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An Effective Method for Seventh-Order Boundary Value Problems

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Abstract: In this paper, we used the Optimal Homotopy Asymptotic Method (OHAM) to find the approximate solution of seventh order linear and nonlinear boundary value problems. The approximate solution using OHAM is compared with Variational Iteration Method (VIM) and exact solutions, an excellent agreement has been observed. The approximate solution of the equations is obtained in terms of convergent series. Low absolute error indicates that OHAM is effective for solving high order linear and nonlinear boundary value problems.

Keywords: Optimal Homotopy Asymptotic Method, Seventh-Order BVP, Absolute Error.

1 Introduction

Consider the seventh-order boundary value problem.

$$\begin{array}{ll} u^{(7)} = f(t,u(t)), & a \le t \le b \\ u^{(i)} = A_i & u^{(j)} = B_j \\ i = 0,1,2,3 & j = 0,1,2 \end{array} \tag{A}$$

Several numerical and semi-analytical methods have been developed for solving high order boundary value problems. For instance, Caglar et al. [1] as well as Siddiqi and Ghazala [2] and Jalil et al. [3] have applied spline functions, the finite difference method by Khan [4], the Adomian Decomposition Method was used by Wazwaz [5], Aslam and Tauseef [6] considered an Iteration Method based on decomposition procedure for the solution of fifth order boundary value problem while Shaowei [7] used Homotopy Perturbation Method. Siddiqi et al. [8] utilized Variational Iteration Method (VIM) for approximate solution of seventh-order boundary value problem. In this paper, the Optimal Homotopy Asymptotic Method is used to investigate seventh-order boundary value problem. In recent years, a lot of attention has been devoted to the study of Optimal Homotopy Asymptotic Method to investigate various scientific models Marinca et al. [9] and Marinca and Nicoles [10]. Islam et al. [11] studied third grade fluid model, Idrees et al. [13], Haq et al. [12], and Mabood et al. [14] provided the approximate solution of eight, sixth and fifth-order boundary value problems via OHAM.

2 Basic Principles of OHAM

The basic principles of OHAM as expounded by Marinca and Nicolae [10]. Consider the following differential equation and boundary condition:

$$L((v(x))+g(x)+N(v(x))=0, \quad B\left(v,\frac{dv}{dx}\right)=0 \tag{1}$$

where L, N are linear and nonlinear operators, x denote independent variable,$v(x)$ is an unknown function, $g(x)$ is a known function and B is a boundary operator. An equation known as a deformation equation is constructed:

$$\begin{aligned}(1-q)[L(\phi(x,q)+g(x)] = H(q)[L(\phi(x,q)+\\ g(x)+N(\phi(x,q))]\end{aligned} \tag{2}$$

$$B\left(v,\frac{dv}{dx}\right)=0$$

where $0 \le q \le 1$ is an embedding parameter, $H(q)$ is auxiliary function such that $H(q) \ne 0$ for $q \ne 0$ and

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$H(0) = 0$, $\phi(x,q)$ is an unknown function. For $q = 0$ and $q = 1$ obviously, $\phi(x,0) = v_0(x)$ and $\phi(x,1) = v(x)$ respectively. Hence, as q increases from 0 to 1 the solution $\phi(x,q)$ varies from the initial approximation $v_0(x)$ to the solution $v(x)$, where $v_0(x)$ is obtained from Eq. (2) for $q = 0$.

$$L((v_0(x)) + g(x) = 0, \quad B\left(v_0, \frac{dv_0}{dx}\right) = 0 \tag{3}$$

We choose the auxiliary function $H(q)$ of the form:

$$H(q) = \sum_{k=1}^{n} q^k C_k \tag{4}$$

where C_i, $i \in N$ are constants which are to be determined. For solution, $\phi(x,q,C_i)$ is expanded in Taylor's series about q and given:

$$\phi(x,q,C_i) = v_0(x) + \sum_{k=1}^{\infty} v_k(x,C_i)q^k, \quad i = 1,2,3\cdots \tag{5}$$

Substituting Eqs. (4) and (5) into Eq. (2), and equating the coefficients of the like powers of q equal to zero, gives the linear equations as described below:
The zeroth order problem is given by Eq. (3), and the first and second order problems are given by the Eqs. (6) and (7) respectively:

$$L(v_1(x)) = C_1 N_0(v_0(x)), \quad B\left(v_1, \frac{dv_1}{dx}\right) = 0 \tag{6}$$

$$\begin{aligned} L(v_2(x)) - L(v_1(x)) &= C_2 N_0(v_0(x)) \\ &+ C_1[L(v_1(x)) + N_1(v_0(x), v_1(x))] \end{aligned} \tag{7}$$

$$B\left(v_2, \frac{dv_2}{dx}\right) = 0$$

The general governing equation for $v_k(x)$ is given by:

$$\begin{aligned} L(v_k(x)) - L(v_{k-1}(x)) &= C_k N_0(v_0(x)) \\ &+ \sum_{i=1}^{k-1} C_i[L(v_{k-i}(x)) + N_{k-i}(v_0(x), \\ &v_1(x), \ldots, v_{k-1}(x)))] \end{aligned} \tag{8}$$

$$B\left(v_k, \frac{dv_k}{dx}\right) = 0, \quad k = 1,2,\cdots$$

where $N_m(v_0(x), v_1(x), v_2(x), ..., v_m(x))$ is the coefficient of q^m in the expansion of $N(\phi(x,q,C_i))$ about the embeding parameter q.

$$\begin{aligned} N(\phi(x,q,C_i)) &= N_0(v_0(x)) \\ &+ \sum_{m=1}^{\infty} N_m(v_0, v_1, v_2, ..., v_m) q^m \end{aligned} \tag{9}$$

It has been observed by previous researchers that the convergence of the series (5) is greatly dependent upon the auxiliary constants $C_1, C_2, \ldots$. In case of convergence at $q = 1$, one has

$$\tilde{v}(x, C_1, \ldots, C_m) = v_0(x) + \sum_{i=1}^{m} v_i(x, C_1, \ldots, C_m) \tag{10}$$

Substituting Eq. (10) into Eq. (1), the general problem results in the following residual:

$$\begin{aligned} R(x, C_1, C_2, \ldots, C_m) &= L(\tilde{v}(x, C_1, C_2, \ldots, C_m)) + g(x) \\ &+ N(\tilde{v}(x, C_1, C_2, \ldots, C_m)) \end{aligned} \tag{11}$$

If $R = 0$, then $\tilde{v}$ will be the exact solution. For nonlinear problems, generally this will not be the case. For determining $C_i (i = 1, 2, \ldots, m)$, a and b are chosen such that the optimum values for C_i are obtained, using the method of least squares:

$$J(C_1, C_2, \ldots, C_m) = \int_a^b R^2(x, C_1, C_2, \ldots, C_m) dx \tag{12}$$

where $R = L(\tilde{v}) + g(x) + N(\tilde{v})$ is the residual and,

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \ldots = \frac{\partial J}{\partial C_m} = 0 \tag{13}$$

With these constants, one can get the approximate solution of order m.

3 Examples

Here we present linear and nonlinear examples.

3.1 Example 1

Consider the seventh-order linear boundary value problem [8].

$$\frac{d^7 u}{dx^7} = u - 7e^x \tag{14}$$

with boundary conditions:
$u(0) = 1$, $u^{(1)}(0) = 0$, $u^{(2)}(0) = -1$, $u^{(3)}(0) = -2$,
$u(1) = 0$, $u^{(1)}(1) = -e$, $u^{(2)}(1) = -2e$
The analytic solution of Eq. (14) is $u(x) = e^x(1 - x)$
Applying the proposed method (OHAM) on Eq. (14), we obtain
Zeroth order problem:

$$u_0^{(7)}(x) = 0 \tag{15}$$

with boundary conditions:
$u_0(0) = 1$, $u_0^{(1)}(0) = 0$, $u_0^{(2)}(0) = -1$, $u_0^{(3)}(0) = -2$,
$u_0(1) = 0$, $u_0^{(1)}(1) = -e$, $u_0^{(2)}(1) = -2e$
Its solution is

$$\begin{aligned} u_0(x) = &\frac{1}{6}(6 - 3x^2 - 2x^3 - 66x^4 + 24ex^4 + 114x^5 \\ &-42ex^5 - 49x^6 + 18ex^6) \end{aligned} \tag{16}$$

First order problem:

$$u_1^{(7)}(x,C_1) = 7e^x C_1 - C_1 u_0 + u_0^{(7)} + C_1 u_0^{(7)} \qquad (17)$$

with boundary conditions:
$u_1(0)=0,\ u_1^{(1)}(0)=0,\ u_1^{(2)}(0)=0,\ u_1^{(3)}(0)=0,$
$u_1(1)=0,\ u_1^{(1)}(1)=0,\ u_1^{(2)}(1)=0$
Its solution is

$$\begin{aligned} u_1(x,C_1) = \frac{1}{259459200}&(-1816214400+1816214400e^x \\ &-1816214400x-908107200x^2-302702400x^3 \\ &+51762149495x^4-19070248875ex^4 \\ &-79005450468x^5+29059424730ex^5 \\ &+32086589269x^6-1180538999ex^6 \\ &-51480x^7+715x^9+143x^{10}+1716x^{11} \\ &-624ex^{11}-1235x^{12}+455ex^{12} \\ &+245x^{13}-90ex^{13})C_1 \end{aligned} \qquad (18)$$

Second order problem:

$$\begin{aligned} u_2^{(7)}(x,C_1,C_2) &= 7e^x C_2 - C_2 u_0 - C_1 u_1 + C_2 u_0^{(7)} \\ &+ u_1^{(7)} + C_1 u_1^{(7)} \end{aligned} \qquad (19)$$

with boundary conditions:
$u_2(0)=0,\ u_2^{(1)}(0)=0,\ u_2^{(2)}(0)=0,\ u_2^{(3)}(0)=0,$
$u_2(1)=0,\ u_2^{(1)}(1)=0,\ u_2^{(2)}(1)=0$
Its solution is a long expression, here few terms are:

$$\begin{aligned} u_2(x,C_1,C_2) = &\frac{1}{2899208226410496000} \\ &(-2029445758487347200C_1+\ldots. \\ &-1005663859200ex^{13}C_2) \end{aligned} \qquad (20)$$

Using Eqs. (16), (18) and (20), the second order approximate solution by OHAM for $q=1$ is

$$\tilde{u}(x,C_1,C_2) = u_0(x) + u_1(x,C_1) + u_2(x,C_1,C_2) \qquad (21)$$

We used least squares method to obtain the unknown convergent constants in $\tilde{u}$. Substituting the values of C_1, C_2 and after simplification of Eq. (21), we obtain the second order approximate solution via OHAM. In table 1 and figure 1, we compare the approximate solution obtained by OHAM with VIM and exact solutions.

Table 1: Comparison of solutions using OHAM, VIM [8]. and exact solution

x	OHAM Solution	Exact Solution	Ab. Error (VIM)	Ab. Error (OHAM)
0.0	1	1	0	0
0.1	0.99465	0.99465	1.22×10^{-15}	7.77×10^{-16}
0.2	0.97712	0.97712	4.44×10^{-16}	3.33×10^{-16}
0.3	0.94490	0.94490	9.99×10^{-16}	1.11×10^{-16}
0.4	0.89509	0.89509	4.55×10^{-15}	1.11×10^{-15}
0.5	0.82436	0.82436	7.32×10^{-15}	5.55×10^{-16}
0.6	0.72884	0.72884	1.02×10^{-14}	2.22×10^{-16}
0.7	0.60412	0.60412	1.22×10^{-14}	3.21×10^{-15}
0.8	0.44510	0.44510	1.50×10^{-14}	2.24×10^{-14}
0.9	0.24595	0.24595	1.06×10^{-14}	1.97×10^{-14}
1.0	0	0	0	0

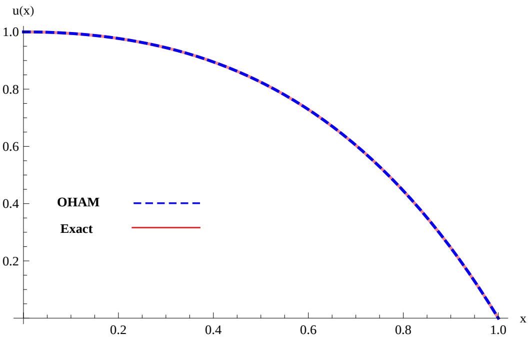

Fig. 1: Comparison of the approximate solution using OHAM with exact

3.2 Example 2

Consider the seventh-order nonlinear boundary value problem [8].:

$$\frac{d^7u}{dx^7} = e^{-x}u^2 \qquad (22)$$

with boundary conditions:
$u(0) = u^{(1)}(0) = u^{(2)}(0) = u^{(3)}(0) = 1,$
$u(1) = u^{(1)}(1) = u^{(2)}(1) = e$
The analytic solution of Eq. (22) is $u(x)=e^x$
Applying the proposed method (OHAM) on Eq. (22), we have
Zeroth order problem:

$$u_0^{(7)}(x) = 0 \qquad (23)$$

with boundary conditions:
$u_0(0) = u_0^{(1)}(0) = u_0^{(2)}(0) = u_0^{(3)}(0) = 1,$
$u_0(1) = u_0^{(1)}(1) = u_0^{(2)}(1) = e$
Its solution is

$$\begin{aligned} u_0(x) = &\frac{1}{6}(6+6x+3x^2+x^3-171x^4+63ex^4+261x^5 \\ &-96ex^5-106x^6+39ex^6) \end{aligned} \qquad (24)$$

First order problem:

$$u_1^{(7)}(x,C_1) = -e^{-x}C_1u_0^2 + u_0^{(7)} + C_1u_0^{(7)} \tag{25}$$

with boundary conditions:
$u_1(0) = u_1^{(1)}(0) = u_1^{(2)}(0) = u_1^{(3)}(0) = 0,$
$u_1(1) = u_1^{(1)}(1) = u_1^{(2)}(1) = 0$
Its solution is a long expression,here few terms are:

$$\begin{aligned} u_1(x,C_1) &= \frac{1}{36}e^{-1-x} \\ &(75452062580398764C_1 + \dots \\ &-8268e^2x^{12}C_1 + 1521e^3x^{12}C_1) \end{aligned} \tag{26}$$

Using Eqs. (24) and (26), the first order approximate solution via OHAM for $q = 1$ is

$$\tilde{u}(x,C_1) = u_0(x) + u_1(x,C_1) \tag{27}$$

We used least squares method to obtain the unknown convergent constant in $\tilde{u}$. Substituting the value of C_1 and after simplification of Eq. (27), we get the first order approximate solution via OHAM. In table 2 and figure 2, we compare the approximate solution obtained via OHAM with VIM and exact solutions.

Table 2: Comparison of solutions using OHAM, VIM [8]. and exact solution

x	OHAM Solution	Exact Solution	Ab. Error (VIM)	Ab. Error (OHAM)
0.0	1	1	0	0
0.1	1.1051	1.1051	3.84×10^{-12}	3.15×10^{-9}
0.2	1.2214	1.2214	1.22×10^{-10}	1.31×10^{-9}
0.3	1.3498	1.3498	2.77×10^{-10}	1.62×10^{-8}
0.4	1.4918	1.4918	7.58×10^{-10}	3.03×10^{-8}
0.5	1.6487	1.6487	1.15×10^{-9}	4.35×10^{-8}
0.6	1.8221	1.8221	1.31×10^{-9}	1.73×10^{-8}
0.7	2.0137	2.0137	1.22×10^{-9}	3.61×10^{-8}
0.8	2.2255	2.2255	6.60×10^{-10}	4.12×10^{-8}
0.9	2.4596	2.4596	1.65×10^{-10}	1.31×10^{-8}
1.0	2.7182	2.7182	1.32×10^{-11}	3.94×10^{-16}

4 Conclusion

In this paper, we reviewed seventh-order linear and nonlinear boundary value problems and applied a new technique known as Optimal Homotopy Asymptotic Method (OHAM). The obtained OHAM solution is then compared with VIM and exact solutions. Excellent agreement has been found. We also observed highly accurate results by even low order approximation which shows the effectiveness of the proposed method.

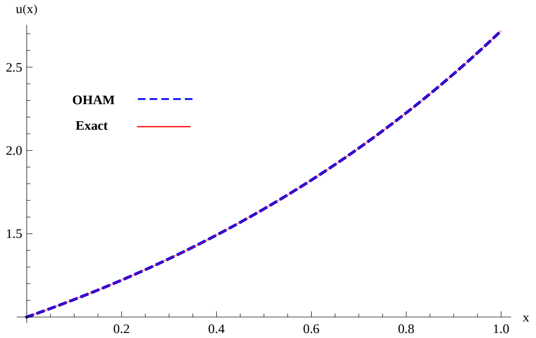

Fig. 2: Comparison of the approximate solution using OHAM with exact

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