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On M-Projective Curvature Tensor of Generalized Sasakian-Space-Forms

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Abstract: The object of the present paper is to characterize generalized Sasakian-Space-Forms satisfying certain curvature conditions on M-projective curvature tensor. In this paper, we study M-projectively semisymmetric, ξ-M-projectively flat, and M-projectively recurrent generalized Sasakian-Space-Forms. Also generalized Sasakian-Space-Forms satisfying W*. S=0 have been studied.

Keywords: Generalized Sasakian -Space-Forms, M-projective curvature tensor, η-Eienstien Manifold, metric.

1 Introduction

In 1971, G. P. Pokhariyal and R. S. Mishra [16]efined a new curvature tensor $W^\star$ on a Riemannian manifold and studied its relativistic significance. The $W^\star$curvature tensor of type (0,4)for $(2n+1)$ Riemannian manifold is defined by

$$'W^\star(X,Y,Z,U) = {}'R(X,Y,Z,U) - \frac{1}{4n}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U) + g(Y,Z)S(X,U) - g(X,Z)S(Y,U)] \quad (1)$$

where S is the Ricci tensor of type (0,2). Such a tensor $W^\star$ is known as $M-$ projective curvature tensor. A manifold whose $M-$projective curvature tensor vanishes at each point of the manifold is known as$M-$projectively flat manifold. Thus this tensor represents the deviation of the manifold from $M-$ projective flatness.Second author [15, 14] defined and studied the properties of $M-$ projective curvature tensor in Sasakian and Kaehler manifolds. He has also shown that it bridges the gap between conformal curvature tensor, con-harmonic curvature tensor and con-circular curvature tensor on one side and $H-$ projective curvature tensor on the other.

Let Mbe an almost contact metric manifold equipped with an almost contact metric structure(ϕ,ξ,η,g) . At each point $p \in M$, decompose the tangent space T_pM into the direct sum $T_pM = \phi(T_pM) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the $1-$ dimensional linear subspace of $T_pM$generated byξ_p. Thus the conformal curvature tensor C is a map

$$C : T_pM \times T_pM \times T_pM \to \phi(T_pM) \oplus \{\xi_p\}, p \in M$$

An almost contact metric manifold M is said to be
(1) conformally symmetric [20] if the projection of the image of Cin $\phi(T_pM)$ is zero,
(2) ξ conformally flat [6] if the projection of the image of C in$\{\xi_p\}$ is zero,
(3)ϕ-conformally flat [7] if the projection of the image of $C|T_pM \times T_pM \times T_pM$in $\phi(T_pM)$ is zero.

Here cases $(1),(2)$, and (3) are synonymous to conformally symmetric,ξ-conformally flat and ϕ-conformally flat. In [20] , it is proved that a conformally symmetric K-contact manifold is locally isometric to the unit sphere. In [6], it is proved that a K-contact manifold is ξ-conformally flat if and only if it is an η -Einstein Sasakian manifold. In [7] some necessary conditions for K-contact manifold to be ϕ-conformally flat are proved. Moreover, in [2] some conditions on conharmonic curvature tensor are studied which has many applications in physics and mathematics on a hypersurface in Semi-Euclidean space E_s^{n+1}. On the other hand a generalized Sasakian-space-form was defined by Alegre et al.[11]. As the almost contact metric manifold $(M^{2n+1},\phi,\xi,\eta,g)$ whose curvature tensor R is given by

$$R = f_1R_1 + f_2R_2 + f_3R_3, \quad (2)$$

Where f_1, f_2, f_3 are some differential functions on M and

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$$R_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y,$$

$$R_2(X,Y)Z = g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z$$

$$R_3(X,Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X - g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \quad (3)$$

for any vector fieldsX,Y,Zon M^{2n+1}. In such a case we denote the manifold as $M(f_1,f_2,f_3)$. This kind of manifold appears as a generalization of the well-known Sasakian-space-forms by taking $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$. It is known that any three-dimensional (α,β)-trans -Sasakian manifold with α,βpending on ξ is a generalized Sasakian-space-form [10]. Alegre et al. has given the results in [9]about B. Y Chen's inequality on submanifolds of generalized complex space-forms and generalized Sasakian- space-forms. Al-Ghefari et al. analyze the CR submanifolds of generalized Sasakian-space-forms [12],[13]. The Structure of a Class of Generalized Sasakian-Space-Forms has studied by De and Majhi [18].Also Some Results on Generalized Sasakian-Space-Forms have studied by U.C. De and A. Sarkar [19]. In [20], Kim studied conformally flat generalized Sasakian- space-forms and locally symmetric generalized Sasakian-space-forms. De and Sarkar [17] have studied generalized Sasakian-space-forms regarding projective curvature tensor. Motivated by the above studies, in the present paper,we study flatness and symmetry property of generalized Sasakian- space-forms regarding $M-$projective curvature tensor. The present paper is organized as follows:

In this paper, we study the $M-$projective curvature tensor of generalized Sasakian-space-forms. In Section 2, some preliminary results are recalled. In Section 3, we study $M-$projective semisymmetric generalized Sasakian-space-forms. ξ-$M-$projectively flat generalized Sasakian-space-forms are studied in Section -4 and necessary and sufficient condition for a generalized Sasakian-space-form to be ξ- $M-$projectively flat is obtained. In Section 5, $M-$projective recurrent generalized Sasakian-space-forms are studied. Section 6 is devoted to the study of generalized Sasakian-space-forms satisfying $W^\star.S = 0$.

2 Preliminaries

If, on an odd-dimensional differentiable manifold M^{2n+1} of differentiability class C^{r+1}, there exists a vector valued real linear function ϕ, a 1-form η, the associated vector field ξ, and the Riemannian metric g satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi(\xi) = 0, \quad (4)$$

$$\eta(\xi) = 1, \quad g(X,\xi) = \eta(X), \quad \eta(\phi X) = 0, \quad (5)$$

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad (6)$$

for arbitrary vector fields X and Y, then (M^{2n+1},g) is said to be an almost contact metric manifold [4], and the structure (ϕ,ξ,η,g) is called an almost contact metric structure to M^{2n+1}. In view of (2.1),(2.2) and (2.3), we have

$$g(\phi X,Y) = -g(X,\phi Y), \quad g(\phi X,X) = 0,$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y). \quad (7)$$

Again we know[10] that in a $(2n+1)$-dimensional generalized Sasakian- space-form

$$\begin{aligned} R(X,Y)Z = & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \end{aligned} \quad (8)$$

for all vector fields X,Y,Zon M^{2n+1}, where Rdenotes curvature tensor of M^{2n+1}

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \quad (9)$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \quad (10)$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3. \quad (11)$$

We also have for a generalized Sasakian-space-forms

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \quad (12)$$

$$R(\xi,X)Y = -R(X,\xi)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X], \quad (13)$$

$$\eta(R(X,Y)Z) = (f_1 - f_3)[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)], \quad (14)$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X), \quad (15)$$

$$Q\xi = 2n(f_1 - f_3)\xi \quad (16)$$

where Q is the Ricci operator, that is,

$$g(QX,Y) = S(X,Y)$$

A generalized Sasakian-space-form is said to be $\eta-$Einstein if its Ricci tensor S is of the form:

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \quad (17)$$

for arbitrary vector fields X and Y, where a and b are smooth functions on M^{2n+1}. For a $2n+1$ -dimensional $(n>1)$ almost contact metric manifold the $M-$projective curvature tensor W^* is given by [5]

$$W^*(X,Y)Z = R(X,Y)Z - \frac{1}{4n}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \quad (18)$$

The M-projective curvature tensorW^*in a generalized Sasakian-space-form satisfies

$$W^*(X,Y)\xi = \frac{2n-1}{2n}(f_1-f_3)[\eta(Y)X-\eta(X)Y] -\frac{1}{4n}[\eta(Y)QX-\eta(X)QY] \quad (19)$$

$$\eta(W^*(X,Y)\xi)=0 \quad (20)$$

$$W^*(\xi,Y)Z = \frac{2n-1}{2n}[g(Y,Z)\xi-\eta(Z)Y] -\frac{1}{4n}[S(Y,Z)\xi-\eta(Z)QY] \quad (21)$$

$$\eta(W^*(\xi,Y)Z) = (\frac{2n-1}{2n})(f_1-f_3)[g(Y,Z)-\eta(Z)\eta(Y)] -\frac{1}{4n}[S(Y,Z)-2n(f_1-f_3)\eta(Y)\eta(Z)] \quad (22)$$

$$\eta(W^*(X,Y)Z) = (\frac{2n-1}{2n})(f_1-f_3)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)] -\frac{1}{4n}[S(Y,Z)\eta(X)-S(X,Z)\eta(Y)] \quad (23)$$

3 M-projectively Semisymmetric Generalized Sasakian-Space-Forms

Definition 1. *A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form is said to be $M-$ projectively semisymmetric [20] if it satisfies $R.W^*$=0, where R is the Riemannian curvature tensor,W^* is the $M-$projective curvature tensor of the space-forms.*

Theorem 1. *A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form is $M-$projectively semisymmetric if and only if $f_1=f_3$*

Proof. Let us suppose that the generalized Sasakian-space-form $M(f_1,f_2,f_3)$ is $M-$projectively semisymmetric. Then we have

$$R(\xi,U).W^*(X,Y)\xi=0 \quad (24)$$

which can be written as

$$R(\xi,U)W^*(X,Y)\xi - W^*(R(\xi,U)X,Y)\xi -W^*(X,R(\xi,U)Y)\xi - W^*(X,Y)R(\xi,U)\xi=0 \quad (25)$$

Using (2.10) the above equation reduces to

$$(f_1-f_3)[g(U,W^*(X,Y)\xi)\xi-\eta(W^*(X,Y)\xi)U -g(X,U)W^*(\xi,Y)\xi)+\eta(X)W^*(U,Y)\xi -g(U,Y)W^*(X,\xi)\xi)+\eta(Y)W^*(X,U)\xi -\eta(U)W^*(X,Y)\xi+W^*(X,Y)U]=0 \quad (26)$$

Now taking the inner product of above equation with ξ and using (2.2) & (2.17), we get

$$(f_1-f_3)[g(U,W^*(X,Y)\xi)+\eta(W^*(X,Y)U)] \quad (27)$$

From the above equation, we have either $f_1=f_3$ or

$$g(U,W^*(X,Y)\xi)+\eta(W^*(X,Y)U)=0 \quad (28)$$

which by using (2.15) & (2.16) gives

$$g(Y,U)\eta(X)-g(X,U)\eta(Y)=0 \quad (29)$$

which is not possible in generalized Sasakian-space-form then from (2.10), we have $R(\xi,U)=0$. Then obviously $R.W^*=0$ is satisfied & proof is completed.

4 $\xi-M-$ projectively Flat Generalized Sasakian-Space-Forms

Definition 2. *A $(2n+1)$ -dimensional $(n>1)$ generalized Sasakian-space-form is said to be $\xi-M-$ projectively flat[6] if $W^*(X,Y)\xi=0$ for all $X,Y\in TM$*

Theorem 2. *A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space -form is $\xi-M-$ projectively flat if and only if it is Einstein manifold.*

Proof. Let us consider that a generalized Sasakian-space-form is $\xi-M-$projectively flat, that is

$$W^*(X,Y)\xi=0$$

Then from (2.16),we have

$$R(X,Y)\xi = \frac{1}{4n}[\eta(Y)QX-\eta(X)QY+S(Y,\xi)X-S(X,\xi)Y \quad (30)$$

By (2.7) & (2.12) becomes

$$2n(f_1-f_3)[\eta(Y)X-\eta(X)Y]=[\eta(Y)QX-\eta(X)QY] \quad (31)$$

Put $Y=\xi$ in (4.2), we get

$$QX=2n(f_1-f_3)X \quad (32)$$

Now taking the inner product of the above equation with U, we get

$$S(X,U)=2n(f_1-f_3)g(X,U) \quad (33)$$

which implies that generalized Sasakian-space-form is Einstein manifold. Conversely, suppose that (4.4) is satisfied. Then from (4.1) and (4.3), we get

$$W^*(X,Y)\xi=0$$

Hence the proof is completed.

5 M-projectively Recurrent Generalized Sasakian-Space-Forms

Definition 3. *A nonflat Riemannian manifold $M^{2n+1}$is said to be $M-$projectively recurrent if its $M-$projective curvature tensor W^*satisfies the condition*

$$\triangledown W^* = A\otimes W^*, \quad (34)$$

where A is nonzero $1-$form[5]

Theorem 3. *A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form is $M-$projectively recurrent if and only if $f_1 = f_3$.*

Proof. Let us define a function $f^2 = g(W^*, W^*)$ on M^{2n+1},where the metric g is extended to the inner product between the tensor fields. Then we have

$$f(Yf) = f^2 A(Y) \tag{35}$$

this implies

$$Yf = f(AY)(f \neq 0) \tag{36}$$

From (5.3), we have

$$X(Yf) - Y(Xf) = \{XA(Y) - YA(X) - A([X,Y])f \tag{37}$$

since Left hand side of (5.4) is identically zero and $f \neq 0$ on M^{2n+1},then

$$dA(X,Y) = 0 \tag{38}$$

that is, $1-$form is closed. Now from

$$(\bigtriangledown_X W^*)(Y,Z)U = A(X)W^*(Y,Z)U \tag{39}$$

we get,

$$(\bigtriangledown_V \bigtriangledown_X W^*)(Y,Z)U = \{VA(X) + A(V)A(X)W^*(Y,Z)U\} \tag{40}$$

From (5.5) & (5.7), we have

$$(R(V,X).W^*)(Y,Z)U = [2dA(V,X)]W^*(Y,Z)U = 0 \tag{41}$$

Thus by theorem 1, we obtain $f_1 = f_3$ and converse follows from retreating the steps.

Corollary 1. *$M-$projectively recurrent generalized Sasakian-space-form is $M-$projectively semi-symmetric.*

Proof. Follows from above theorem & theorem 1.

6 Generalized Sasakian-Space-Forms satisfying $W^*.S = 0$

Theorem 4. *A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form satisfying $W^*.S = 0$ is an $\eta-$Einstein manifold.*

Proof. Let us consider generalized Sasakian-space-form M^{2n+1} satisfying

$$W^*(\xi, X).S = 0$$

Thus we can write

$$S(W^*(\xi,X)Y,Z) + S(Y, W^*(\xi,X)Z) = 0 \tag{42}$$

By (2.18) above equation reduces to

$$\begin{aligned}
&2(2n-1)(f_1-f_3).2n(f_1-f_3)[g(X,Y)\\
&-g(X,Z)\eta(Y) - \eta(Y)S(X,Z) - \eta(Z)S(X,Y)]\\
&-[2n(f_1-f_3)\{S(X,Y)\eta(Z) + S(X,Z)\eta(Y)\}\\
&-\eta(Y)S(QX,Z) + \eta(Z)S(QX,Y)] = 0
\end{aligned} \tag{43}$$

Putting $Z = \xi$ in (6.2), we get

$$\begin{aligned}
S(QX,Y) = {} & 2n(f_1-f_3)[\{2(2n-1)(f_1-f_3)-1\}S(X,Y)\\
&+2(2n-1)(f_1-f_3)g(X,Y)]\\
&-[(2n(f_1-f_3)+1).2(2n+1)(f_1-f_3)\eta(X)\eta(Y)]
\end{aligned} \tag{44}$$

In view of (2.7) the above equation reduces to

$$\begin{aligned}
S(X,Y) = {} & \frac{2(f_1-f_3)}{k}[2.2n(2n-1)(f_1-f_3)g(X,Y)\\
&+n(3f_2 + (2n-1)f_3)\\
&-(2n(f_1-f_3)+1).(2n-1)\eta(X)\eta(Y)]
\end{aligned} \tag{45}$$

where,

$$k = [(2nf_1 + 3f_2 - f_3) - 2n(f_1-f_3)\{2(2n-1)(f_1-f_3)-1\}]$$

Thus the proof is completed.

7 Conclusion

In this paper we have characterized generalized Sasakian-space-forms satisfying certain curvature conditions on M-projective curvature tensor.

We have found that

- A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form is M-projectively semisymmetric if and only if $f_1 = f_3$.
- If a $(2n+1)-$ dimensional $(n>1)$ generalized Sasakian-space-form is $M-$projectively flat then $3f_2 = (1-2n)f_3$ & $f_1 = 0$.
- If $f_1 = \frac{3f_2}{1-2n} = f_3$ then a $(2n+1)-$dimensional $(n>1)$ generalized Sasakian-space-form is $M-$projectively flat.
- A $(2n+1)-$dimensional $(n>1)$ generalized Sasakian-space -form is $\xi - M-$ projectively flat if and only if it is Einstein manifold.
- A $(2n+1)-$dimensional $(n>1)$ generalized Sasakian-space-form is $M-$ projectively recurrent if and only if $f_1 = f_3$.
- $M-$projectively recurrent generalized Sasakian-space-form is $M-$projectively semi-symmetric.
- A $(2n+1)-$dimensional $(n>1)$ generalized Sasakian-space-form satisfying W*.S = 0 is an $\eta-$Einstein manifold.

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