Mathematical Sciences Letters
An International Journal

Essentially Copied Topological Spaces

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Received: 1 Jan. 2012; Revised: 13 May 2012; Accepted: 1 June 2012

Abstract: We introduce the notion of an essential copy in a topological space. Then we present a classification of topological spaces based on this notion. In addition, we obtain some results regarding this classification.

Keywords: Essential copy, D-space.

1 Introduction

Current progress of topology continues to show its applications in different disciplines such as chemistry, information systems, quantum physics, biology and dynamical systems, see [5], [6], [3] and [7]. The basic concept that lies behind all in topology is the idea of being homeomorphic, which fits with the ideas of congruence and similarity. This idea plays an important role in different applications. For instance, in [4] Latecki, Conrad, and Gross worked on the problem of recognizing the properties of real objects based on their digital image obtained by some sampling device like a CCD camera. They derived a topological model of a well-composed digital image which guarantees that a real object and its digital image are topologically equivalent (homeomorphic). Also, in [1] Bazin and Pham explored the use of topological information as a prior and proposed a segmentation framework based on both topological and statistical atlases of brain anatomy. Their method guarantees strict topological equivalence between the segmented image and the atlas, and relies only weakly on a statistical atlas of shape.

The main purpose of the present paper is to introduce the notion of an essential copy in a topological space, which is simply an open set that is homeomorphic to the whole space. This notion enables us to introduce a new classification of topological spaces which measures in some sense how a topological space copies itself within its open subspaces. The outline of this paper goes as follows: in Section 2, we introduce the definition of ecopied and 0-ecopied topological spaces, and present some examples of them. Then we obtain some results regarding these spaces. In Secion 3, we introduce a classification of ecopied spaces and give some properties. In Section 4, we study denseness in ecopied spaces and study the relationship between some ecopied spaces and D-spaces. We close this paper with Section 5 where we study separation axioms in ecopied spaces.

Thought this paper, R denotes the set of real numbers, τ_u denotes the usual topology on R or R^n for $n \geq 2$, and τ_{indisc} denotes the indiscrete topology. Also, τ_{lr} denotes the left ray topology on R, τ_{disc} denotes the discrete topology, and N denotes the set of natural numbers.

2 0-ecopied topological spaces

In this section we introduce the definition of essentially copied and 0-copied topological spaces. Then we discuss some properties of these spaces and present some examples. We start with the definition of an essential copy in a topological space.

Definition 2.1. Let (X,τ) be a topological space. An essential copy (simply: ecopy) of (X,τ) is an open proper subset C of X that is homeomorphic to X. The set of all ecopies of the topological space (X,τ) is denoted by $EC(X,\tau)$.

Definition 2.2. The topological space (X,τ) is said to be essentially copied (simply: ecopied) if the set $EC(X,\tau)$ is non-empty. The topological space (X,τ) is said to be 0-ecopied if it is not ecopied.

Theorem 2.1. The homeomorphic image of an ecopy is an ecopy.

Proof. Let $f:(X_1,\tau_1)\to(X_2,\tau_2)$ be a homeomorphism, and let C_1 be an ecopy of X_1. Then there is a homeomorphism $g_1:(C_1,\tau_{C_1})\to(X_1,\tau_1)$. Let $C_2=f(C_1)$. Note that C_2 is an open proper subset of X_2. Moreover, $f\circ g_1\circ f^{-1}{}_{C_2}:(C_2,\tau_{C_2})\to(X_2,\tau_2)$ is a homeomorphism. Hence C_2 is an ecopy of X_2.

Corollary 2.1. The property of being an ecopied topological space is a topological property.

Theorem 2.2. If C_1 is an ecopy of (X,τ), and C_2 is an ecopy of C_1, then C_2 is an ecopy of (X,τ).

Proof. Note that C_2 is an open proper subset of X since C_1 is. Moreover, if $g_1:(C_1,\tau_{C_1})\to(X,\tau)$, and $g_2:(C_2,\tau_{C_2})\to(C_1,\tau_{C_1})$ are homeomorphisms, then $g_1\circ g_2:(C_2,\tau_{C_2})\to(X,\tau)$ is a homeomorphism too. Hence C_2 is an ecopy of X.

Theorem 2.3. If C_0 is an ecopy of the ecopied topological space (X,τ), then there exists a sequence of ecopies $\{C_n:n\in\mathrm{N}\}$ such that $C_{n+1}\subset C_n\subset C_0$.

Proof. Note From Corollary 2.1, it follows that (C_0,τ_{C_0}) is ecopied and hence it has an ecopy C_1. Using Theorem 2.2, it follows that C_1 is an ecopy of (X,τ). Similarly, (C_1,τ_{C_1}) has an ecopy, call it C_2. If we continue in this way the proof will end up by the required sequence.

Corollary 2.2. An ecopied topological space has infinitely many ecopies.

Corollary 2.3. Let (X,τ) be a topological space. If τ is finite then (X,τ) is 0-ecopied.

Theorem 2.4. A compact Hausdorff space that consists of finitely many connected components is 0-ecopied.

Proof. Let (X,τ) be a compact Hausdorff topological space with a finite number of connected components $A_1,A_2,\ldots,A_n$, and suppose to the contrary that there exists an ecopy C of (X,τ). Then C is a compact subset of the Hausdorff topological space (X,τ), and hence C is a closed subset of X. Now, C is clopen in the connected topological space (X,τ) and hence C is the union of a proper sub-collection of $\{A_1,A_2,\ldots,A_n\}$, which cannot be homeomorphic to X, a contradiction.

Corollary 2.4. A compact connected Hausdorff topological space is 0-ecopied.

Corollary 2.5. A compact connected subspace of (R^n, τ_u) is 0-ecopied.

Theorem 2.5. A connected subspace of (R, τ_u) is 0-ecopied if and only if it is compact.

Proof. A connected subspace I of R is an interval. If I is an open interval, then I is an ecopy of (R, τ_u) and by Corollary 2.1, it is ecopied. If $I = (a,b]$, then $\left(\frac{a+b}{2}, b\right]$ is an ecopy of I. If $I = (-\infty, a]$, then $(a-1, a]$ is an ecopy of I. If $I = [a,b]$, then a proposed copy of I should be connected and compact, say $[c,d] \subset [a,b]$, which fails to be open in $[a,b]$. Hence the only connected 0-ecopied subspace of (R, τ_u) is the closed bounded interval $[a,b]$, which is compact.

The following example shows that a 0-ecopied connected subspace of (R^2, τ_u) need not be compact in general.

Example 2.1. (A 0-ecopied connected subspace of R^2 that is not compact). Let $A \subset R^2$ be defined by $A = P \cup S$, where $P = \{(0,0)\}$ and $S = \left\{\left(x, \sin\left(\frac{1}{x}\right)\right) : x \in (0,1]\right\}$. Note that A is connected, not compact (not closed). Next, we show that A is 0-ecopied. If C is an ecopy of A, then C is a connected, not path-connected proper subset of A homeomorphic to A. Hence $C = A \cap \{(x,y) \in R^2 : x \le a\}$ for some $a \in (0,1)$, which is not open in A.

3 Classification of essentially copied topological spaces

In this section we introduce a classification of essentially copied topological spaces. Then we obtain some results concerning this classification.

Definition 3.1. For an ecopied topological space (X, τ), denote the set
$\{A \subseteq EC(X, \tau) : A \neq \phi$ and $A_1 \cap A_2 = \phi$ for each $A_1, A_2 \in A$ with $A_1 \neq \mathrm{A}_2\}$
by $PDEC(X, \tau)$.

Theorem 3.1. Let (X, τ) be an ecopied topological space. If $A \in PDEC(X, \tau)$ with $|A| = 2$; then for every natural number n, there exists $A_n \in PDEC(X, \tau)$ such that $|A_n| = 2^n$.

Proof. By mathematical induction. From the assumption the result is true for $n = 1$. Now let k be any natural number such that there exists $A_k \in PDEC(X, \tau)$ with $|A_k| = 2^k$, say $A_k = \{C_1, C_2, ..., C_{2^k}\}$. For each $1 \le i \le k$, $(C_i, \tau_{C_i}) \cong (X, \tau)$ and hence it has two disjoint ecopies, say C_{i_1} and C_{i_2}. Let $A_{k+1} = \{C_{i_1}, C_{i_2} : 1 \le i \le 2^k\}$. Applying Theorem 2.2 we get that $A_{k+1} \subseteq EC(X, \tau)$. Therefore, $A_{k+1} \in PDEC(X, \tau)$ with $|A_{k+1}| = 2^{k+1}$.

Corollary 3.1. Let (X,τ) be an ecopied topological space. If $A \in PDEC(X,\tau)$ with $|A| = 2$, then there exists $B \in PDEC(X,\tau)$ with $|B| = \vartheta\circ$.

The previous result leads us to the following definitions.

Definition 3.2. Let (X,τ) be an ecopied topological space and let $\alpha = 1$ or an infinite cardinal number. Then (X,τ) is called α-ecopied if there exists $A_0 \in PDEC(X,\tau)$ such that $|A_0| = \alpha$ and $|A| \le \alpha$ for every $A \in PDEC(X,\tau)$. If (X,τ) is an α-ecopied for which $\alpha = \vartheta\circ$ then we say that (X,τ) is denumerably ecopied, and if $\alpha > \vartheta\circ$, then we say that (X,τ) is uncountablly ecopied. If (X,τ) is either denumerably ecopied or uncountablly ecopied then we call it infinitely ecopied.

Definition 3.3 The least cardinality of a dense set of a topological space (X,τ) is called the density of the space and is denoted by $d(X,\tau)$.

Theorem 3.2. If (X,τ) is an α-ecopied topological space where $\alpha = 1$ or α is an infinite cardinal number, then $\alpha \le d(X,\tau)$.

Proof. Let $A \in PDEC(X,\tau)$. Choose a dense subset $D \subseteq X$ such that $d(X,\tau) = |D|$. For every $B \in A$ choose $x(B) \in B \cap D$. Then
$|A| = |\{x(B) : B \in A\}| \le \alpha \le |D| = d(X,\tau)$.

Corollary 3.2. A separable ecopied topological space is either 1-ecopied or $\vartheta\circ$-ecopied.

Corollary 3.3. For every natural number n, (R^n, τ_u) is $\vartheta\circ$-ecopied.

Example 3.1. The set R with the left ray topology is a 1-ecopied topological space.

Example 3.2. Let α be any infinite cardinal number and let X be a set with $|X| = \alpha$. We are going to show that (X, τ_{disc}) is an α-ecopied topological space. Choose a bijection $f : X \to X \times X$. Let $C_x = f^{-1}(X \times \{x\})$, $x \in X$. Note that
$\{C_x : x \in X\} \in PDEC(X, \tau_{disc})$, $|\{C_x : x \in X\}| = \alpha$ and $d(X, \tau_{disc}) = \alpha$.
Therefore, by Theorem 3.2, it follows that (X, τ_{disc}) is α-ecopied.

Theorem 3.3. 1-ecopied and α-ecopied are topological properties.

Remark 3.1. 1-ecopied are not separable in general as the set R with the co-countable topology is a 1-ecopied topological space that is not separable.

The following is an example of an $\vartheta\circ$-ecopied topological space that is not separable.

Example 3.3. Let $X_1 = R$, $X_2 = (R \times \{1\}) \cup (R \times \{2\})$, τ_1 be the usual topology on X_1, τ_2 be the discrete topology on X_2, $X = X_1 \cup X_2$, and τ be the topology on X having $\tau_1 \cup \tau_2$ as a base. Since (X_1, τ_1) is connected and (X_2, τ_2) is a discrete topological space, it follows that C is an ecopy of (X, τ) if and only if $C = C_1 \cup C_2$ where $C_1 \in EC(X_1, \tau_1) \cup \{X_1\}$ and $C_2 \in EC(X_2, \tau_2) \cup \{X_2\}$. Since $(0,1) \cup R \times \{1\}$ and $(2,3) \cup R \times \{2\}$ are disjoint ecopies of (X, τ), then by Corollary 3.1 it follows that there exists $B \in PDEC(X, \tau)$ such that $|B| = \vartheta\circ$. Moreover, let $A \in PDEC(X, \tau)$. For each $D \in A$, there exists an open interval $I(D)$ and a subset $J(D) \subseteq X_2$ with $|J(D)| = |X_2|$ such that $D = I(D) \cup J(D)$. It is clear that $\{I(D) : D \in A\} \in PDEC(X_1, \tau_1)$. Thus, by Corollary 3.2, it follows that $|A| \le \vartheta\circ$. Hence (X, τ) is $\vartheta\circ$-ecopied. On the other hand, if (X, τ) is separable then the open subspace (X_2, τ_2) is separable, which is impossible. Therefore, (X, τ) is not separable.

4 Densely and basically essentially copied topological spaces

Definition 4.1. A topological space (X, τ) is said to be a D-space if every non-empty open subset of X is dense.

Remark 4.1. D-spaces are not ecopied in general as (R, τ), where $\tau = \{\phi, X, \{0\}\}$, is a D-space that is a 0-ecopied.

Definition 4.2. A topological space (X, τ) is said to be:

(1) Ecopied D-space if it is D-space and ecopied.

(2) Densely ecopied if every ecopy of (X, τ) is dense.

Theorem 4.1.

(1) Every ecopied D-space is densely ecopied.

(2) Every densely ecopied topological space is 1-ecopied.

Proof.

(1) Let (X, τ) be an ecopied D-space topological space and let C be an ecopy of (X, τ). Since (X, τ) is D-space, it follows that the non-empty open set C is dense. Therefore, (X, τ) is densely ecopied.

(2) If (X, τ) is a densely ecopied topological space then any ecopy of (X, τ) must intersect all non-empty open subsets of X and so it intersects all ecopies of (X, τ). Therefore, (X, τ) is 1-ecopied.

The following examples show, respectively, that each of the implications in Theorem 4.2 is not true in general.

Example 4.1. Consider the set $X = R \cup \{a, b\}$ with the topology τ having

$B = \{U \subseteq R : 0 \in U\} \cup \{\{a\}, \{b\}\}$

as a base. Let $A = \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \cup \{a, b\}$. Then $A \in \tau$. Moreover, it is not difficult to see that the function

$f : (A, \tau_A) \to (X, \tau)$, where

$$f(x)=\begin{Bmatrix} \tan x & \text{if } x\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right) \\ x & \text{if } x\in\{a,b\} \end{Bmatrix},$$

is a homeomorphism. Therefore, $A\in EC(X,\tau)$ and hence (X,τ) is ecopied. Also, it is not difficult to see that the dense subset $\{0,a,b\}$ of X is contained in any ecopy of X and hence (X,τ) is densely ecopied. On the other hand, since $\{a\}$ is open in X that is not dense, it follows that (X,τ) is not densely D-space.

Example 4.2. Consider the set $X=R\cup(N\times N)$ with the topology τ having

$B=\{U\subseteq R:0\in U\}\cup\{\{x\}:x\in N\times N\}$

as a base. Let $A=X-\{(1,1)\}$. Then $A\in\tau$. Choose a bijection $g:N\times N\rightarrow N\times N-\{(1,1)\}$. Define $f:(X,\tau)\rightarrow(A,\tau_A)$ where

$$f(x)=\begin{Bmatrix} x & \text{if } x\in R \\ g(x) & \text{if } x\in N\times N \end{Bmatrix}.$$

Then f is a homeomorphism. Therefore, $A\in EC(X,\tau)$ and hence (X,τ) is ecopied. Moreover, since $\{(1,1)\}$ is a non-empty open subset of X and $\{(1,1)\}\cap A=\phi$, then A is an ecopy of (X,τ) which is not dense. This shows that (X,τ) is not densely ecopied. Also, it is not difficult to see that every ecopy of (X,τ) must contain 0 and hence (X,τ) is 1-ecopied .

Definition 4.3. A topological space (X,τ) is said to be:

(1) Totally ecopied if every non-empty proper subset of X is an ecopy.
(2) Basically ecopied if τ has a basis of copies.

Remark 4.2.

(1) Every totally ecopied topological space is basically ecopied.
(2) Basically ecopied topological space are not totally ecopied in general as (R^n,τ_u) is a basically ecopied topological space that is not totally ecopied.
(3) Totally ecopied topological space are not infinitely ecopied in general as (R,τ_{lr}) is a totally ecopied topological space that is 1-ecopied .

Lemma 4.1. [2] For the topological space (R,τ_s), the Sorgenfrey space, a subset $A\subseteq R$ is homeomorphic to R if and only if A has no isolated points and is both an G_δ and an F_σ subset of R .

Example 4.3. (The Sorgenfrey line is an example of an $\vartheta\circ$-ecopied topological space that is totally ecopid). If C is a nonempty open subset of (R,τ_s), then there exists a family $\{[a_\alpha,b_\alpha):\alpha\in\Delta\}$ such that $C=\bigcup_{\alpha\in\Delta}[a_\alpha,b_\alpha)$. Since (R,τ_s) hereditarily Lindelöf, S then there exists a countable subset $\Omega\subseteq\Delta$ such that $C=\bigcup_{\alpha\in\Omega}[a_\alpha,b_\alpha)$. Since $[a_\alpha,b_\alpha)$ is closed in R for every $\alpha\in\Omega$, it follows that C is an F_σ. Also, it is clear that C has no isolated points and is G_δ. Therefore, by Lemma 4.1, it follows that $C\in(R,\tau_s)$ and hence (R,τ_s) is totally ecopid.

On the other hand, since (R, τ_s) has two disjoint open sets, then by Theorem 3.2, it is infinitely ecopied. Since (R, τ_s) is separable, then by Corollary 3.2, (R, τ_s) is $\vartheta\circ$-ecopied.

Theorem 4.2. Let (X, τ) be a basically ecopied topological space. Then the following are equivalent.

(1) (X, τ) is D-space.

(2) (X, τ) is densely ecopied.

(3) (X, τ) is 1-ecopied.

Proof. (1) =) (2) and (2) =) (3) follow from Theorem 4.1. To prove (3) =) (1), suppose to the contrary that there exists a non-empty open set U in X that is not dense. Then there exists a non-empty open set V in X such that $U \cap V = \phi$. Now by the assumption that (X, τ) is basically ecopied, it follows that there exist $C_1, C_2 \in EC(X, \tau)$ such that $C_1 \subseteq U$ and $C_2 \subseteq V$ and hence $C_1 \cap C_2 = \phi$, a contradiction.

Corollary 4.1. Let (X, τ) be a totally ecopied topological space. Then the following are equivalent.

(1) (X, τ) is D-space.

(2) (X, τ) is densely ecopied.

(3) (X, τ) is 1-ecopied.

5 Ecopied topological spaces and separation axioms

Example 5.1. (An example of a 1-ecopied D-space that is not a T_0-space) Consider the set $X = R$ with the topology $\tau = \{U \subseteq R : [-1,1] \subseteq U\}$. Let $A = \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. Then $A \in \tau$. Moreover, it is easy to see that the function $f : (A, \tau_A) \to (X, \tau)$, where

$$f(x) = \begin{cases} x & \text{if } |x| \le 1 \\ \tan x & \text{if } 1 < |x| < \frac{\pi}{2} \end{cases}$$

is a homeomorphism. Therefore, $A \in EC(X, \tau)$ and hence (X, τ) is ecopied. Also, it is clear that any two copies of (X, τ) must intersect and (X, τ) is an D-space and not a T_0-space.

Remark 5.1. Every D-space is not Hausdorff.

Theorem 5.1. Every basically ecopied 1-ecopied topological space in not Hausdorff.

Proof. Follows from Theorem 4.2 and Remark 5.1.

Remark 5.2. The property Hausdorff in Theorem 5.1 cannot be replaced by the property "T_1". In fact, any infinite set with the cofinite topology is a totally ecopied 1-ecopied topological space that is T_1.

We close this section with answering the following question:

Is it true that every basically ecopied infinitely ecopied topological space is Hausdorff?

The following example gives a negative answer.

Example 5.2. The product topological space of the topological spaces (R, τ_u) and (R, τ_{indisc}) is a basically ecopied infinitely ecopied topological space that is not Hausdorff.

6 Conclusion

The notion of an essential copy in a topological space plays an important role in classifying topological spaces. This notion measures in some sense how a topological space copies itself within its open subspaces. In addition, many topological concepts can be studied in different classes of ecopied topological spaces, whose definition is based on this new notion of essential copy.

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