

The error signals, plotted in Fig. 4, show that the error signals reach to zero state in the range of $[-2.5, 2]$ within 3.5 seconds, while the error signals in [21] are synchronized after 20 seconds. Thus, the time delay is almost 16.5 seconds.

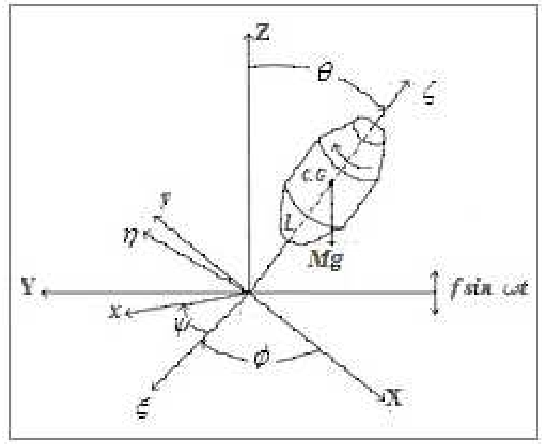

Fig. 1: A schematic diagram for symmetric gyro

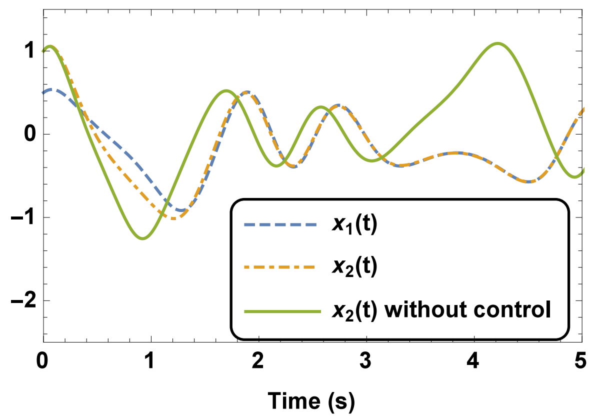

Fig. 2: Time series of the trajectories $x_1(t)$ and $x_2(t)$

4 Non-identical synchronization between chaotic gyro and DHDVP oscillators

The Doubled-Hump Duffing-Van der Pol oscillator (DHDVP) describes periodically a self-excited oscillator. The DHDVP oscillator has wide applications in engineering, physics, electronics and neurology etc. The DHDVP model has been extensively studied in the single-well, double-well and double-hump configuration [24]. The normalized form of the chaotic DHDVP is described as follows:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \left\{ \begin{matrix} \begin{bmatrix} 0 & 1 \\ -a & b \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ g\cos\varphi t \end{bmatrix} - \\ \begin{bmatrix} 0 & 0 \\ x^2(t)\,y(t) & x^3(t) \end{bmatrix} \begin{bmatrix} b \\ \eta \end{bmatrix} \end{matrix} \right\} \qquad (18)$$

where $[x(t), y(t)]^T \in R^2$ are the state variables and a, b, η, g, and φ are the parameters of the chaotic DHDVP.

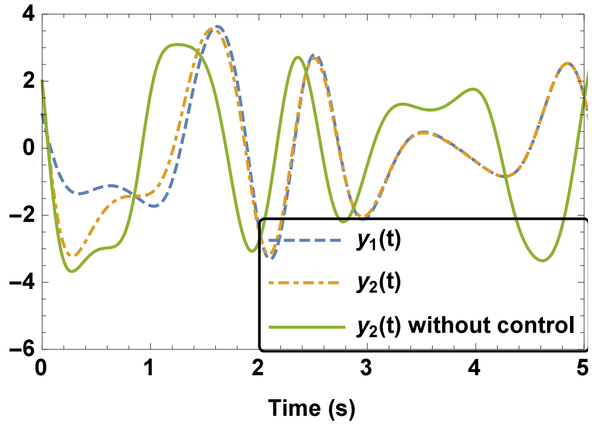

Fig. 3: Time series of the trajectories $y_1(t)$ and $y_2(t)$

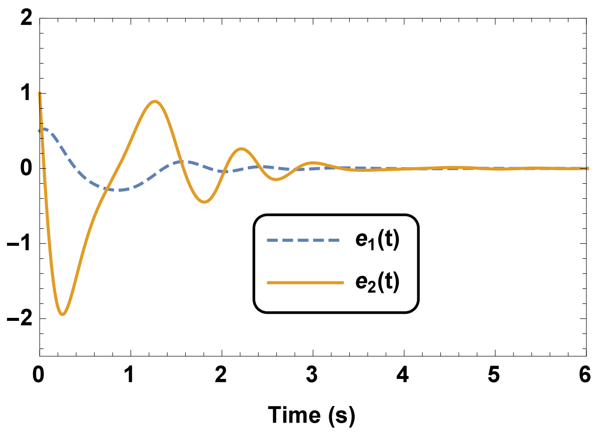

Fig. 4: Time series of the trajectories $e_1(t)$ and $e_2(t)$ of the error system tend to zero

4.1 Problem statement

In this sub-section of the paper, the main objective is to achieve the non-identical synchronization between the chaotic gyro and chaotic DHDVP oscillators. The chaotic gyro is assumed to be the master system, denoted by subscript 1, and the DHDVP is assumed to be the slave system, denoted by the subscript 2. The master-slave system configuration for the chaotic gyro and DHDVP oscillators is described as follows:

(Master Gyro)

$$\begin{cases} \begin{bmatrix}\dot{x}_1(t)\\ \dot{y}_1(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -c_1\end{bmatrix}\begin{bmatrix}x_1(t)\\ y_1(t)\end{bmatrix} + \\ \begin{bmatrix}0 & 0 & 0\\ h(x_1(t)) & \sin x_1(t) & -y_1^3(t)\end{bmatrix}\begin{bmatrix}1\\ \beta+f\sin\omega t\\ c_2\end{bmatrix}\end{cases}$$

(Slave DHDVP) (19)

$$\begin{cases} \begin{bmatrix}\dot{x}_2(t)\\ \dot{y}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -a & b\end{bmatrix}\begin{bmatrix}x_2(t)\\ y_2(t)\end{bmatrix} - \\ \begin{bmatrix}0 & 0\\ x_2^2(t)y_2(t) & x_2^3(t)\end{bmatrix}\begin{bmatrix}b\\ \eta\end{bmatrix} + \begin{bmatrix}0\\ g\cos\varphi t\end{bmatrix} + \mu(t)\end{cases}$$

where $[x_1(t),\, y_1(t)]^T \in R^2$ and $[x_2(t),\, y_2(t)]^T \in R^2$ are the state variables, $(\alpha^2,\beta,c_1,c_2,\omega,f)$ and (a,b,η,φ,g) are the parameters of the master and slave oscillators, respectively with $(a>0, b>0)$. $\mu(t) = [0,\mu_2(t)]^T \in R^2$ is the control input. The controller $\mu(t) \in R^2$ is to be determined for the purpose of synchronizing two non-identical chaotic gyro and DHDVP oscillators with different parameters and initial conditions.

The error dynamical system for the synchronization scheme (19) is given as follows:

$$\begin{bmatrix}\dot{e}_1(t)\\ \dot{e}_2(t)\end{bmatrix} = \begin{bmatrix} \begin{bmatrix}0 & 1\\ -a & b\end{bmatrix}\begin{bmatrix}e_1(t)\\ e_2(t)\end{bmatrix} + \begin{bmatrix}0 & 0\\ -a & (b+c_1)\end{bmatrix}\begin{bmatrix}x_1(t)\\ y_1(t)\end{bmatrix} - \\ \begin{bmatrix}0 & 0 & 0\\ h(x_1(t)) & \sin x_1(t) & y_1^3(t)\end{bmatrix}\begin{bmatrix}1\\ \beta+f\sin\omega t\\ -c_2\end{bmatrix} - \\ \begin{bmatrix}0 & 0\\ x_2^2(t)y_2(t) & x_2^3(t)\end{bmatrix}\begin{bmatrix}b\\ \eta\end{bmatrix} + \begin{bmatrix}0\\ g\cos\varphi t\end{bmatrix} + \begin{bmatrix}\mu_1(t)\\ \mu_2(t)\end{bmatrix}\end{bmatrix} \tag{20}$$

Objective 4.1. The states of the two coupled oscillators (19) are globally exponentially synchronized in the sense that,

$$\lim_{t\to\infty}\|e_1(t)\| = \lim_{t\to\infty}\|x_2(t)-x_1(t)\| = 0,$$
$$\lim_{t\to\infty}\|e_2(t)\| = \lim_{t\to\infty}\|y_2(t)-y_1(t)\| = 0,$$

Objective 4.2. The closed-loop (20) is globally exponentially stable.

Theorem 4.1. The GES of master and slave oscillators (19) is accomplished by using the following active controller:

$$\begin{bmatrix}\mu_1(t)\\ \mu_2(t)\end{bmatrix} = \begin{bmatrix} \begin{bmatrix}0 & 0\\ a & -(b+c_1)\end{bmatrix}\begin{bmatrix}x_1(t)\\ y_1(t)\end{bmatrix} + \\ \begin{bmatrix}0 & 0 & 0\\ h(x_1(t)) & \sin x_1(t) & y_1^3(t)\end{bmatrix}\begin{bmatrix}1\\ \beta+f\sin\omega t\\ -c_2\end{bmatrix} + \\ \begin{bmatrix}0 & 0\\ x_2^2(t)y_2(t) & x_2^3(t)\end{bmatrix}\begin{bmatrix}b\\ \eta\end{bmatrix} - \begin{bmatrix}0\\ g\cos\varphi t\end{bmatrix} + \begin{bmatrix}v_1(t)\\ v_2(t)\end{bmatrix}\end{bmatrix} \tag{21}$$

The structure of the sub-controller matrix $\begin{bmatrix}v_1(t)\\ v_2(t)\end{bmatrix}$ is constructed in the same way as described in sub-section 3.2.

Proof. Using (21), rewriting the system of Eq. (20), we get:

$$\begin{bmatrix}\dot{e}_1(t)\\ \dot{e}_2(t)\end{bmatrix} = \left[\begin{bmatrix}0 & 1\\ -a & b\end{bmatrix} - \begin{bmatrix}k_{11} & k_{12}\\ k_{21} & k_{22}\end{bmatrix}\right]\begin{bmatrix}e_1(t)\\ e_2(t)\end{bmatrix}$$

$$\begin{bmatrix}\dot{e}_1(t)\\ \dot{e}_2(t)\end{bmatrix} = \begin{bmatrix}0-k_{11} & 1-k_{12}\\ -a-k_{21} & b-k_{22}\end{bmatrix}\begin{bmatrix}e_1(t)\\ e_2(t)\end{bmatrix} \quad (22)$$

Since, $\eta_1(t) = 0 \Rightarrow v_1(t) = 0$ and by considering $k_{11} = k_{12} = 0$, rewriting system of equations (22), we get:

$$\begin{bmatrix}\dot{e}_1(t)\\ \dot{e}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -a-k_{21} & b-k_{22}\end{bmatrix}\begin{bmatrix}e_1(t)\\ e_2(t)\end{bmatrix} \quad (23)$$

$$\Rightarrow \dot{e}(t) = Me(t) \quad (24)$$

where,

$$M = \begin{bmatrix}0 & 1\\ -a-k_{21} & b-k_{22}\end{bmatrix} \quad (25)$$

The coefficient matrix $M \in R^{2\times 2}$ (25) of the closed-loop system (24) is Hurwitz, if the real part of eigenvalues of the coefficient matrix $M \in R^{2\times 2}$ is negative. Then, the closed-loop system (23) is globally exponentially stable.

According to the Lyapunov stability theory and Routh-Hurwitz criterion, all the eigenvalues of the system matrix $M \in R^{2\times 2}$ are negative if the following conditions are satisfied:

$$\begin{cases} -a-k_{21} = \Delta \\ b-k_{22} = \Delta \end{cases} \quad (26)$$

where, Δ is a numerical value which is set equal to -2.

Numerical value of the coefficients of the linear controller gain matrix is selected as follows:

$$K = \begin{bmatrix}0 & 1\\ -a+\Delta & b-\Delta\end{bmatrix} \quad (27)$$

With this choice of the controller gain matrix, the conditions (26) are satisfied. This completes the proof.

4.2 Numerical simulation

Numerical results are furnished to justify the efficiency of the proposed approach. The parameters for the chaotic gyro [21] are selected as, $\alpha^2 = 100, \beta = 1, c_1 = 0.5, c_2 = 0.05, \omega = 2, f = 35.5$, with initial conditions: $[x_1(0), y_1(0)]^T = [0.5, 1]^T$. For the chaotic DHDVP [24], the parameters are set as $a = 4.5, b = 0.1, \eta = -0.79, \varphi = 0.675, g = 0.079$ with initial conditions being taken as $[x_2(0), y_2(0)]^T = [1.3, 1.4]^T$.

All simulation procedures are coded and executed using the software, Mathematica™ 10 to solve the systems of differential equations (19) and (20) using the parameters and initial conditions mentioned above for the chaotic Gyro and DHDVP oscillators. The simulation results are shown in Figs. 5 to 7. The time histories of the state vectors of the synchronized gyro and DHDVP oscillators and the state trajectories of the unsynchronized gyro and DHDVP oscillators are shown in Figs. 5 & 6. These results demonstrate that the states of the slave oscillator converge to that of the master oscillator under the synthesized control action (21), while the uncontrolled state trajectories are completely different than the master state trajectories when the controller is deactivated. According to theorem 4.1 and conditions

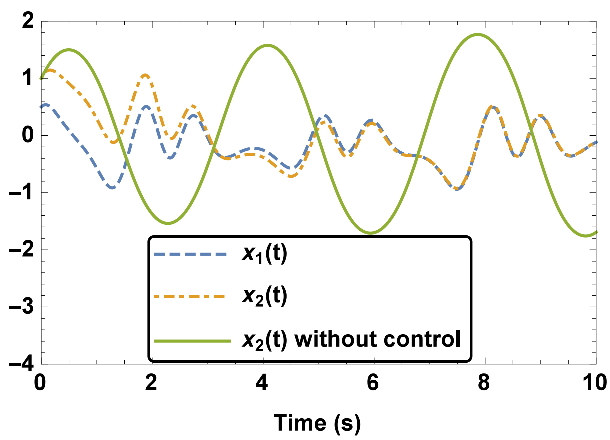

Fig. 5: Time series of the trajectories $x_1(t)$ and $x_2(t)$

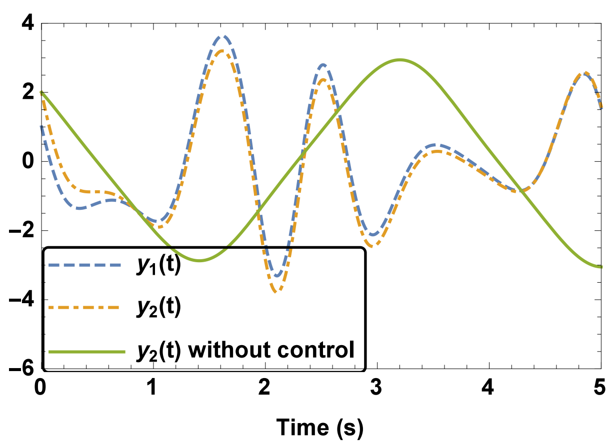

Fig. 6: Time series of the trajectories $y_1(t)$ and $y_2(t)$

(27), the two eigenvalues of the closed-loop system (23) under the control action (21) are $(-1 \pm i)$, which confirms that the closed-loop system (23) is globally exponentially stable.

The convergence of the error states for the non-identical synchronization is depicted in Fig. 7. It is observed that the error signals reach to zero state in the range of $[-0.5, 1]$ within 10 seconds. This result depicts that the investigated controllers are more robust to the accidental mismatches in the transmitter and receiver, which is helpful in certain physical applications such as marine and aeronautical sciences, etc.

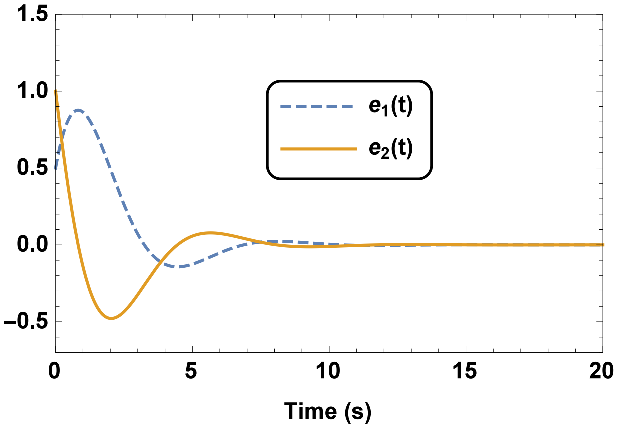

Fig. 7: Time series of the error vectors $e_1(t)$ and $e_2(t)$

4.3 *Synchronization stability analysis in the presence of unknown external disturbances*

Depending upon the characteristics of external disturbance signals, robustness of the proposed active controller is investigated in the presence of unknown external disturbance. For the purpose of numerical simulations, the following disturbance signals,

$$d_m(t) = 0.2 - 0.2\cos(180t)$$
$$d_s(t) = 0.2 + 0.3\sin(150t)$$

are assumed to be present in the master and slave oscillators respectively. Here, a general disturbance is considered that does not belong to the $L_2(0,\infty)$. This means that the time-varying disturbance is not required to have known bounds.

The error signals, for two identical gyros and two non-identical gyros & DHDVP chaotic oscillators, are plotted in Figs. 8 to 9. The plots show that exact synchronization is achieved regardless of the effect of unknown external disturbances.

The small control effort and time-varying nature of the external disturbances, in problem formulation and control design procedure, makes the proposed algorithm to be more effective.

5 Conclusions

In this paper, the synchronization problem of two identical chaotic gyros and two non-identical chaotic gyro

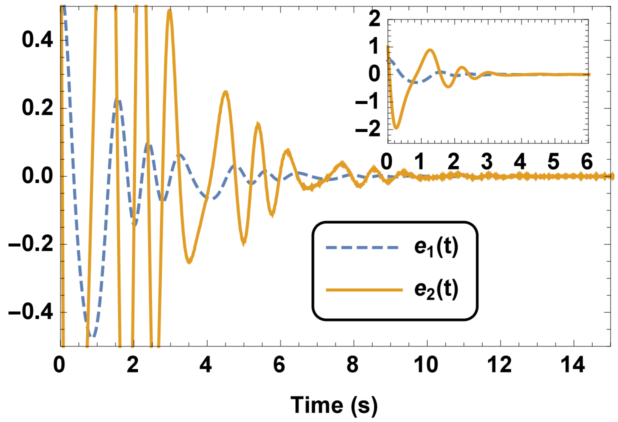

Fig. 8: Time series of the synchronized error states for two identical chaotic gyros under the effect of unknown external disturbance

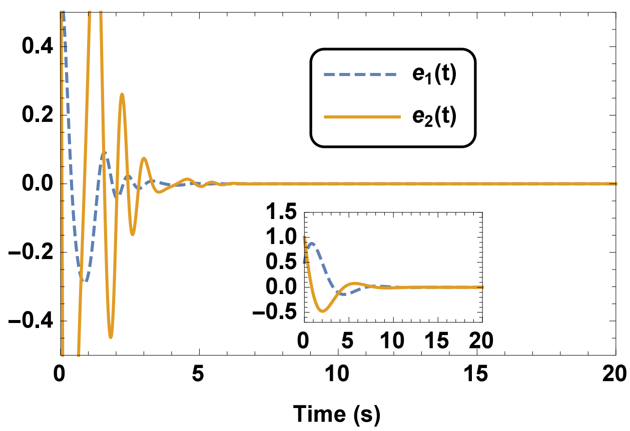

Fig. 9: Time series of the synchronized error states for two non-identical chaotic gyro and DHDVP oscillators under the effect of unknown external disturbances

and DHDVP oscillators have been investigated. Based on the Lyapunov stability theory and Routh-Hurwitz criterion and using the active control technique, a single control function is designed to synchronize two identical and two non-identical parametrically excited chaotic oscillators. In the proposed method, in contrast to the conventional active control techniques, the number of controllers are numerically equal to the dimension of the error system for synchronization of two identical and two non-identical chaotic oscillators. This considerably reduces the controller complexity and synchronization cost. The proposed control strategies, synchronizing two identical as well as two non-identical chaotic oscillators, are robust against the effects of unknown time-varying external disturbances.

Since gyros are widely employed in various applications in the field of engineering and other physical sciences, results of this research work provides vast applications in navigation, aeronautics and aerospace engineering fields.

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