Information Sciences Letters
An International Journal

http://dx.doi.org/10.12785/isl/040205

Boundary Constrained Controllability for Hyperbolic Systems

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Received: 8 Aug. 2014, Revised: 29 Mar. 2015, Accepted: 30 Mar. 2015
Published online: 1 May 2015

Abstract: The purpose of this work is to extend the concept of regional controllability to the case where the desired state is between two prescribed profiles, only on a boundary subregion Γ of the system evolution domain Ω . We'll characterize the minimum energy control that satisfied the out put constraints and which is limited mainly to systems described by hyperbolic partial differential equations. This problem is solved using Lagrangian approach and the obtained results are illustrated numerically.

Keywords: Minimum energy, hyperbolic systems, optimal control, Lagrangian multipliers, boundary subregion.

1 Introduction

Solving problems related to real applications in biological, economical or mechanical fields, need methods developed rigorously and with precision. Applied mathematics and control theory formulate these phenomena in a distributed system using PDEs which keep for each parameter its true physical meaning. Hence in the field of analysis and control of these systems, several studies have been developed particularly on notions of controllability, stability by duality observability and detectability, etc. These various concepts have been widely studied and leads to a vast and disparate literature [3], [4].
For distributed parameter systems, the controllability concept consists in steering a system from a initial state to a prescribed one defined on a spatial domain Ω. Later the term of regional controllability has been used to refer to control problems in which the target of our interest is not fully specified as a state, but refers only to a smaller region of the system domain. This concept has been developed and interesting results have been obtained, in particular, the possibility to reach a state only on an internal subregion [5] or on a part of the boundary [12].
There are many reasons for studying this kind of problem, one of them is the mathematical model of a real system which is obtained from measures or from approximation techniques and is very often affected by disturbances, and the solution of such a system is approximately known. For these reasons we are here interested in introducing the concept of controllability with constraints, which the aim is to steer a system from an initial state to a final one between two prescribed functions given only on a part of the boundary subregion $\partial\Omega$ of the geometric area Ω where the system is considered.
This work is a contribution to the enlargement of the regional controllability with constraints [1,2,13], limited mainly to systems described by hyperbolic partial differential equations and focussing only on a boundary part of the system evolution domain Ω. This paper is organized as follows, in section 2, we introduce the notion of constrained boundary regional controllability of hyperbolic systems, we provide results on this type of controllability and we give some definitions and properties related to this notion. Then in section 3, we solve the problem of minimum energy control using approach devoted to the computation of the optimal control problem for the hyperbolic equations excited by a boundary zone actuator. The last section illustrate the obtained result numerically

2 Constrained boundary controllability

Let Ω be an open bounded and regular subset of $\mathbb{R}^n (n > 1)$ with a boundary $\partial\Omega$. For $T > 0$, let $Q = \Omega \times]0,T[$ and $\Sigma = \partial\Omega \times]0,T[$, we consider the following hyperbolic system

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$$\begin{cases} \frac{\partial^2 y}{\partial t^2}(x,t) - Ay(x,t) = Bu(t) & Q \\ y(x,0) = y_0(x), \frac{\partial y}{\partial t}(x,0) = y_1(x) & \Omega \\ \frac{\partial y}{\partial \nu_A}(\xi,t) = 0 & \Sigma \end{cases} \quad (1)$$

Where A is a second-order elliptic linear symmetric operator, which generates a strongly continuous semi-group $(S(t))_{t\geq 0}$ [10], $\frac{\partial y}{\partial \nu_A}(\xi,t)$ denotes the co-normal with respect to A, $B \in \mathscr{L}(I\!R^p, H^1(\Omega))$, $u \in U = L^2(0,T,I\!R^p)$ (p depends on the number of the considered actuators) and (y_0,y_1) in the state space $\mathscr{X} = H^2(\Omega)\times H^1(\Omega)$. We design by $Z_u(T) = (y_u(T), \frac{\partial y_u}{\partial t}(T)) \in \mathscr{X}$ the solution of (1) [7].

For $\Gamma \subseteq \partial\Omega$ let consider

$$\begin{aligned} \chi_\Gamma : H^{\frac{3}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega) &\to H^{\frac{3}{2}}(\Gamma)\times H^{\frac{1}{2}}(\Gamma) \\ (z,z') &\mapsto \chi_\Gamma(z,z') = (\bar{\chi}_\Gamma z, \tilde{\chi}_\Gamma z') \end{aligned}$$

with

$$\begin{aligned} \tilde{\chi}_\Gamma : H^{\frac{1}{2}}(\partial\Omega) &\to H^{\frac{1}{2}}(\Gamma) \\ z &\mapsto \tilde{\chi}_\Gamma z = z_{|\Gamma} \end{aligned}$$

$$\begin{aligned} \bar{\chi}_\Gamma : H^{\frac{3}{2}}(\partial\Omega) &\to H^{\frac{3}{2}}(\Gamma) \\ z' &\mapsto \bar{\chi}_\Gamma z' = z'_{|\Gamma} \end{aligned}$$

While χ_Γ^* (resp. $\tilde{\chi}_\Gamma^*$ and $\bar{\chi}_\Gamma^*$) is the adjoint operator of χ_Γ (resp. $\tilde{\chi}_\Gamma$ and $\bar{\chi}_\Gamma$) which is the restriction operator.

Let's consider the trace operator

$$\begin{aligned} \gamma : H^2(\Omega)\times H^1(\Omega) &\to H^{\frac{3}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega) \\ (z_1,z_2) &\mapsto \gamma(z_1,z_2) = (\gamma_0 z_1, \gamma_0 z_2) \end{aligned}$$

with $\gamma_0 : H^m(\Omega) \to H^{m-\frac{1}{2}}(\partial\Omega)$ $(m = 1,2)$ denotes the trace operator of order zero which is linear, continuous, and surjective, while γ^* (resp. γ_0^*) is the adjoint operator of γ (resp. γ_0). Let $\alpha_1(.)$ and $\beta_1(.)$(resp. $\alpha_2(.)$ and $\beta_2(.)$) two given functions from $H^{\frac{3}{2}}(\partial\Omega)$ (resp. $H^{\frac{1}{2}}(\partial\Omega)$) such that $\alpha_1(.) \leq \beta_1(.)$ (resp. $\alpha_2(.) \leq \beta_2(.)$) a.e on Γ.

Throughout the paper we set

$$I := [\alpha_1(.),\beta_1(.)]\times[\alpha_2(.),\beta_2(.)] = \{(y(.),\frac{\partial y}{\partial t}(.)) \in H^{\frac{3}{2}}(\Gamma)\times H^{\frac{1}{2}}(\Gamma) \mid \alpha_1(.) \leq y(.) \leq \beta_1(.) \text{ and } \alpha_2(.) \leq \frac{\partial y}{\partial t}(.) \leq \beta_2(.) \text{ a. e on } \Gamma\}$$

Let H be the operator from $U \to H^2(\Omega)\times H^1(\Omega)$, for $u \in U$, defined by:

$$Hu = \int_0^T S(T-s)Bu(s)ds$$

We recall that an actuator is conventionally defined by a couple (D,f), Where $D \subset \overline{\Omega}$ is the geometric support of the actuator and f is the spatial distribution of the action on the support D (see [5]).

In the case of a pointwise actuator (internal or boundary) $D = \{b\}$ and $f = \delta(b-.)$, where δ is the Dirac mass concentrated in b, and the actuator is then denoted by (b,δ_b). For definitions and the properties of strategic actuators we refer to [4,6].

Definition 1.
We say that the system (1) is $[\alpha_1(.),\beta_1(.)]\times[\alpha_2(.),\beta_2(.)]$-Controllable on Γ if

$$(Im\chi_\Gamma\gamma H)\cap([\alpha_1(.),\beta_1(.)]\times[\alpha_2(.),\beta_2(.)]) \neq \emptyset$$

Remark.
The above definition is equivalent to say that:
The system (1) is $[\alpha_1(.),\beta_1(.)]\times[\alpha_2(.),\beta_2(.)]$-Controllable on Γ at the time T if there exists $u \in U$ such that:

$$\alpha_1(.) \leq \bar{\chi}_\Gamma\gamma_0 y_u(T) \leq \beta_1(.)$$
and
$$\alpha_2(.) \leq \tilde{\chi}_\Gamma\gamma_0\frac{\partial y_u}{\partial t}(T) \leq \beta_2(.) \text{ a.e on } \Gamma$$

Definition 2.
The actuator (D,f) is $[\alpha_1(.),\beta_1(.)]\times[\alpha_2(.),\beta_2(.)]$-strategic on Γ, if the excited system (1) is $[\alpha_1(.),\beta_1(.)]\times[\alpha_2(.),\beta_2(.)]$- Controllable on Γ.

Remark. 1.The above definition means that we are interested only in the transfer of system (1) to a position (resp. speed) between $\alpha_1(.)$ and $\beta_1(.)$ (resp. $\alpha_2(.)$ and $\beta_2(.)$) on Γ
2.A system which is controllable on Γ [14] is $[\alpha_1(.),\beta_1(.)]\times[\alpha_2(.),\beta_2(.)]-$controllable on Γ
3.A system (1) which is $[\alpha_1(.),\beta_1(.)]\times[\alpha_2(.),\beta_2(.)]$-Controllable on Γ_1 is $[\alpha_1(.),\beta_1(.)]\times[\alpha_2(.),\beta_2(.)]$-Controllable for any $\Gamma_2 \subseteq \Gamma_1$.
4.The control u depends on the time variable, but it also implicitly depends on Γ

The $[\alpha_1(.),\beta_1(.)]\times[\alpha_2(.),\beta_2(.)]$-controllability on Γ may be characterized by the following result:

Proposition 1.
The system (1) is $[\alpha_1(.),\beta_1(.)]\times[\alpha_2(.),\beta_2(.)]$-Controllable on Γ if and only if

$$(Ker\chi_\Gamma + Im\gamma H)\cap([\alpha_1(.),\beta_1(.)]\times[\alpha_2(.),\beta_2(.)]) \neq \emptyset$$

Proof.
We suppose that the system (1) is $[\alpha_1(.),\beta_1(.)]\times[\alpha_2(.),\beta_2(.)]$- Controllable on Γ which is equivalent to

$$(Im\chi_\Gamma\gamma H)\cap([\alpha_1(.),\beta_1(.)]\times[\alpha_2(.),\beta_2(.)]) \neq \emptyset$$

so there exists $z \in ([\alpha_1(.),\beta_1(.)]\times[\alpha_2(.),\beta_2(.)])$, and $u \in U$ such that $\chi_\Gamma\gamma Hu = \chi_\Gamma z$ which gives $\chi_\Gamma(z-\gamma Hu) = 0$. Let's consider $z_1 = z - \gamma Hu$ and $z_2 = \gamma Hu$, then $z = z_1 + z_2$ with $z_1 \in ker(\chi_\Gamma)$ and $z_2 \in Im\gamma H$ which prove that
$z \in (Ker\chi_\Gamma + Im\gamma H)$.

Conversely, we suppose $(Ker\chi_\Gamma + Im\gamma H) \cap ([\alpha_1(.),\beta_1(.)] \times [\alpha_2(.),\beta_2(.)]) \neq \emptyset$ then there exists $z \in ([\alpha_1(.),\beta_1(.)] \times [\alpha_2(.),\beta_2(.)])$ such that $z \in (Ker\chi_\Gamma + Im\gamma H)$, so $z = z_1 + z_2$, with $\chi_\Gamma z_1 = 0$ and $\exists \; u \in U \mid z_2 = \gamma Hu$. It follows that $\chi_\Gamma z = \chi_\Gamma z_1 + \chi_\Gamma z_2 = \chi_\Gamma z_2 = \chi_\Gamma \gamma Hu$ and $\chi_\Gamma z \in (Im\chi_\Gamma \gamma H)$ which prove

$$(Im\chi_\Gamma \gamma H) \cap ([\alpha_1(.),\beta_1(.)] \times [\alpha_2(.),\beta_2(.)]) \neq \emptyset$$

then system (1) is $[\alpha_1(.),\beta_1(.)] \times [\alpha_2(.),\beta_2(.)]$-Controllable on Γ

3 Minimum energy control

The purpose of this section is to explore the Lagrangian approach devoted to the computation of the optimal control problem for the hyperbolic equation excited by a boundary zone actuator which steers the system (1) from $(y_0,y_1) \in H^2(\Omega) \times H^1(\Omega)$ to a final state (p^d,v^d) such that $\alpha_1(.) \leq \bar{\chi}_\Gamma \gamma_0 p^d \leq \beta_1(.)$ and $\alpha_2(.) \leq \tilde{\chi}_\Gamma \gamma_0 v^d \leq \beta_2(.)$ on a subregion Γ.
More precisely we are interested to the following minimization problem

$$\begin{cases} \inf \mathscr{J}(u) = \int_0^T ||u(t)||^2_{\mathbb{R}^p} dt \\ u \in U_{ad}^\Gamma \end{cases} \tag{2}$$

where

$$U_{ad}^\Gamma = \{u \in U \mid \alpha_1(.) \leq \bar{\chi}_\Gamma \gamma_0 y_u(T) \leq \beta_1(.) \text{ and } \alpha_2(.) \leq \tilde{\chi}_\Gamma \gamma_0 \frac{\partial y_u}{\partial t}(T) \leq \beta_2(.)\},$$

is the set of admissible controls.
The following result ensure the existence and the uniqueness of the solution of problem (2).

Proposition 2.
If the system (1) is $[\alpha_1(.),\beta_1(.)] \times [\alpha_2(.),\beta_2(.)]$-Controllable on Γ then the problem (2) has a unique solution u^*.

Proof.
If the system (1) is $[\alpha_1(.),\beta_1(.)] \times [\alpha_2(.),\beta_2(.)]$-Controllable on Γ then U_{ad} is a non-empty subset of the reflexive U, then the mapping $u \mapsto (y_u(T), \frac{\partial y_u}{\partial t}(T))$ is linear, so U_{ad}^Γ is convex, and to prove that U_{ad}^Γ is closed, we consider a sequence $(u_n)_n$ in U_{ad}^Γ such that u_n converge strongly to u in U. Since $\chi_\Gamma \gamma H$ is continuous, then $\chi_\Gamma \gamma H u_n$ converges strongly to $\chi_\Gamma \gamma Hu$ in $H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$, and $\chi_\Gamma \gamma(y_{u_n}(T), \frac{\partial y_{u_n}}{\partial t}(T)) \in [\alpha_1(.),\beta_1(.)] \times [\alpha_2(.),\beta_2(.)]$ which is closed so U_{ad}^Γ is closed.
Furthermore $\lim_{\|u\| \to +\infty} \mathscr{J}(u) = +\infty$ and the mapping $u \mapsto \frac{1}{2} \parallel u \parallel^2$ is continue and strictly convex then (2) has a unique solution.

We consider the problem (2), when the system is excited by one zone actuator (D, f). The following result gives a useful characterisation of problem (2):

Proposition 3.
If the actuator (D, f) is $[\alpha_1(.),\beta_1(.)] \times [\alpha_2(.),\beta_2(.)]$-strategic on Γ then the solution of (2) is given by :

$$u^* = -(\chi_\Gamma \gamma H)^*(\lambda_1^*, \lambda_2^*) \tag{3}$$

Where $(\lambda_1^*,\lambda_2^*)$ is the solution of:

$$\begin{cases} (p^{d^*}, v^{d^*}) = P_I[\rho(\lambda_1^*, \lambda_2^*) + (p^{d^*}, v^{d^*})] \\ (p^{d^*}, v^{d^*}) + \; R_\Gamma(\lambda_1^*, \lambda_2^*) = \chi_\Gamma \gamma S(T)(y_0, y_1) \end{cases} \tag{4}$$

While
$P_I : H^{3/2}(\Gamma) \times H^{1/2}(\Gamma) \longrightarrow [\alpha_1(.),\beta_1(.)] \times [\alpha_2(.),\beta_2(.)]$ *denotes the projection operator, $\rho > 0$ and* $R_\Gamma = (\chi_\Gamma \gamma H)(\chi_\Gamma \gamma H)^*$.

Proof.
If the actuator (D, f) is $[\alpha_1(.),\beta_1(.)] \times [\alpha_2(.),\beta_2(.)]$-strategic on Γ then $U_{ad}^\Gamma \neq \emptyset$ and (2) has a unique solution. The problem (2) is equivalent to the following saddle point problem:

$$\begin{cases} \inf \mathscr{J}(u) \\ (u, p^d, v^d) \in V \end{cases} \tag{5}$$

Where

$$V = \{(u, p^d, v^d) \in U \times I \mid \chi_\Gamma \gamma(y_u(T), \frac{\partial y_u}{\partial t}(T)) = (p^d, v^d)$$

To study this constraints, we'll use a Lagrangian functional and steers the problem (5) to a saddle point problem.
We associate to the problem (5) the Lagrangian functional defined by:

$$\forall (u, p^d, v^d, \lambda_1, \lambda_2) \in U \times I \times H^{3/2}(\Gamma) \times H^{1/2}(\Gamma),$$

$$L(u, p^d, v^d, \lambda_1, \lambda_2) = \frac{1}{2} \parallel u \parallel^2 + \langle \lambda_1, \bar{\chi}_\Gamma \gamma_0 y_u(T) - p^d \rangle + \langle \lambda_2, \tilde{\chi}_\Gamma \gamma_0 \frac{\partial y_u}{\partial t}(T) - v^d \rangle$$

Let recall that $(u^*, p^{d^*}, v^{d^*}, \lambda_1^*, \lambda_2^*)$ is a saddle point of L if:

$$\max_{\substack{\lambda_1 \in L^2(\omega) \\ \lambda_2 \in L^2(\omega)}} L(u^*, p^{d^*}, v^{d^*}, \lambda_1, \lambda_2) = L(u^*, p^{d^*}, v^{d^*}, \lambda_1^*, \lambda_2^*) = \min_{\substack{u \in U \\ (p^d, v^d) \in I}} L(u, p^d, v^d, \lambda_1^*, \lambda_2^*)$$

the proof is presented in three steps.
• Step 1
$U \times I$ is non-empty, closed and convex subset. The Functional L satisfies conditions

–$(u, p^d, v^d) \mapsto L(u, p^d, v^d, \lambda_1, \lambda_2)$ is convex and lower semi-continuous for all $(\lambda_1, \lambda_2) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$.

–$(\lambda_1,\lambda_2)\mapsto L(u,p^d,v^d,\lambda_1,\lambda_2)$ is concave and upper semi-continuous for all $(u,p^d,v^d)\in U\times I$

Moreover there exists $(\lambda_1^0,\lambda_2^0)\in H^{3/2}(\Gamma)\times H^{1/2}(\Gamma)$ such that

$$\lim_{\|(u,p^d,v^d)\|\to+\infty} L(u,p^d,v^d,\lambda_1^0,\lambda_2^0)=+\infty \quad (6)$$

And there exists $(u_0,p_0^d,v_0^d)\in U\times I$ such that

$$\lim_{\|(\lambda_1,\lambda_2)\|\to+\infty} L(u_0,p_0^d,v_0^d,\lambda_1,\lambda_2)=-\infty \quad (7)$$

Then, the functional L admits a saddle point. For more details we refer to [8].

• Step 2

Let $(u^*,p^{d^*},v^{d^*},\lambda_1^*,\lambda_2^*)$ be a saddle point of L and prove that u^* is the solution of (2). We have

$$L(u^*,p^{d^*},v^{d^*},\lambda_1,\lambda_2)\le L(u^*,p^{d^*},v^{d^*},\lambda_1^*,\lambda_2^*)\le L(u,p^d,v^d,\lambda_1^*,\lambda_2^*)$$

For all $(u,p^d,v^d,\lambda_1,\lambda_2)\in U\times I\times H^{3/2}(\Gamma)\times H^{1/2}(\Gamma)$

From the first inequality

$$L(u^*,p^{d^*},v^{d^*},\lambda_1,\lambda_2)\le L(u^*,p^{d^*},v^{d^*},\lambda_1^*,\lambda_2^*)$$

it follows that :

$$\begin{aligned}&\langle\lambda_1,\bar{\chi}_\Gamma\gamma_0 y_{u^*}(T)-p^{d^*}\rangle+\langle\lambda_2\tilde{\chi}_\Gamma\gamma_0\frac{\partial y_{u^*}}{\partial t}(T)-v^{d^*}\rangle\le\\&\langle\lambda_1^*,\bar{\chi}_\Gamma\gamma_0 y_{u^*}(T)-p^{d^*}\rangle+\langle\lambda_2^*,\tilde{\chi}_\Gamma\gamma_0\frac{\partial y_{u^*}}{\partial t}(T)-v^{d^*}\rangle\end{aligned}$$

which implies that $\bar{\chi}_\Gamma\gamma_0 y_{u^*}(T)=p^{d^*}$ and $\tilde{\chi}_\Gamma\gamma_0\frac{\partial y_{u^*}}{\partial t}(T)=v^{d^*}$, hence $\bar{\chi}_\Gamma\gamma_0 y_{u^*}(T)\in[\alpha_1(.),\beta_1(.)]$ and $\tilde{\chi}_\Gamma\gamma_0\frac{\partial y_{u^*}}{\partial t}(T)\in[\alpha_2(.),\beta_2(.)]$.

From the second inequality if follows that :

$$\begin{aligned}&\frac{1}{2}\|u^*\|^2+\langle\lambda_1^*,\bar{\chi}_\Gamma\gamma_0 y_{u^*}(T)-p^{d^*}\rangle+\langle\lambda_2^*,\tilde{\chi}_\Gamma\gamma_0\frac{\partial y_{u^*}}{\partial t}(T)-v^{d^*}\rangle\\&\le\frac{1}{2}\|u\|^2+\langle\lambda_1^*,\bar{\chi}_\Gamma\gamma_0 y_u(T)-p^d\rangle+\langle\lambda_2^*,\tilde{\chi}_\Gamma\gamma_0\frac{\partial y_u}{\partial t}(T)-v^d\rangle\end{aligned}$$

and $(p^d,v^d)\in[\alpha_1(.),\beta_1(.)]\times[\alpha_2(.),\beta_2(.)]$.

Since $\bar{\chi}_\Gamma\gamma_0 y_{u^*}(T)=p^{d^*}$ and $\tilde{\chi}_\Gamma\gamma_0\frac{\partial y_{u^*}}{\partial t}(T)=v^{d^*}$ we have,

$$\begin{aligned}\frac{1}{2}\|u^*\|^2\le\frac{1}{2}\|u\|^2&+\langle\lambda_1^*,\bar{\chi}_\Gamma\gamma_0 y_u(T)-p^d\rangle\\&+\langle\lambda_2^*,\tilde{\chi}_\Gamma\gamma_0\frac{\partial y_u}{\partial t}(T)-v^d\rangle\end{aligned}$$

taking $p^d=\bar{\chi}_\Gamma\gamma_0 y_u(T)\in[\alpha_1(.),\beta_1(.)]$ and $v^d=\tilde{\chi}_\Gamma\gamma_0\frac{\partial y_u}{\partial t}(T)\in[\alpha_2(.),\beta_2(.)]$, we obtain $\frac{1}{2}\|u^*\|^2\le\frac{1}{2}\|u\|^2$ which implies that u^* is the minimum energy.

• Step 3

$(u^*,p^{d^*},v^{d^*},\lambda_1^*,\lambda_2^*)$ is a saddle point of L then the following assumptions hold:

$$\langle u^*,u-u^*\rangle+\langle(\lambda_1^*,\lambda_2^*),\chi_\Gamma\gamma H(u-u^*)\rangle=0\quad\forall u\in U \quad (8)$$

$$\langle(\lambda_1^*,\lambda_2^*),(p^d,v^d)-(p^{d^*},v^{d^*})\rangle\le 0\quad\forall(p^d,v^d)\in I \quad (9)$$

$$\begin{aligned}\langle(\lambda_1,\lambda_2)-(\lambda_1^*,\lambda_2^*),\chi_\Gamma\gamma(y_{u^*}(T),\frac{\partial y_{u^*}}{\partial t}(T))-(p^{d^*},v^{d^*})\rangle=0,\\ \forall(\lambda_1,\lambda_2)\in H^{3/2}(\Gamma)\times H^{1/2}(\Gamma)\end{aligned} \quad (10)$$

(Details on the sadlle point theory and its applications can be found for instance in [6, 11]).

From (8) we deduce that (3) and (10) is equivalent to $(p^{d^*},v^{d^*})=\chi_\Gamma\gamma S(T)(y_0,y_1)+\chi_\Gamma\gamma H(u^*)$, and with (3) the second part of (4) is obtained. From the inequality (9) we obtain $\langle(\rho(\lambda_1^*,\lambda_2^*)+(p^{d^*},v^{d^*}))-(p^{d^*},v^{d^*}),(p^d,v^d)-(p^{d^*},v^{d^*})\rangle\le 0$, for all $(p^d,v^d)\in I$, which is equivalent to the first part of (4).

Corollary 1.

If the system (1) is exactly controllable on Γ, and ρ convenably chosen, then the system (4) has only one solution $(\lambda_1^*,\lambda_2^*,p^{d^*},v^{d*})$.

Proof.

The regional controllability on Γ implies that $(\chi_\Gamma\gamma H)^*$ and R_Γ are bijective, so if $(u^*,p^{d^*},v^{d*},\lambda_1^*,\lambda_2^*)$ is a saddle point of L then the system (4) is equivalent to

$$\begin{aligned}(\lambda_1^*,\lambda_2^*)&=R_\Gamma^{-1}(\chi_\Gamma\gamma S(T)(y_0,y_1)+(p^{d^*},v^{d^*}))\\(p^{d^*},v^{d^*})&=P_I(-\rho R_\Gamma^{-1}(p^{d^*},v^{d^*})+\rho R_\Gamma^{-1}\chi_\Gamma\gamma S(T)(y_0,y_1)\\&\quad+(p^{d^*},v^{d^*}))\end{aligned} \quad (11)$$

It follows that $(p^{d^*},v^{d^*})\in[\alpha_1(.),\beta_1(.)]\times[\alpha_2(.),\beta_2(.)]$ is a fixed point of the function

$$\begin{aligned}&F_\rho: I\to I\\&(Z_1,Z_2)\mapsto(P_I(-\rho R_\Gamma^{-1}(Z_1,Z_2)\\&\qquad+\rho R_\Gamma^{-1}\chi_\Gamma\gamma S(T)(y_0,y_1)+(Z_1,Z_2))\end{aligned} \quad (12)$$

The operator R_Γ^{-1} is coercive, then there exists $m>0$ such that

$\langle R_\Gamma^{-1}(Z_1,Z_2),(Z_1,Z_2)\rangle\ge m\,\|(Z_1,Z_2)\|^2$

$\forall\,(Z_1,Z_2)\in H^{3/2}(\Gamma)\times H^{1/2}(\Gamma)$

It follows that

$$\begin{aligned}&\|F_\rho(Z_1,Z_2)-F_\rho(Y_1,Y_2)\|^2\\&\le(1+\rho^2\|R_\Gamma^{-1}\|^2-2\rho m)\,\|(Z_1,Z_2)-(Y_1,Y_2)\|^2\end{aligned}$$

for all (Z_1,Z_2) and (Y_1,Y_2) in $[\alpha_1(.),\beta_1(.)]\times[\alpha_2(.),\beta_2(.)]$, then we deduce that if

$$0<\rho<\frac{2m}{\|R_\Gamma^{-1}\|^2}$$

Then F_ρ is contractant, which implies the uniqueness of p^{d^*}, v^{d^*}, λ_1^* and λ_2^*.

Remark.

1. If $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$ we find the notion of exact regional controllability and the solution of (2) is given by

$$u^*(t) = (\chi_\Gamma \gamma H)^* R_\Gamma^{-1}((\alpha_1, \alpha_2) - \chi_\Gamma \gamma S(T)(y_0, y_1))$$

2. Similar results can be obtained in pointwise actuator case.

4 Numerical approach

4.1 Algorithm

From proposition (3) the solution of the problem (1) arises to compute the saddle points of L, which is equivalent to solving the problem

$$\inf_{(u,p^d,v^d)\in U\times I} \Big(\sup_{(\lambda_1,\lambda_2)\in H^{3/2}(\Gamma)\times H^{1/2}(\Gamma)} L(u, p^d, v^d, \lambda_1, \lambda_2) \Big) \quad (13)$$

To achieve this we shall use the following algorithm which is based on Uzawa one

1. Choose:
   .The subregion Γ, the actuator (D, f) and a precision threshold ε small enough.
   .Functions $(p_0^d, v_0^d) \in [\alpha_1(.), \beta_1(.)] \times [\alpha_2(.), \beta_2(.)]$ and $(\lambda_1^0, \lambda_2^0) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$
2. Repeat
   $.u_n = -(\chi_\Gamma \gamma H)^*(\lambda_1^n, \lambda_2^n)$,
   $.p_n^d = P_{[\alpha_1(.),\beta_1(.)]}(\rho \lambda_1^{n-1} + p_{n-1}^d)$,
   $.v_n^d = P_{[\alpha_2(.),\beta_2(.)]}(\rho \lambda_2^{n-1} + v_{n-1}^d)$
   $.\lambda_1^n = \lambda_1^{n-1} + (\tilde{\chi}_\Gamma \gamma_0 y_{u_{n-1}}(T) - p_{n-1}^d)$
   $.\lambda_2^n = \lambda_2^{n-1} + (\tilde{\chi}_\Gamma \gamma_0 \dfrac{\partial y_{u_{n-1}}}{\partial t}(T) - v_{n-1}^d)$
   Until $\| p_n^d - p_{n-1}^d \|_{H^{3/2}(\Gamma)} + \| v_n^d - v_{n-1}^d \|_{H^{1/2}(\Gamma)} \leq \varepsilon$

4.2 Example

Here we give a numerical example that test the efficiency of the previous algorithm. For this, let us consider a two-dimensional system defined on
$\Omega =]0,1[\times]0,1[$, described by the hyperbolic equation and excited by a pointwise actuator:

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2}(x,t) - \Delta y(x,t) = \delta(x,b)u(t) & \Omega\times]0,T[\\ y(x,0) = 0, \ \dfrac{\partial y}{\partial t}(x,0) = 0 & \Omega \\ \dfrac{\partial y}{\partial \nu_A}(\xi,t) = 0 & \partial\Omega\times]0,T[\end{cases} \quad (14)$$

Where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \text{ and } \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

We take T=1, $(b_1, b_2) = (0.15, 0.85)$ (location of the pointwise actuator),
$\alpha_1(x_1,x_2) = \alpha_2(x_1,x_2) = x_1 + x_2^2(1-x_2)^2$ and $\beta_1(x_1,x_2) = \beta_2(x_1,x_2) = x_1 + 5x_2^2(1-x_2)^2$.
Applying the previous algorithm for the global case where $\Gamma =]0,1[\times\{0\}$, we obtain the following results:

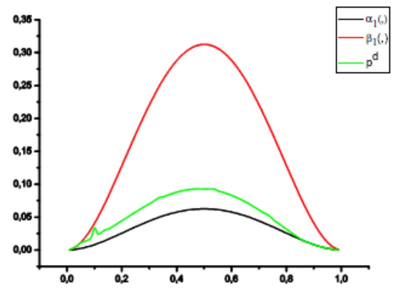

Fig. 1: Reached position between $\alpha_1(.)$ and $\beta_1(.)$ on Γ

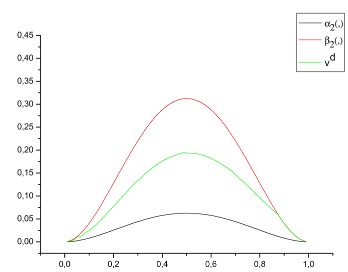

Fig. 2: Reached speed between $\alpha_2(.)$ and $\beta_2(.)$ on Γ

From figure 1 and 2, we note that the reached position (resp. speed) is between $\alpha_1(.)$ and $\beta_1(.)$ (resp. $\alpha_2(.)$ and $\beta_2(.)$) in the subregion Γ, the location of the actuator is $[\alpha_1(.), \beta_2(.)] \times [\alpha_2(.), \beta_2(.)]$-strategic and the reached position (resp. speed) is obtained with reconstruction error $\varepsilon \simeq 1.26 \times 10^{-4}$ and cost $\| u^* \|^2 \simeq 1.69$

For the regional case where $\Gamma =]0.3, 0.75[\times\{0\}$ we obtain:

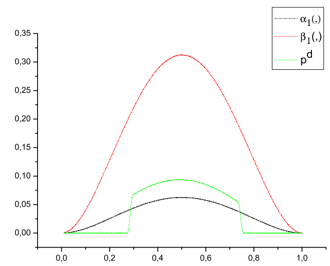

Fig. 3: Reached position between $\alpha_1(.)$ and $\beta_1(.)$ on Γ

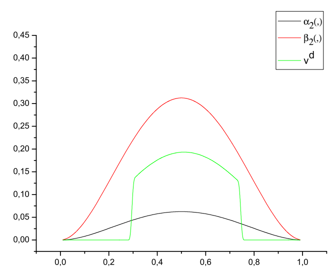

Fig. 4: Reached speed between $\alpha_2(.)$ and $\beta_2(.)$ on Γ

Also figure 3 and 4 show clearly that in this regional case the position (resp. speed) is between $\alpha_1(.)$ and $\beta_1(.)$ (resp. $\alpha_2(.)$ and $\beta_2(.)$) which means that the $[\alpha_1(.),\beta_1(.)] \times [\alpha_2(.),\beta_2(.)]$- Controllability is satisfied with reconstruction error $\varepsilon \simeq 3.21 \times 10^{-5}$ and cost $\| u^* \|^2 \simeq 2.37 \times 10^{-1}$.

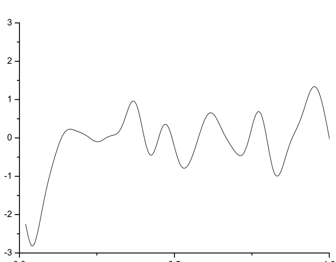

Fig. 5: Evolution of the control function

Figure 5 shows the evolution of the optimal control u^* which steers the system from the initial position (resp. speed) to the desired one between $\alpha_1(.)$ and $\beta_1(.)$ (resp. $\alpha_2(.)$ and $\beta_2(.)$)

5 Conclusion

We have developed an extension of the notion of controllability for hyperbolic systems with constraints in the boundary case, we characterized the optimal control using Lagrangian approach and interesting results are obtained and illustrated with numerical example and simulations. Future works aim to extend this notion of regional controllability with constrained to the case of the gradient.

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