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Some New Classes of Quasi Split Feasibility Problems

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Abstract: In thispaper, we introduce and consider a new problem of finding $u \in K(u)$ such that $Au \in C$, where $K : u \to K(u)$ is a closed convex-valued set in the real Hilbert space H_1, C is closed convex set in the real Hilbert space H_2 respectively and A is linear bounded self-adjoint operator from H_1 and H_2. This problem is called the quasi split feasibility problem. We show that the quasi feasibility problem is equivalent to the fixed point problem and quasi variational inequality. These s alternative equivalent formulations are used to consider the existence of a solution of the quasi split feasibility problem. Some special cases are also considered. Problems considered in this paper may open further research opportunities in these fields.

Keywords: Quasi split feasibility problem, existence of a solution.

1 Introduction

The split feasibility problems, introduced and studied by Censor and Elfying [4], have played a fundamental and significant part in the study of several unrelated problems. These problems arise in diverse fields of pure and applied sciences including image reconstruction, medica sciences (medical image), signal processing, image denoising and decomposition, see [1,2,3,4,5,6,13,26]. It has been shown [1,2,3,4,5,6,13,21,26,28,29,30] that the split feasibility problems are equivalent to fixed point problems, variational inequalities and optimization problems. These equivalent alternative formulations of the split feasibility problems have been used to study the existence of a solution as well as to develop various numerical methods. In the formulation of split feasibility problem, the underlying convex sets do not depend on the solution. This fact has motivated us to consider a class of split feasibility problem, which is called quasi split feasibility problem. We would like to emphasized that such type of quasi split feasibility problems have not been investigated up to now. It has been shown that the quasi split feasibility problems are equivalent to the fixed point problems and quasi variational inequalities. These equivalent formulations are used to study the existence of a solution of the quasi split feasibility problem. This result is new and original. Several special cases are also discussed. Our results continue to hold for these cases. Some iterative methods for finding the approximate solutions of the quasi split feasibility problems are suggested. It is expected that the ideas and techniques of this paper may stimulate further research in this area. The interested readers may find new and novel applications of quasi split feasibility problems in image reconstruction, medical imaging and related fields.

2 Preliminaries

Let H be real Hilbert space, whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. Let K be a non-empty, closed and convex set in H.
We now recall some basic concepts and results, which are needed.

Definition 1. *An operator T is said to be strongly monotone, if there exists a constant $\alpha > 0$ such that*

$$\langle Tu - Tv, u - v \rangle \geq \alpha \|u - v\|^2, \quad \forall u, v \in H.$$

Definition 2. *An operator T is said to expanding, if and only if*

$$\|Tu - Tv\| \geq \|u - v\|, \quad \forall u, v \in H.$$

From Definition 2.1 and Definition 2.2, it follows that every strongly monotone operator is expanding, but the converse is not true.

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Definition 3. *An operator T is said to be Lipschitz continuous, if nad only if, there exists a constant $\beta > 0$ such that*

$$\|Tu - Tv\| \leq \beta \|u - v\|, \quad \forall u, v \in H.$$

Lemma 1. *For a given $z \in H$, $u \in K$, a closed and convex set, satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in K,$$

if and only if

$$u = P_K z,$$

where P_K is the projection of H onto the closed and convex set K.

From Lemma 1, we have

$$\langle P_K z - z, v - P_K z \rangle \geq 0, \quad \forall v \in K.$$

It is well known that the projection operator P_K is nonexpansive and monotone, that is,

$$\|P_K u - P_K v\| \leq \|u - v\|, \quad \forall u, v \in H$$

and

$$\langle P_K u - P_K v, u - v \rangle \geq 0, \quad \forall u, v \in H.$$

Let $K : u \to K(u)$ be a point-to-set mapping which associates closed and convex set K in a real Hilbert space H_1 of an element of $u \in H_1$ and let C be a closed and convex set in the real Hilbert space H_2. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $P_{K(u)}$ be a projection of H_1 onto the closed and convex-valued set $K(u)$ and P_C be projection of H_2 onto the closed and convex set C respectively.
We consider the problem of finding

$$u \in K(u) \text{ such that } Au \in C. \tag{1}$$

The problem (1) is called the quasi or implicit split feasibility problem. Such type of problem are connected with set-valued optimization problem, which arise in signal and medical image reconstruction. We note that, if $K(u) = K$, a closed convex set in H_1, then problem (1) reduces to finding

$$u \in K \text{ such that } Au \in C, \tag{2}$$

which is known as the split feasibility problem. These problems have been studied extensively in recent years and have application in various areas of engineering, physical and mathematical sciences such as medical image reconstruction and remote sensing. For the applications, numerical methods and other aspects of the split feasibility problem (1), see [1,2,3,4,5,6,21,26,27,28,29,30] and the references therein.
It is well known that the split feasibility problem is equivalent to the fixed point problem of finding $u \in K$ such that

$$u = P_K[u - \rho A^*(1 - P_C)Au], \tag{3}$$

where P_K is the projection of H_1 onto the closed and convex set K and P_C is the projection of H_2 onto the closed and convex set C respectively. Here A^* is the adjoint of operator A. The equivalence between problems (2) and (3) have been used to develop several numerical methods for solving the split feasibility problems (2). See, for example, [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30].
Using Lemma 1, one can show that the fixed point problem (3) is equivalent to finding such that

$$\langle A^*(I - P_C)Au, v - u \rangle \geq 0, \quad \forall c \in K, \tag{4}$$

which is called the variational inequalities. It is worth mention that the variational inequalities were introduced and studied by Stampacchia [27] in 1964. For the recent applications, formulation, numerical results and generalizations, see [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30] and the references therein.

3 Main Results

In this section, we first show that the quasi split feasibility problem (1) can be viewed as a minimization problem. We also establish the equivalence between the quasi split feasibility problem (1) and the fixed point problems. These equivalent formulation are used to study the existence of a solution of problem (1).
From (1), it is clear that $u \in K(u)$ implies that there exists $w \in C$ such that

$$Au - w = Au - P_C Au = 0, \tag{5}$$

where P_C is the projection of H_2 onto the closed and convex set C. This formulation enables us to consider the minimization problem as:

$$\min_{u \in K(u)} I[v] = \frac{1}{2}\|Av - P_C Av\|^2. \tag{6}$$

Here the functional $I[v]$ is differentiable and,

$$\nabla I(u) = A^*(I - P_C)Au. \tag{7}$$

It is well known that ∇I is Lipschitz continuous with constant $\beta = \|A\|^2$, [28] that is

$$\begin{aligned} \|\nabla I(u) - \nabla I(v)\| &\leq \|A\|^2 \|u - v\| \\ &= \beta \|u - v\|, \quad \forall u, v \in H_1. \end{aligned} \tag{8}$$

One can easily show that the minimum $u \in K(u)$ of the functional $I[v]$, defined by (6), is equivalent to finding $u \in K(u)$ such that

$$\langle A^*(I - P_C)Au, v - u \rangle \geq 0, \quad \forall v \in K(u), \tag{9}$$

which is called the quasi variational inequality. For the formulation and numerical results of the quasi variational

inequalities, see [7,8,9,10,12,13,14,16,19,22,25].
We now show that the quasi split feasibility problem (2) is equivalent to fixed point problem.
For a constant $\rho > 0$, one can rewrite (5) in the following equivalent form as:

$$\rho A^*(I-P_C)Au = 0. \tag{10}$$

Since $u \in K(u)$, from (9), it follows that

$$u = P_{K(u)}[u-\rho A^*(I-P_C)Au], \tag{11}$$

which is fixed point formulation of the quasi split feasibility problem (1).
Using Lemma 2.1, one can show that the problem (10) and (11) are equivalent. For the sake of completeness, we state it as.

Lemma 2. *The element $u \in K(u) : Au \in C$ is a solution of (8), if and only if $u \in K(u) : Au \in C$ is a solution of fixed point problem (11).*

Lemma 2 implies that problems (1), (11) and (9) are equivalent. These equivalent formulations of the quasi split feasibility problems (1) play an important and fundamental part in the study of the existence of a solution of (1) and in the development of the iterative methods. In this paper, we use the fixed point formulation to investigate the existence of a solution of the quasi split feasibility problem. This is one of the main motivation of this paper.
Using (10), we define the mapping $F(u)$ as

$$F(u) = P_{K(u)}[u-\rho A^*(I-P_C)Au]. \tag{12}$$

To prove the existence of a solution of (1), it is enough to show that the mapping $F(u)$ defined by (12) has a fixed point satisfying (1). In passing, we remark that the projection operator $P_{K(u)}$ is not nonexpensive operator. However, it is well known that the projection operator $P_{K(u)}$ satisfies the Lipschitz type continuity condition.
In this direction, we have the following assumption.

Assumption 1. The projection operator $P_{K(u)}$ satisfies the condition

$$\|P_{K(u)}w - P_{K(v)}w\|| \leq \mu\|u-v\||, \quad \forall u,v,w \in H,$$

where $\mu > 0$ is a constant.
This assumption has been used in the study of the existence of a solution of the quasi variational inequalities. For the application of Assumption 1, see [13,14,16,19,25].

Lemma 3. *For any operator A, we have*

$$\langle A^*(I-P_C)Au - A^*(I-P_C)Av, u-v\rangle \geq 0, \quad \forall u,v \in H_1.$$

Proof. $\forall u \neq v \in H_1$, consider

$$\begin{aligned}
&\langle A^*(I-P_C)Au - A^*(I-P_C)Av, u-v\rangle \\
&= \langle Au - P_CAu - (A^*v - P_CAv), Au - Av\rangle \\
&= \|Au-Av\|^2 - \langle P_CAu - P_CAv, Au-Av\rangle \\
&\geq \|Au-Av\|^2 - \|Au-Av\|^2 \\
&\geq 0.
\end{aligned}$$

□

We now prove the existence of solution of the quasi split feasibility problem (1) under some suitable conditions and this is the main motivation of our next result.

Theorem 1. *Let A be the bounded linear operator and let Assumption 1 hold. If there exists a constant $\rho > 0$ such that*

$$\left\|\rho - \frac{1}{\beta^2}\right\| < \frac{\sqrt{\mu(2-\mu)}}{\beta^2}, \mu < 1, \tag{13}$$

then there exists a solution of the quasi split feasibility problem (1).

Proof. Let $u \in K(u) : Au \in C$. Then, from Lemma 2, it follows that the problem (1) is equivalent to fixed point problem (12). In order to prove the existence of a solution of (1), it is enough to show that the mapping $F(u)$ defined by (12) has a fixed point satisfying (1).
Consider

$$\begin{aligned}
&\|F(u_2) - F(u_2)\| \\
&= \|P_{K(u_1)}[u_1 - \rho A^*(I-P_C)Au_1] \\
&- P_{K(u_2)}[u_2 - \rho A^*(I-P_C)Au_2]\| \\
&\leq \|P_{K(u_1)}[u_1 - \rho A^*(I-P_C)Au_1] \\
&- P_{K(u_2)}[u_1 - \rho A^*(I-P_C)Au_1]\| \\
&+ \|P_{K(u_2)}[u_1 - \rho A^*(I-P_C)Au_1] \\
&- P_{K(u_2)}[u_2 - \rho A^*(I-P_C)Au_2]\| \\
&\leq \mu\|u_1-u_2\| + \|u_1-u_2-\rho(A^*(I-P_C)Au_1) \\
&- A^*(I-P_C)Au_2\|,
\end{aligned} \tag{14}$$

where we have used Assumption 1.

Now, from Lemma 3 and (8), we have

$$\begin{aligned}
&\|u_1-u_2-\rho(A^*(I-P_C))Au_1 - A^*(I-P_C)Au_2\|^2 \\
&\leq \|u_1-u_2\|^2 \\
&- 2\rho\langle A^*(I-P_C)Au_1 \\
&- A^*(I-P_C)Au_2, u_1-u_2\rangle \\
&+ \rho^2\|A^*(I-P_C)Au_1 - A^*(I-P_C)Au_2\|^2 \\
&\leq \|u_1-u_2\|^2 + \rho^2\beta^2\|u_1-u_2\|^2 \\
&= (1+\rho^2\beta^2)\|u_1-u_2\|^2.
\end{aligned} \tag{15}$$

From (14) and (15), we obtain

$$\begin{aligned}
&\|F(u_1) - F(u_2)\| \\
&\leq (u + \sqrt{1+\rho^2\beta^2})\|u_1-u_2\| \\
&= \theta\|u_1-u_2\|,
\end{aligned}$$

where

$$\theta = \mu + \sqrt{1+\rho^2\beta^2}. \tag{16}$$

From (13), it follows that $\theta < 1$. This implies that the mapping $F(u)$ defined by (12) is a contraction and consequently, it has unique fixed point $F(u) = u \in K(u)$ satisfying the problem (1), the required result. □

Remark. In many applications, the convex-valued set $K(u)$ has the following form:

$$K(u) = m(u) + K, \tag{17}$$

where m is a point-to-point and K is a closed convex set. Consequently, quasi split feasibility problem (1) becomes: Find

$$u \in m(u) + K : Au \in C, \tag{18}$$

which is called the implicit split feasibility problem.
One can easily show that problem (18) is equivalent to the fixed point problem of type:

$$u = m(u) + P_K[u - m(u) - \rho A^*(I - P_C)Au],$$

which is equivalent to finding $u \in m(u) + K$ such that

$$\langle A^*(I - P_C)Au, v - u\rangle \geq 0, \quad \forall v \in m(u) + K. \tag{19}$$

Problem (19) is called the implicit variational inequality. Consequently, one can associate the mapping $F_1(u)$ with problem (18) as

$$F_1(u) = m(u) + P_K[u - m(u) - \rho A^*(I - P_C)Au],$$

Using the technique of Theorem 1, one can prove the existence of a solution of the implicit split feasibility problems (18).

We now use the fixed point formulation (11) to suggest the following iterative methods for solving (1).

Algorithm 1. For a given u_0, find the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = P_{K(u_n)}[u_n - \rho A^*(I - P_C)Au_n], \quad n = 0, 1, 2, \ldots$$

Algorithm 1 is called the explicit iterative method.

Algorithm 2. For a given u_0, find the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = P_{K(u_n)}[u_n - \rho A^*(I - P_C)Au_{n+1}], \quad n = 0, 1, 2, \ldots$$

Algorithm 2 is an implicit iterative method.
To implement this method, we use the predictor-corrector technique. We use Algorithm 1 as the predictor and Algorithm 2 as the corrector. Thus, we have the following two-step iterative method.

Algorithm 3. For a given u_0, find the approximate solution u_{n+1} by the iterative scheme

$$y_n = P_{K(u_n)}[u_n - \rho A^*(I - P_C)Au_n],$$
$$u_{n+1} = P_{K(u_n)}[u_n - \rho A^*(I - P_C)Ay_n], \quad n = 0, 1, 2, \ldots$$

Algorithm 3 is an extragradient type method. Such type of methods are due to Kopelevich [11].

4 Conclusions

In this paper, we have introduced and studied a new class of split feasibility problems, which is called the quasi split feasibility problem. It is shown that quasi split feasibility problems are equivalent to fixed point problems, quasi variational inequality and optimization problems. We have studied the existence of a solution of the quasi split feasibility problem under some suitable conditions. Some special cases are also discussed. Results and ideas of this paper will be a starting point for the further research.

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