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Fixed point Theorem Satisfying Weak Integral Type Contraction in Generalized Metric Spaces

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Abstract: In this article, we establish a unique fixed point theorem for $\psi_{\int\phi}$ weakly integral type contraction in the context of complete G-metric space. Our established result extend some well known results in literature. Suitable example is also given for the usability of the derived result.

Keywords: Fixed point, Complete G-metric space, weak integral type contraction.

1 Introduction

Poincare, in 1886 introduced the concept of fixed point. Later on a French mathematician Frechet initiated the concept of metric space in 1906. Combining these two different concepts a new field of fixed point theory originated called Metric Fixed Point Theory. Metric fixed point theory is an important mathematical discipline because of its applications in areas such as variational and linear inequalities, optimization, and approximation theory. It is used for existence of fixed point in differential equations, matrix equations and integral equations.
In 1922, Banach [3] proved a theorem known as Banach contraction principle. Banach contraction principle states "A contraction mapping in a complete metric space has a unique fixed point". After that many authors generalized this principle in various spaces for following different contractive conditions. The concept of weak contraction is initiated by Alber and Gurre [1] in 1997, as a generalization of contraction and established the existence of fixed points for a self map in a Hilbert space. Rhoades extended this concept to metric spaces and defined ϕ-weak contraction as following:
A self map T on metric space (X,d) is said to be ϕ-weak contraction if there exists a map $\phi:\mathbb{R}^+\to\mathbb{R}^+$ with $\phi(0)=0$ and $\phi(t)>0$ for all $t>0$ such that

$$d(Tx,Ty)\le d(x,y)-\phi(d(x,y)),\ \ \forall\, x,y\in X.$$

Rhoades [14] proved the following theorem.

Theorem 1.*Weak contractive self map in a complete metric space has a unique fixed point.*

Dutta and Choudhury [7] generalized the concept of weak contraction as a (ψ,ϕ) weak contraction and established the following result.

Theorem 2.*Let T be a self map on complete metric space (X,d) satisfying the following inequality*

$$\psi(d(Tx,Ty))\le\psi(d(x,y))-\phi(d(x,y)),\ \ \forall\, x,y\in X.$$

where $\psi,\phi:\mathbb{R}^+\to\mathbb{R}^+$ is monotonic non-decreasing and continuous function such that $\psi(0)=0=\phi(0)$, $\psi(t)>0$ and $\phi(t)>0$ for $t>0$. Then T has a unique fixed point.

Zhang and Song [15] proved the following theorem for two self map in a complete metric space.

Theorem 3.*Let S, T be a self map on complete metric space (X,d) such that $\forall\, x,y\in X$,*

$$d(Tx,Sy)\le M(x,y)-\phi(M(x,y)),\ \ \forall\, x,y\in X,$$

where

$$M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Sy),\tfrac{1}{2}[d(x,Sy)+d(y,Tx)]\},$$

and $\phi:\mathbb{R}^+\to\mathbb{R}^+$ is lower semi continuous function with $\phi(0)=0$ and $\phi(t)>0$ for $t>0$. Then S and T has a unique common fixed point.

Dragan Doric [5] extend the result of Zhang and Song in the following way:

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Theorem 4. *A self map T and S on complete metric space (X,d) such that $\forall\, x,y \in X$,*

$$\psi(d(Tx,Sy)) \le \psi(M(x,y)) - \phi(M(x,y)),$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Sy), \frac{1}{2}[d(x,Sy)+d(y,Tx)]\},$$

$\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous monotone non-decreasing function and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is lower semi-continuous function with $\psi(0) = 0 = \phi(0)$ and $\psi(t) > 0$, $\phi(t) > 0$ for $t > 0$. Then S and T has a common fixed point.

In 2005 Mustafa and Sims [12], introduced a new concept of generalized metric spaces, called G-metric spaces. In such spaces every triplet of elements is assigned to a non-negative real number, based on the notion of G-metric spaces after that many researcher extend the known contractions in G-metric space one of these is (ψ,ϕ) weak contraction see[6,9].
In 2002, the famous Banach fixed point theorem was extended by Branciari [4] for a mapping of integral type. Branciari established the following fixed point theorem:

Theorem 5. *If T be a self- map of a complete metric space (X,d) such that $\forall\, x,y \in X$*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le \eta \int_0^{d(x,y)} \varphi(t)dt, \quad \eta \in [0,1),$$

where, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable, non-negative mapping which is summable on each compact subset of $\mathbb{R}^+$ such that $\int_0^{\varepsilon} \phi(t)dt > 0$ for each $\varepsilon > 0$. Then T has a unique fixed point.

This result was more generalized and extend by many authors either by relaxing the condition of contractivity or changing the underlying space or sometimes both for the study of the existence of fixed points and common fixed points for different mapping in complete metric space see [10,11] and references therein.
Luong and Thuan [13] proved the following theorem for $\psi_{\int\phi}$ weakly contractive condition.

Theorem 6. *Let T be $\psi_{\int\phi}$ weakly contractive self map on complete metric space (X,d) and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be Lebesgue integrable mapping such that for each $x,y \in X$,*

$$\psi\Big(\int_0^{d(Tx,Ty)} \varphi(t)dt\Big) \le \psi\Big(\int_0^{d(x,y)} \varphi(t)dt\Big) - \phi\Big(\int_0^{d(x,y)} \varphi(t)dt\Big),$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing function, $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a lower semi continuous and nondecreasing function such that $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$. Then T has a unique fixed point.

Hassen ayadi [2] prove the following common fixed point theorem for integral type contraction in generalized metric spaces.

Theorem 7. *Let (X,G) be a complete G-metric space and $f,g : X \to X$ be a mapping such that*

$$\int_0^{G(fx,fy,fz)} \phi(t)dt \le \alpha \int_0^{G(gx,gy,gz)} \phi(t)dt,$$

$\forall\, x,y,z \in X$ where $\alpha \in [0,1)$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable, non-negative mapping which is summable on each compact subset of $\mathbb{R}^+$ such that $\int_0^{\varepsilon} \phi(t)dt > 0$ for each $\varepsilon > 0$. If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Recently Gupta *et al* [8] established the following result for weak contraction.

Theorem 8. *Let (X,d) be a complete metric space and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be Lebesgue integrable mapping. $T : X \to X$ such that $\forall\, x,y \in X$*

$$\psi\Big(\int_0^{G(Tx,Ty)} \varphi(t)dt\Big) \le \psi\Big(\int_0^{M(x,y)} \varphi(t)dt\Big) - \phi\Big(\int_0^{M(x,y)} \varphi(t)dt\Big),$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable, non-negative mapping which is summable on each compact subset of $\mathbb{R}^+$ such that $\int_0^{\varepsilon} \phi(t)dt > 0$ for each $\varepsilon > 0$. $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing function and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a lower semi continuous and nondecreasing function such that $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$ where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{[d(x,Tx)+d(y,Ty)]}{2}\right\}.$$

Then T has a unique fixed point.

Through out the paper $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{N}_0$ will denote the set of all non-negative real numbers, the set of positive integer and The set of non-negative integer respectively.
Let

$$\Phi = \Big\{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+, \ \varphi \text{ is Lebesgue integrable, summable on each compact subset of } \mathbb{R}^+ \text{ and } \int_0^{\varepsilon} \varphi(t)dt > 0 \text{ for each } \varepsilon > 0\Big\}.$$

In this paper, using the concept of $\psi_{\int\phi}$ weakly integral type contraction a fixed point theorem in complete G-metric space is investigated.

2 Preliminaries

Definition 1.[12] *Let X be a non-empty set and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the conditions:*

1. $G(x,y,z) = 0$ *implies that* $x = y = z \; \forall \, x,y,z \in X$;
2. $0 < G(x,x,y) \; \forall \, x,y \in X$ *with* $x \neq y$;
3. $G(x,x,y) \leq G(x,y,z) \; \forall \, x,y,z \in X$ *with* $y \neq z$;
4. $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$ *(symmetry in three variables);*
5. $G(x,y,z) \leq G(x,a,a) + G(a,y,z) \; \forall \, x,y,z,a \in X$.

Then it is called G-metric on X and the pair (X,G) is a G-metric space.

*Example 1.*Let $X = \mathbb{R}^+$ and $G : X \times X \times X \to \mathbb{R}^+$ be the function defined is follow

$$G(x,y,z) = \max\{|x-y|, |y-z|, |z-x|\},$$

for all $x,y,z \in X$. Then G is G-metric on X.

Definition 2.*Let (X,G) be a G-metric space and let x_n be a sequence in X. A point $x \in X$ is said to be the limit of the sequence x_n if*

$$\lim_{n,m \to \infty} G(x_n, x_m, x) = 0$$

and the sequence x_n is said to be G-convergent to X.

Definition 3.*A sequence x_n is called a G-Cauchy sequence if for every $\varepsilon > 0$, there is a positive integer $\mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n,m,l > \mathbb{N}$.*

Definition 4.*A metric space (X,G) is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in (X,G) is G-convergent in X.*

Proposition 1.[12] *Let (X,G) be a G-metric space. Then the following statement holds:*

1. $|G(x,y,z) - G(x,y,a)| \leq \max\{G(a,z,z), G(z,a,a)\}$;
2. $G(x,y,y) \leq 2G(y,x,x) \; \forall \, x,y,z,a \in X$.

Lemma 1.[10] *Let $\varphi \in \Phi$ and $\{r_n\}_{n \in \mathbb{N}}$ is a non-negative sequence with $\lim_{n \to \infty} r_n = a$. Then*

$$\lim_{n \to \infty} \int_0^{r_n} \varphi(t)dt = \int_0^a \varphi(t)dt.$$

Lemma 2.[10] *Let $\varphi \in \Phi$ and $\{r_n\}_{n \in \mathbb{N}}$ is a non-negative sequence. Then*

$$\lim_{n \to \infty} \int_0^{r_n} \varphi(t)dt = 0 \Leftrightarrow \lim_{n \to \infty} r_n = 0.$$

3 Main results

Theorem 9.*Let (X,G) be a complete G-metric space and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a Lebesgue integrable mapping. $T : X \to X$ such that $\forall \, x,y,z \in X$*

$$\psi\Big(\int_0^{G(Tx,Ty,Tz)} \varphi(t)dt\Big) \leq \psi\Big(\int_0^{M(x,y,z)} \varphi(t)dt\Big) - \phi\Big(\int_0^{M(x,y,z)} \varphi(t)dt\Big), \quad (1)$$

where $\varphi \in \Phi$ and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and non-decreasing function and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a lower semi continuous and non-decreasing function such that $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$, where

$$M(x,y,z) = \max\Big\{G(x,Tx,y), G(x,Tx,z), G(x,y,z), G(y,Ty,Ty), G(z,Tz,Tz), \frac{[G(x,Ty,Tz) + G(Tx,y,z)]}{2}\Big\}.$$

Then T has a unique fixed point.

*Proof.*Take x be arbitrary point in X define a sequence $x_n = Tx_{n-1}$ for $n = 0,1,2\ldots\ldots$ using (1) for each $n \in \mathbb{N}_0$ we get

$$\psi\Big(\int_0^{G(x_n,x_{n+1},x_{n+1})} \varphi(t)dt\Big) = \psi\Big(\int_0^{G(Tx_{n-1},Tx_n,Tx_n)} \varphi(t)dt\Big) \leq \psi\Big(\int_0^{M(x_{n-1},x_n,x_n)} \varphi(t)dt\Big) - \phi\Big(\int_0^{M(x_{n-1},x_n,x_n)} \varphi(t)dt\Big) (2)$$

which implies that,

$$\psi\Big(\int_0^{G(x_n,x_{n+1},x_{n+1})} \varphi(t)dt\Big) \leq \psi\Big(\int_0^{M(x_{n-1},x_n,x_n)} \varphi(t)dt\Big).$$

By using ψ function we have

$$\int_0^{G(x_n,x_{n+1},x_{n+1})} \varphi(t)dt \leq \int_0^{M(x_{n-1},x_n,x_n)} \varphi(t)dt. \quad (3)$$

Now, by using the rectangle property of G-metric, we have

$$\begin{aligned} M(x_{n-1},x_n,x_n) &= \max\Big\{G(x_{n-1},Tx_{n-1},x_n), G(x_{n-1},Tx_{n-1},x_n), G(x_{n-1},x_n,x_n), G(x_n,Tx_n,Tx_n), G(x_n,Tx_n,Tx_n) \\ &\quad \frac{G(x_{n-1},Tx_n,Tx_n) + G(Tx_{n-1},x_n,x_n)}{2}\Big\} \\ &= \max\Big\{G(x_n,x_{n+1},x_{n+1}), G(x_{n-1},x_n,x_n), \frac{G(x_{n-1},x_{n+1},x_{n+1})}{2}\Big\}. \end{aligned}$$

As, $\dfrac{G(x_{n-1},x_{n+1},x_{n+1})}{2} \leq \dfrac{G(x_n,x_{n+1},x_{n+1}) + G(x_{n-1},x_n,x_n)}{2}$,

So

$$M(x_{n-1},x_n,x_n) = \max\Big\{G(x_n,x_{n+1},x_{n+1}), G(x_{n-1},x_n,x_n)\Big\},$$

Assume that $G(x_n,x_{n+1},x_{n+1}) > G(x_{n-1},x_n,x_n)$ clearly $G(x_n,x_{n+1},x_{n+1}) > 0$ Therefore $\phi(\int_0^{G(x_n,x_{n+1},x_{n+1})}\varphi(t)dt) > 0$ then by (2) we have

$$\psi\Big(\int_0^{G(x_n,x_{n+1},x_{n+1})}\varphi(t)dt\Big) \le \psi\Big(\int_0^{G(x_n,x_{n+1},x_{n+1})}\varphi(t)dt\Big) - \phi\Big(\int_0^{G(x_n,x_{n+1},x_{n+1})}\varphi(t)dt\Big),$$

, hence

$$\psi\Big(\int_0^{G(x_n,x_{n+1},x_{n+1})}\varphi(t)dt\Big) < \psi\Big(\int_0^{G(x_n,x_{n+1}.x_{n+1})}\varphi(t)dt\Big),$$

it is not possible. So by using (3)

$$\psi\Big(\int_0^{G(x_n,x_{n+1},x_{n+1}))}\varphi(t)dt\Big) \le \psi\Big(\int_0^{G(x_n,x_{n-1},x_{n-1})}\varphi(t)dt\Big).$$

Since ψ is monotone non-decreasing, we get

$$\int_0^{G(x_n,x_{n+1},x_{n+1})}\varphi(t)dt \le \int_0^{G(x_n,x_{n-1},x_{n-1})}\varphi(t)dt.$$

Therefore there exist $l \ge 0$ such that

$$\lim_{n\to\infty}\int_0^{G(x_n,x_{n+1},x_{n+1})}\varphi(t)dt = l. \quad (4)$$

we claim that $l > 0$. Taking limit $n \to \infty$ in (2) using (4) we get

$$\psi(l) \le \psi(l) - \phi(l) < \psi(l).$$

Which is contradiction. Therefore $l = 0$. So we have

$$\lim_{n\to\infty}\int_0^{G(x_{n-1},x_n,x_n)}\varphi(t)dt = 0,$$

by Lemma 2

$$\lim_{n\to\infty} G(x_{n-1},x_n,x_n) = 0, \quad (5)$$

using Proposition 1 we can write

$$\lim_{n\to\infty} G(x_n,x_{n-1},x_{n-1}) = 0. \quad (6)$$

Now, we show that $\{x_n\}$ is a G-Cauchy sequence . Suppose that, $\{x_n\}$ is not a G-Cauchy sequence. Then, there exist $\varepsilon > 0$ and subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that,

$$G(x_{m(k)},x_{m(k)},x_{n(k)}) \ge \varepsilon, \forall\, k \in \mathbb{N}. \quad (7)$$

Furthermore, corresponding to $m(k)$ one can choose $n(k)$ such that, it is the smallest integer with $n(k) > m(k)$ satisfying (7) then,

$$G(x_{m(k)},x_{m(k)},x_{n(k)-1}) < \varepsilon, \forall\, k \in \mathbb{N} \quad (8)$$

Using equation (7) and rectangular property of G- metric space, we have

$$\varepsilon \le G(x_{m(k)},x_{m(k)},x_{n(k)}) = G(x_{n(k)},x_{m(k)},x_{m(k)}) \le G(x_{m(k)},x_{m(k)},x_{n(k)-1}) + G(x_{n(k)-1},x_{n(k)-1},x_{n(k)}), \quad (9)$$

$$G(u_{m(k_1)-1},u_{m(k_1)-1},u_{n(k_1)-1}) = G(u_{n(k_1)-1},u_{m(k_1)-1},u_{m(k_1)-1}) \le G(u_{m(k_1)-1},u_{m(k_1)-1},u_{m(k_1)}) + G(u_{m(k_1)},u_{m(k_1)},u_{n(k_1)}) + G(u_{n(k_1)-1},u_{n(k_1)},u_{n(k_1)}), \quad (10)$$

and

$$G(x_{m(k)-1},x_{m(k)-1},x_{n(k)-1}) = G(x_{n(k)-1},x_{m(k)-1},x_{m(k)-1}) \le G(x_{m(k)-1},x_{m(k)-1},x_{m(k)}) + G(x_{m(k)},x_{m(k)},x_{n(k)}) + G(x_{n(k)-1},x_{n(k)},x_{n(k)}), \quad (11)$$

Using limit $k \to \infty$ in (9), (10) and (11), we get

$$\lim_{k\to\infty} G(x_{m(k)},x_{m(k)},x_{n(k)}) = \lim_{k\to\infty} G(x_{m(k)-1},x_{m(k)-1},x_{n(k)-1}) = \varepsilon, \quad (12)$$

Consider

$$\psi\Big(\int_0^{\varepsilon}\varphi(t)dt\Big) \le \psi\Big(\int_0^{G(x_{m(k)},x_{m(k)},x_{n(k)}}\varphi(t)dt\Big) \le \psi\Big(\int_0^{M(x_{m(k)-1},x_{m(k)-1},x_{n(k)-1})}\varphi(t)dt\Big) - \phi\Big(\int_0^{M(x_{m(k)-1},x_{m(k)-1},x_{n(k)-1})}\varphi(t)dt\Big), \quad (13)$$

where

$$M(x_{m(k)-1},x_{m(k)-1},x_{n(k)-1}) = \max\Big\{G(x_{m(k)-1},Tx_{m(k)-1},x_{m(k)-1}) \; G(x_{m(k)-1},Tx_{m(k)-1},x_{n(k)-1}), G(x_{m(k)-1},x_{m(k)-1},x_{n(k)-1}) \; G(x_{m(k)-1},Tx_{m(k)-1},Tx_{m(k)-1}), G(x_{n(k)-1},Tx_{n(k)-1},Tx_{n(k)-1}) \; \frac{G(x_{m(k)-1},Tx_{m(k)-1},Tx_{n(k)-1}) + G(Tx_{m(k)-1},x_{m(k)-1},x_{n(k)-1}}{2}\Big\}$$

$$= \max\Big\{G(x_{m(k)-1},x_{m(k)},x_{m(k)-1}), G(x_{m(k)-1},x_{m(k)},x_{n(k)-1}), G(x_{m(k)-1},x_{m(k)-1},x_{n(k)-1}), G(x_{m(k)-1},x_{m(k)},x_{m(k)}), G(x_{n(k)-1},x_{n(k)},x_{n(k)}), \frac{G(x_{m(k)-1},x_{m(k)},x_{n(k)}) + G(x_{m(k)},x_{m(k)-1},x_{n(k)-1}}{2}\Big\}.$$

Now

$$\int_0^{M(x_{m(k)-1},x_{m(k)-1},x_{n(k)-1})}\varphi(t)dt = \int_0^{\max\{A,B,C,D,E,F\}}\varphi(t)dt,$$

where

$A = G(x_{m(k)-1},x_{m(k)},x_{m(k)-1}), B = G(x_{m(k)-1},x_{m(k)},x_{n(k)-1}),$

$$C = G(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1})$$

$$D = G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}), E = G(x_{n(k)-1}, x_{n(k)}, x_{n(k)})$$

$$F = \frac{G(x_{m(k)-1}, x_{m(k)}, x_{n(k)}) + G(x_{m(k)}, x_{m(k)-1}, x_{n(k)-1})}{2}$$

$$\int_0^{M(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1})} \varphi(t)dt = \max\Big\{\int_0^A \varphi(t)dt, \int_0^B \varphi(t)dt, \int_0^C \varphi(t)dt, \int_0^D \phi(t)dt, \int_0^E \varphi(t)dt, \int_0^F \varphi(t)dt\Big\}, \quad (14)$$

$$G(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1}) = G(x_{n(k)-1}, x_{m(k)}, x_{m(k)-1}) \leq G(x_{m(k)-1}, x_{m(k)-1}, x_{m(k)}) + G(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}), \quad (15)$$

and

$$G(x_{m(k)-1}, x_{m(k)}, x_{n(k)}) = G(x_{n(k)}.x_{m(k)}, x_{m(k)-1}) \leq G(x_{n(k)}, x_{m(k)}, x_{m(k)}) + G(x_{m(k)}, x_{m(k)}, x_{m(k)-1}), \quad (16)$$

Applying limit $k \to \infty$ in (15) and (16) using (6) and (12), we have

$$\lim_{k\to\infty} G(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1}) = \varepsilon, \quad (17)$$

$$\lim_{k\to\infty} G(x_{m(k)-1}, x_{m(k)}, x_{n(k)}) = \varepsilon. \quad (18)$$

Applying limit $k \to \infty$ in (14), using (6), (12), (17), (18), we get

$$\lim_{k\to\infty} \int_0^{M(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1})} \phi(t)dt = \int_0^{\varepsilon} \phi(t)dt \quad (19)$$

Taking limit of (13) using (19) and lower semi continuity of ϕ we have

$$\psi\Big(\int_0^{\varepsilon} \varphi(t)dt\Big) \leq \psi\Big(\int_0^{\varepsilon} \varphi(t)dt\Big) - \phi\Big(\int_0^{\varepsilon} \varphi(t)dt\Big) < \psi\Big(\int_0^{\varepsilon} \varphi(t)dt\Big)$$

which is contradiction. Therefore x_n is a G-Cauchy sequence. By a G-completeness of X, $x_n \to x$ in X we claim that x is a fixed point. Consider

$$\psi\Big(\int_0^{G(x_{n+1}, Tx, Tx)} \varphi(t)dt\Big) = \psi\Big(\int_0^{G(Tx_n, Tx, Tx)} \varphi(t)dt\Big) \leq \psi\Big(\int_0^{M(x_n, x, x)} \varphi(t)dt\Big) - \phi\Big(\int_0^{M(x_n, x, x)} \varphi(t)dt\Big), \quad (20)$$

Where

$$M(x_n, x, x) = \max\Big\{G(x_n, Tx_n, x), G(x_n, Tx_n, x), G(x_n, x, x), G(x, Tx, Tx), G(x, Tx, Tx), \frac{[G(x_n, Tx, Tx) + G(Tx_n, x, x)]}{2}\Big\}$$
$$= \max\Big\{G(x_n, x_{n+1}, x), G(x_n, x, x), G(x, Tx, Tx), \frac{[G(x_n, Tx, Tx) + G(x_{n+1}, x, x)]}{2}\Big\},$$

Applying limit $n \to \infty$, we have

$$\lim_{n\to\infty} M(x_n, x, x) = G(x, Tx, Tx). \quad (21)$$

Taking limit in (20) and using (21) one can get

$$\psi\Big(\int_0^{G(x,Tx,Tx)} \varphi(t)dt\Big) \leq \psi\Big(\int_0^{G(x,Tx,Tx)} \varphi(t)dt\Big) - \phi\Big(\int_0^{G(x,Tx,Tx)} \varphi(t)dt\Big),$$

if $G(x, Tx, Tx) > 0$ then

$$\psi\Big(\int_0^{G(x,Tx,Tx)} \varphi(t)dt\Big) \leq \psi\Big(\int_0^{G(x,Tx,Tx)} \varphi(t)dt\Big) - \phi\Big(\int_0^{G(x,Tx,Tx)} \varphi(t)dt\Big) < \psi\Big(\int_0^{G(x,Tx,Tx)} \varphi(t)dt\Big),$$

Which is contradiction. So $G(x, Tx, Tx) = 0$ implies $x = Tx$.

Uniqueness: Now we prove that x is the unique fixed point of T. Suppose it is not then there exist y such that $Ty = y$ and $x \neq y$.

$$\psi\Big(\int_0^{G(x,x,y)} \varphi(t)dt\Big) = \psi\Big(\int_0^{G(Tx,Tx,Ty)} \varphi(t)dt\Big) \leq \psi\Big(\int_0^{M(x,x,y)} \varphi(t)dt\Big) - \phi\Big(\int_0^{M(x,x,y)} \varphi(t)dt\Big), \quad (22)$$

$$M(x, x, y) = \max\Big\{G(x, Tx, x), G(x, Tx, y), G(x, x, y), G(x, Tx, Tx), G(y, Ty, Ty), \frac{G(x, Tx, Ty) + G(Tx, x, y)}{2}\Big\}$$
$$= \max\Big\{G(x, Tx, x), G(x, x, y), G(y, Ty, Ty), \frac{G(x, Tx, Ty) + G(Tx, x, y)}{2}\Big\}$$
$$= G(x, x, y).$$

Using (22) we have

$$\psi\Big(\int_0^{G(x,x,y)}\varphi(t)dt\Big) < \psi\Big(\int_0^{G(x,x,y)}\varphi(t)dt\Big).$$

We arrive at contradiction. This proves the uniqueness and hence the result.

Corollary 1. *Let (X,d) be a complete G-metric space and $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ be Lebesgue integrable mapping. $T:X\to X$ such that $\forall\, x,y,z\in X$*

$$\int_0^{G(Tx,Ty,Tz)}\varphi(t)dt \le \int_0^{M(x,y,z)}\varphi(t)dt - \phi\Big(\int_0^{M(x,y,z)}\varphi(t)dt\Big). \tag{23}$$

Where $\varphi\in\Phi$ and $\psi:\mathbb{R}^+\to\mathbb{R}^+$ is continuous and non-decreasing function and $\phi:\mathbb{R}^+\to\mathbb{R}^+$ is a lower semi continuous and non-decreasing function such that $\phi(t)=0$ if and only if $t=0$ and $\phi(t)>0$ if $t>0$ where

$$M(x,y,z)=\max\Big\{G(x,Tx,y),G(x,Tx,z), G(x,y,z),G(y,Ty,Ty),G(z,Tz,Tz), \frac{G(x,Ty,Tz)+G(Tx,y,z)}{2}\Big\}.$$

Then T has a unique fixed point.

Proof. The proof follow by taking $\psi(t)=t$ in Theorem 3.1.

Corollary 2. *Let (X,d) be a complete G-metric space and $T:X\to X$ such that $\forall\, x,y,z\in X$*

$$\psi\Big(G(Tx,Ty,Tz)\Big) \le \psi\Big(M(x,y,z)\Big) - \phi\Big(M(x,y,z)\Big),$$

where $\psi:\mathbb{R}^+\to\mathbb{R}^+$ is continuous and non-decreasing function and $\phi:\mathbb{R}^+\to\mathbb{R}^+$ is a lower semi continuous and non-decreasing function such that $\psi(t)=0=\phi(t)$ if and only if $t=0$ where

$$M(x,y,z)=\max\Big\{G(x,Tx,y),G(x,Tx,z),G(x,y,z), G(y,Ty,Ty),G(z,Tz,Tz), \frac{G(x,Ty,Tz)+G(Tx,y,z)}{2}\Big\}.$$

Then T has a unique fixed point.

Proof. The proof follow by taking $\phi(t)=1$ in Theorem 9.

Example 2. Let $X=[1,2]$ be endowed with G-metric

$$G(x,y,z)=\max\{|x-y|,|y-z|,|z-x|\}, \forall\, x,y,z\in X.$$

Assume $x\le y\le z$. Let $T:X\to X$, $\psi:\mathbb{R}^+\to\mathbb{R}^+$, $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ and $\phi:\mathbb{R}^+\to\mathbb{R}^+$ defined by,

$$T(x)=\begin{cases}1 & \forall\, x\in[1,2),\\ \frac{5}{4} & \forall\, x=2.\end{cases}$$

$$\psi(t)=t\ \ \forall\ t\in\mathbb{R}^+, \phi(t)=\frac{t}{2}\ \ \forall\ t\in\mathbb{R}^+, \quad \varphi(t)=3t^2\ \ \forall\ t\in\mathbb{R}^+.$$

Now we have,

$$\psi\Big(\int_0^{G(Tx,Ty,Tz)}\varphi(t)dt\Big)=\psi\int_0^{\frac{1}{4}}\varphi(t)dt=\psi(\frac{1}{64})=\frac{1}{64}$$

$$\psi\Big(\int_0^{M(x,y,z)}\varphi(t)dt\Big)-\phi\Big(\int_0^{M(x,y,z)}\varphi(t)dt\Big)=\psi(1)-\phi(1)=\tfrac{1}{2}$$

Implies that

$$\psi\Big(\int_0^{G(Tx,Ty,Tz)}\varphi(t)dt\Big)\le\psi\Big(\int_0^{M(x,y,z)}\varphi(t)dt\Big) - \phi\Big(\int_0^{M(x,y,z)}\varphi(t)dt\Big),$$

All conditions of of Theorem 9 are satisfied. Thus T has a unique fixed point.

4 Conclusion

The established results generalize some results of [8] and [13] in the setting of generalized metric spaces

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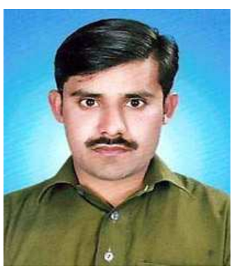

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