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Mathematical Analysis and Numerical Simulation of Chaotic Noninteger Order Differential Systems with Riemann-Liouville Derivative

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Abstract: The present paper investigates stability analysis and numerical treatment of chaotic fractional differential system in Riemann-Liouville sense. Simulation results show that chaotic phenomena only occur if the reaction or local dynamics of such system are coupled or nonlinear in nature. Illustrative examples that currently interest economists, engineers, mathematicians and physicists are employed, to describe the points and queries that may arise.

Keywords: Chaotic phenomena, Fractional derivative, Riemann-Liouville operator, Numerical experiments.

1 Introduction

The study of fractional calculus, which has more than 300 years of history, is not new in the pieces of literature [1–3]. Fractional calculus is mostly considered as the generalization of integer-order derivatives and integrals to equivalent fractional order cases. Because of its different appearance in the fields of science and engineering with various applications in biological systems, feedback amplifiers, finance, fluid mechanics, groundwater process, generalized voltage dividers, capacitor theory, electrode-electrolyte interface models, fractional order predator-prey systems, fractional neurons models, fitting experimental data, medical, geo-hydrology, motion control, analysis of some special functions, and chaotic phenomena [2, 4–11], many authors have been actively involved in its development.

Recent developments in numerical integration solvers and computer simulation methods have enabled researchers to model various chaotic equations with desired properties. Several research papers based on the study of chaos and chaotic systems have been widely reported in scientific community since the inception of the so-called Lorenz attractor generated by atmospheric convection was discovered [12]. Chaotic models have been derived in the form of maps, as well as ordinary and partial differential equations. Recently, they have representation in the form of fractional order differential equations with presence of chaos. Many chaotic systems with different types of chaotic attractors have been formulated. For example, the Lorenz-like systems, Sprott systems, no-equilibria chaotic systems, Chua system, Rossler, system, Lu system Jerk systems, Chen system, and fractional-order chaotic systems [13–19].

Some systems that are modelled with fractional concept do not have exact analytical solutions and the analytical solution is too involved to be useful. As a result, we seek an approximate and numerical techniques [17, 20]. Various numerical methods developed to solve fractional differential equations have been reported, see [21]. The present paper is outlined as follows: In Section Two, some useful preliminaries in terms of definition and properties of fractional calculus in Riemann-Liouville sense are presented. Section Three addresses the stability analysis and numerical approximation techniques. Some examples of chaotic systems in literature are illustrated in Section Four, where the classical time derivative is replaced with the Riemann-Liouville fractional order derivative. Conclusion is presented in the last section.

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2 Preliminaries

Some useful preliminaries in terms of definitions and properties of the theory of fractional calculus [1–3, 17, 22] are presented in this section.

The left- and right-hand Riemann-Liouville fractional derivatives of order $\gamma > 0$ for function $y(t) \in C^1([a,b],\mathbb{R}^n)$ are defined as

$$\begin{aligned} {}^{RL}\mathscr{D}^{\gamma}_{a,t}y(t) &= \frac{d^n}{dt^n}\left[\mathscr{D}^{\gamma-n}_{a,t}y(t)\right] \\ &= \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_a^t (t-\xi)^{n-\gamma-1}y(\xi)d\xi \end{aligned}$$

and

$$\begin{aligned} {}^{RL}\mathscr{D}^{\gamma}_{t,b}y(t) &= (-1)^n\frac{d^n}{dt^n}\left[\mathscr{D}^{\gamma-n}_{t,b}y(t)\right] \\ &= \frac{(-1)^n}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_t^b (\xi-t)^{n-\gamma-1}y(\xi)d\xi, \end{aligned}$$

respectively, where n is an integer which satisfies $n-1 \leq \gamma < n$.

The Riemann-Liouville fractional derivative of order γ for a function $y(t)$ is defined as

$$ {}^{RL}\mathscr{D}^{\gamma}_{a,t}y(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_a^t (t-\xi)^{n-\gamma-1}y(\xi)d\xi, $$

where $n-1 \leq \gamma < n$, and $\Gamma(\cdot)$ denotes the usual Gamma function.

The Riemann-Liouville fractional integral of order γ for a function $y(t)$ is defined by

$$ {}^{RL}\mathscr{I}^{\gamma}_{0}y(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-\xi)^{\gamma-1}y(\xi)d\xi,\ \ 0<\gamma<\infty, $$

where $\Gamma(\cdot)$ denotes the Euler's gamma function.

The left and right Caputo derivatives with fractional order $\gamma > 0$ of function $y(t)$, for $t \in (a,b)$ are defined as

$$\begin{aligned} {}^{C}\mathscr{D}^{\gamma}_{a,t}y(t) &= \mathscr{D}^{\gamma-n}_{a,t}\left[u^{(n)}(t)\right] \\ &= \frac{1}{\Gamma(n-\gamma)}\int_a^t (t-\xi)^{n-\gamma-1}y^{(n)}(\xi)d\xi, \end{aligned}$$

and

$$\begin{aligned} {}^{C}\mathscr{D}^{\gamma}_{t,b}y(t) &= (-1)^n\mathscr{D}^{\gamma-n}_{t,b}\left[y^{(n)}(t)\right] \\ &= \frac{(-1)^n}{\Gamma(n-\gamma)}\int_t^b (\xi-t)^{n-\gamma-1}y^{(n)}(\xi)d\xi \end{aligned}$$

respectively, where n is an integer which satisfies $n-1 < \gamma < n$.

The Caputo fractional order derivative of a function $y(t) \in C^1([0,b],\mathbb{R}^n)$; $b > 0$ with $\gamma \in (0,1]$ is defined as

$$ {}^{C}\mathscr{D}^{\gamma}_{0,t}y(t) = \frac{1}{\Gamma(n-\gamma)}\int_0^t (t-\xi)^{n-\gamma-1}y^{n}(\xi)d\xi, $$

for all $t \in [0,b]$.

The Grunwald-Letnikov fractional derivative of order $\gamma > 0$ of a function $u(t)$ is defined as [17]

$$ {}^{GL}\mathscr{D}^{\gamma}_{0,t}y(t) = \lim_{h\to 0}\frac{1}{h^{\gamma}}\sum_{k=0}^{\left[\frac{t-\gamma}{h}\right]}(-1)^k\binom{\gamma}{k}y(t-kh), $$

where h is the time-step.

The Laplace transform of the Riemann-Liouville fractional operator is given as

$$ \int_0^{\infty} e^{-pt\,RL}\mathscr{D}^{\gamma}_{a,t}y(t)dt = p^{\gamma}Y(p) - \sum_{k=0}^{n-1}p^k\left[\mathscr{D}^{\gamma-k-1}\right]_{t=a}\ \ n-1\leq\gamma<n. $$

The one-parameter Mittag-Leffler function $E_\gamma(z)$ is defined by

$$E_\gamma(z) = \sum_{k=0}^{\infty} \frac{\omega^k}{\Gamma(\gamma k+1}.$$

The corresponding two-parameter Mittag-Leffler function is defined as

$$E_{\gamma,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\gamma+\beta)}$$

where $\beta > 0, \gamma > 0$ and $z \in C$. With $\beta = 1$, we have the following useful properties [6]

$$E_\gamma(z) = E_{\gamma,1}(z), \;\; E_{1,1} = e^z.$$

The Laplace transform of the two-parameter Mittag-Leffler function is

$$\int_0^\infty e^{-pt} t^{k\gamma+\beta-1} E^{(k)}_{\gamma,\beta}(\pm a t^\gamma) dt = \frac{k! p^{\gamma-\beta}}{(p^\gamma \pm a)^{k+1}} \tag{1}$$

where $\mathbf{R}(s) > |a|^{1/n}$.

The zero solution of fractional differential

$${}^{RL}\mathscr{D}^\gamma_{t_0,t} y(t) = \mathbf{F}(y(t),t), \tag{2}$$

with fractional order $0 < \gamma \leq 1$ is stable if for any y_k, $k = 0$, there exists $\varepsilon > 0$ such that $\|t\| < \varepsilon$ for every $t > t_0$. Then, we said the zero solution of fractional differential equation (2) is asymptotically stable if $\|y(t)\| \to 0$ as $t \to \infty$.

Lemma 21 Let $x(t)$ and $z(t)$ be continuous functions on t, t_0 and $x(t) \geq 0$, with r_1 and r_2 being positive constants, if

$$z(t) \leq r_1 + \int_{t_0}^t \{x(\xi)z(\xi) + r2\} d\xi,$$

then

$$z(t) \leq (r_1 + r_2(t_1 - t_0)) \exp \int_{t_0}^t x(\xi) d\xi, \;\; t_0 \leq t \leq t_1.$$

3 Mathematical analysis of fractional system and method of approximation

3.1 Stability analysis

In this paper, we consider both the autonomous and non-autonomous time-fractional systems represented by

$${}^{RL}\mathscr{D}^\gamma_{0,t} y(t) = \mathbf{F}(t, y(t)), \; 0 < \alpha < 1 \tag{3}$$

and

$${}^{RL}\mathscr{D}^\gamma_{0,t} y(t) = \mathbf{L}y(t) + \mathbf{F}(t, y(t)), \; 0 < \alpha < 1 \tag{4}$$

subject to initial condition

$${}^{RL}\mathscr{D}^{\gamma-1}_{0,t} y(t) = y_0, \tag{5}$$

where $y \in \mathbf{R}^n$, and $\mathbf{L} \in \mathbf{R}^{n\times n}, \mathbf{F}(t) : [0,\infty] \to \mathbf{R}^{n\times n}$ is a continuous t matrix.

Theorem 31 If all the eigenvalues of F satisfy

$$|\arg(\lambda(F))| > \frac{\gamma\pi}{2}. \tag{6}$$

Then the zero solution of fractional differential system (3) is asymptotically stable.

Proof. See [17] for details.

Theorem 32 Assume $\|t^{\gamma-1}E_{\gamma,\gamma}(\mathbf{L}t^{\gamma})\| \leq Ae^{-\alpha t}$ for $0 \leq t < \infty$, $\alpha > 0$ and

$$\int_0^{\infty} \|\mathbf{F}(t)\| dt$$

is bounded. This implies that $\int_0^{\infty} \|\mathbf{F}(t)\| dt \leq B$, with $A, B > 0$, then we deduce that the solution of (4) is asymptotically stable.

Proof. Applying the Laplace and inverse Laplace transforms, the given non-autonomous system (4) and its initial condition transform into

$$y(t) = t^{\gamma-1}E_{\gamma,\gamma}(\mathbf{L}t^{\gamma})y_0 + \int_0^t (t-\xi)^{\gamma-1}E_{\gamma,\gamma}(\mathbf{L}(t-\xi)^{\gamma})\mathbf{F}(\xi)y(\xi)d\xi. \tag{7}$$

Taking the norms, the above-mentioned example leads to

$$\|y(t)\| \leq \|t^{\gamma-1}E_{\gamma,\gamma}(\mathbf{L}t^{\gamma})\|\|y_0\| + \int_0^t \|(t-\xi)^{\gamma-1}E_{\gamma,\gamma}(\mathbf{L}(t-\xi)^{\gamma})\|\|\mathbf{F}(\xi)\|\|y(\xi)\|d\xi. \tag{8}$$

Keep in mind that

$$\|y(t)\| \leq Ae^{-\alpha t}\|y_0\| + \int_0^t Ae^{-\alpha(t-\xi)}\|\mathbf{F}(\xi)\|\|y(\xi)\|d\xi. \tag{9}$$

Next, we multiply both sides of (9) by term $e^{\alpha t}$ to obtain

$$e^{\alpha t}\|y(t)\| \leq A\|y_0\| + \int_0^t Ae^{\alpha t}\|\mathbf{F}(\xi)\|\|y(\xi)\|d\xi.$$

With reference to Lemma 21, we set $e^{\alpha t}\|y(t)\| = z(t)$, to have

$$e^{\alpha t}\|y(t)\| \leq (A\|y_0\|)\exp\left(A\int_0^t \|\mathbf{F}(t)\|dt\right), \tag{10}$$

Again, we multiply both sides of (10) by $e^{-\alpha t}$ to get

$$\|y(t)\| \leq (A\|y_0\|)\exp\left(A\int_0^t \|\mathbf{F}(t)\|dt\right)e^{-\alpha t}, \tag{11}$$

then

$$\|y(t)\| \leq (A\|y_0\|)e^{AB-\alpha t}, \quad \text{so that } \|y(t)\| \to 0, \text{ as } t \to \infty.$$

Hence, the solution of fractional differential system (4) is asymptotically stable.

3.2 *Numerical approximation techniques*

In this section, we follow the idea reported in [17] and present the numerical approximation technique for the Riemann-Liouville operator.

We recall from definition,

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(1-\gamma)}\int_0^t (t-\xi)^{-\gamma}F(\xi)d\xi, \\ {}^{RL}\mathscr{D}_{0,t}^{\gamma}\{F(t)\} &= \frac{d}{dt}y(t), \\ \frac{d}{dt}y(t) &= \frac{y(t_{j+1}-y(t_j))}{\Delta t}, \end{aligned} \tag{12}$$

where

$$y(t_{j+1}) = \frac{1}{\Gamma(1-\gamma)}\int_0^{t_{j+1}} (t_{j+1}-\xi)^{-\gamma}F(\xi)d\xi$$

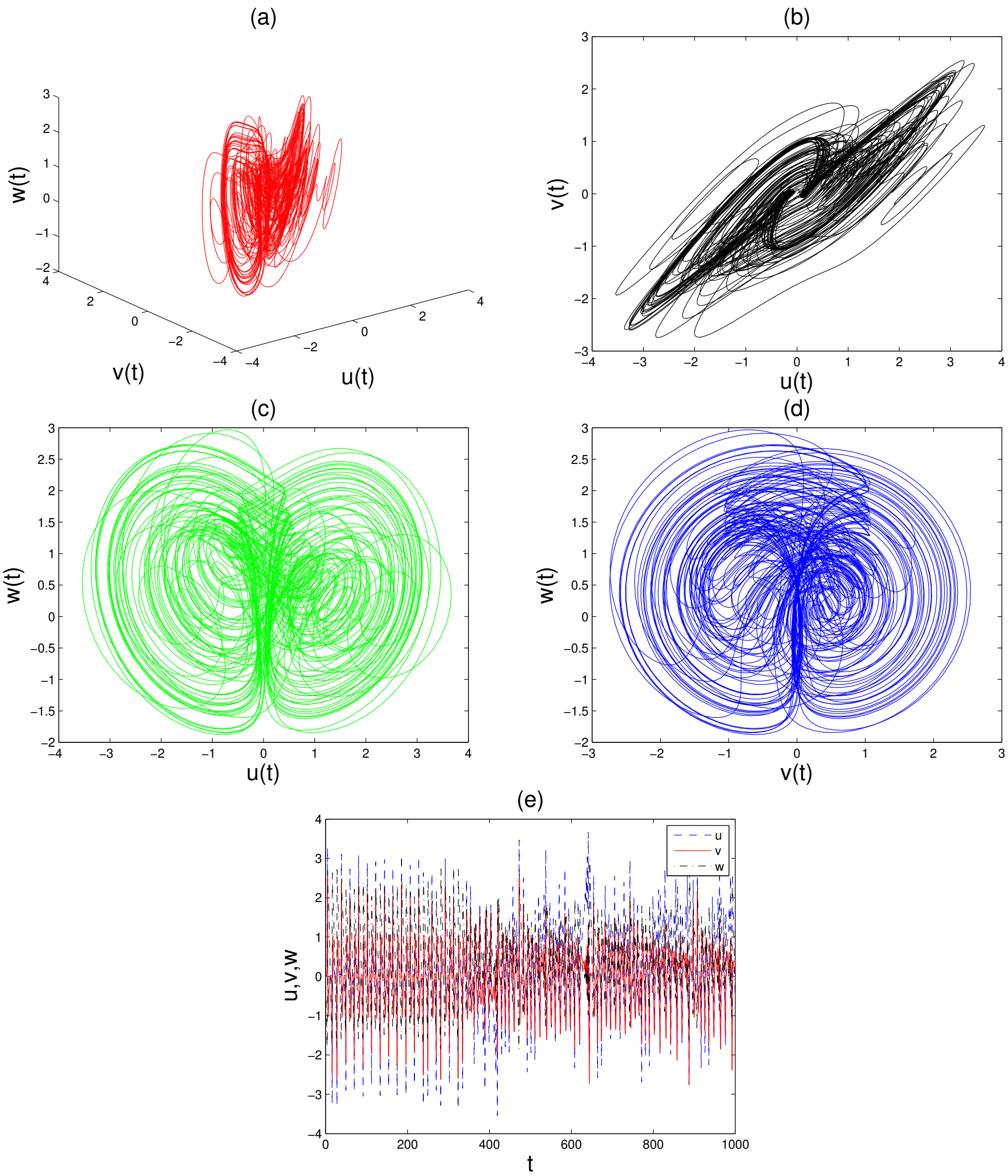

Fig. 1: Various phase portraits for fractional differential system (19)at $\gamma = 0.77$.

and

$$y(t_j) = \frac{1}{\Gamma(1-\gamma)} \int_0^{t_j} (t_j - \xi)^{-\gamma} F(\xi) d\xi$$

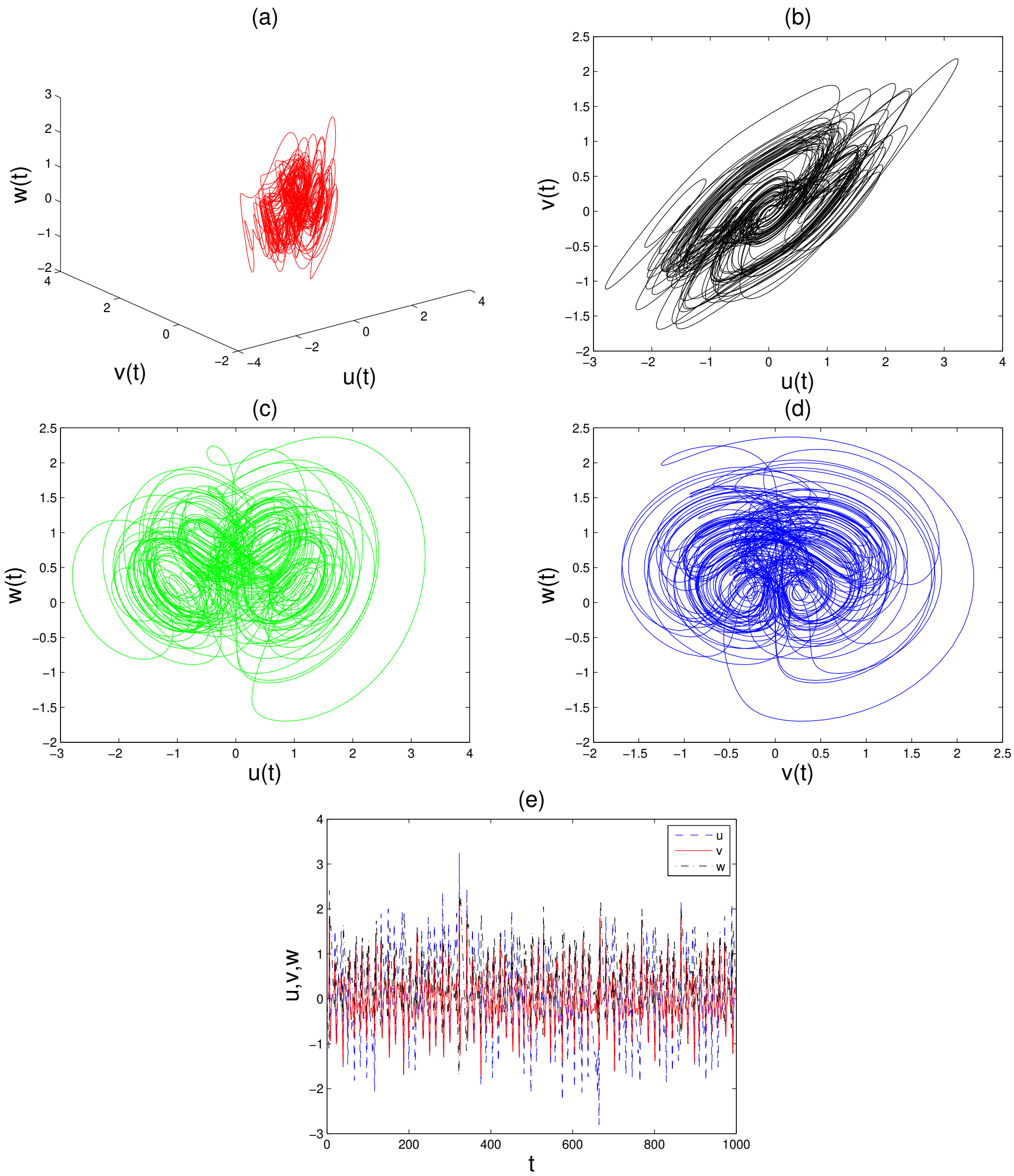

Fig. 2: Various phase portraits for fractional differential system (19)at $\gamma = 0.85$.

Thus, we present the following numerical approximation:

$$
\begin{aligned}
y(t_{j+1}) &= \frac{1}{\Gamma(1-\gamma)} \int_0^{t_{j+1}} (t_{j+1}-\xi)^{-\gamma} F(\xi) d\xi, \\
&= \frac{1}{\Gamma(1-\gamma)} \sum_{s=0}^{j} \int_{t_s}^{t_{s+1}} (t_{j+1}-\xi)^{-\gamma} \frac{F(t_{s+1})+F(t_s)}{2} d\xi, \\
&= \frac{1}{\Gamma(1-\gamma)} \sum_{s=0}^{j} \frac{F(t_{s+1})+F(t_s)}{2} \int_{t_s}^{t_{s+1}} (t_{j+1}-\xi)^{-\gamma} d\xi, \\
&= \frac{1}{\Gamma(2-\gamma)} \sum_{s=0}^{j} \frac{F(t_{s+1})+F(t_s)}{2} \left[(t_{j+1}-t_{s+1})^{1-\gamma} - (t_{j+1}-t_s)^{1-\gamma}\right],
\end{aligned}
\tag{13}
$$

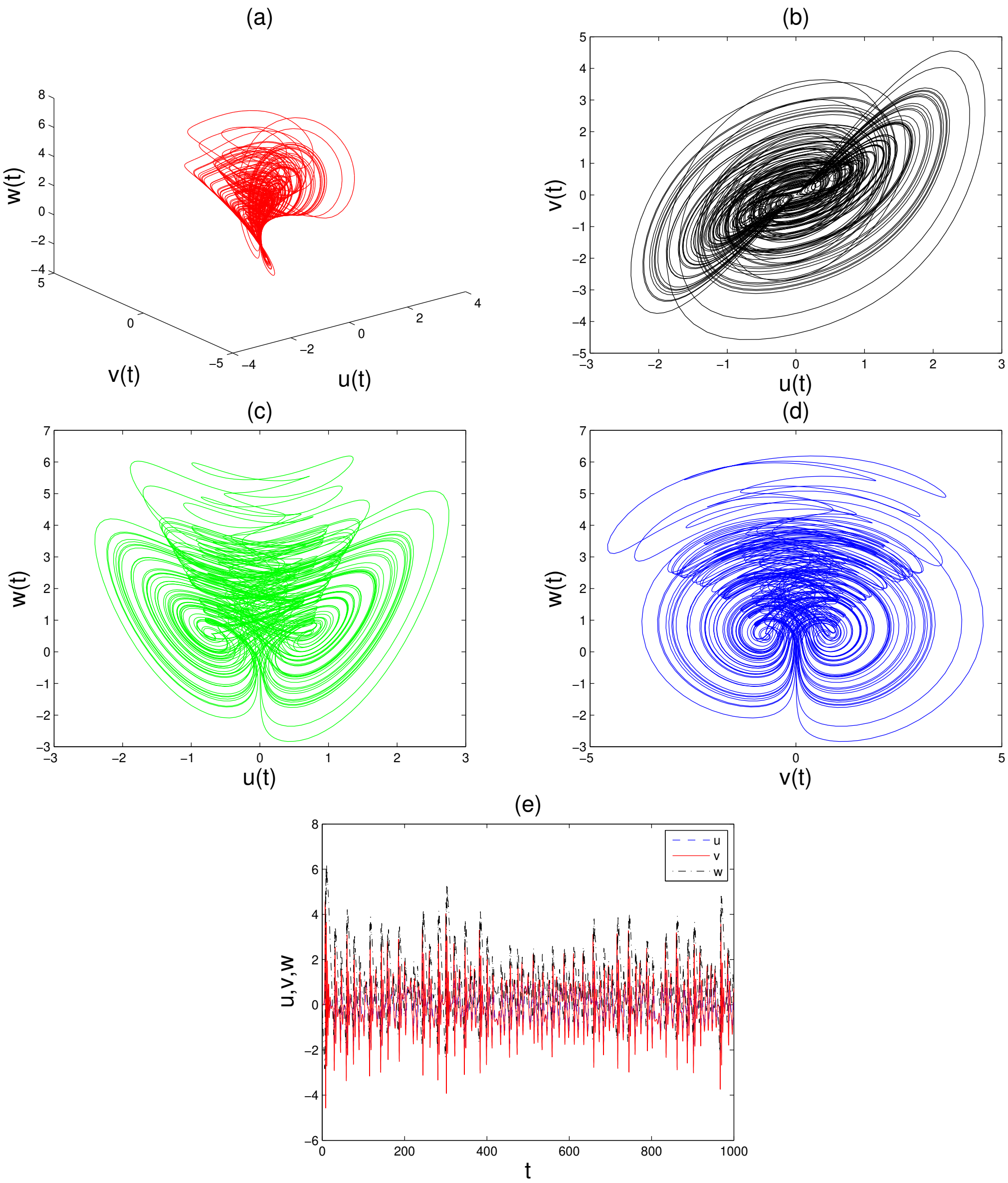

Fig. 3: Various phase portraits for classical-order differential system (19) obtained with $\gamma = 1.00$.

Similarly,

$$y(t_j) = \frac{1}{\Gamma(1-\gamma)} \int_0^{t_j} (t_j - \xi)^{-\gamma} F(\xi) d\xi,$$
$$= \frac{1}{\Gamma(2-\gamma)} \sum_{s=1}^{j} \frac{F(t_s) + F(t_{s-1})}{2} \left[(t_j - t_{s-1})^{1-\gamma} - (t_j - t_s)^{1-\gamma} \right] + \mathscr{O}(\Delta t). \tag{14}$$

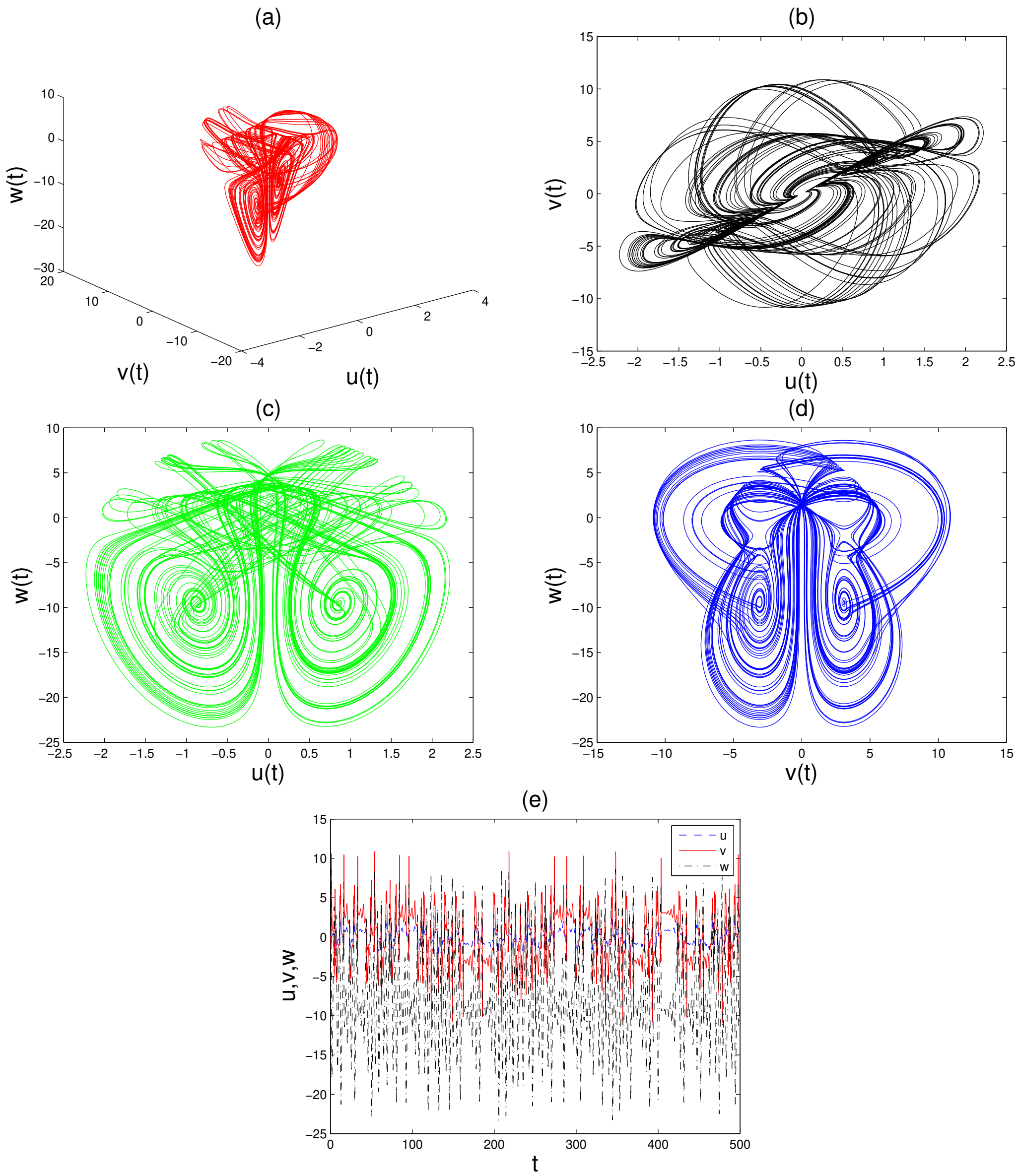

Fig. 4: Fractional attractors of system (21) with $\gamma = 0.89$.

Thus

$$
\begin{aligned}
{}^{RL}\mathscr{D}^{\gamma}_{0,t}\{y(t)\} = \frac{1}{\Delta t \Gamma(2-\gamma)} &\left\{ \sum_{s=0}^{j} \frac{F(t_{s+1})+F(t_s)}{2} \left[(t_{j+1}-t_{s+1})^{1-\gamma} - (t_{j+1}-t_s)^{1-\gamma} \right] \right. \\
&\left. - \sum_{s=1}^{j} \frac{F(t_s)+F(t_{s-1})}{2} \left[(t_j-t_{s-1})^{1-\gamma} - (t_j-t_s)^{1-\gamma} \right] \right\} \\
&+ \mathbb{E}_{\gamma,j},
\end{aligned} \tag{15}
$$

where

$$
\mathbb{E}_{\gamma,j} = \frac{1}{\Delta t \Gamma(1-\gamma)} \left\{ \sum_{s=0}^{j} \int_{t_s}^{t_{s+1}} \frac{F(y)-F(t_{s+1})}{(t_{j+1}-y)^{\gamma}} dy - \sum_{s=0}^{j-1} \int_{t_s}^{t_{s+1}} \frac{F(y)-F(t_{s+1})}{(t_j-y)^{\gamma}} dy \right\}.
$$

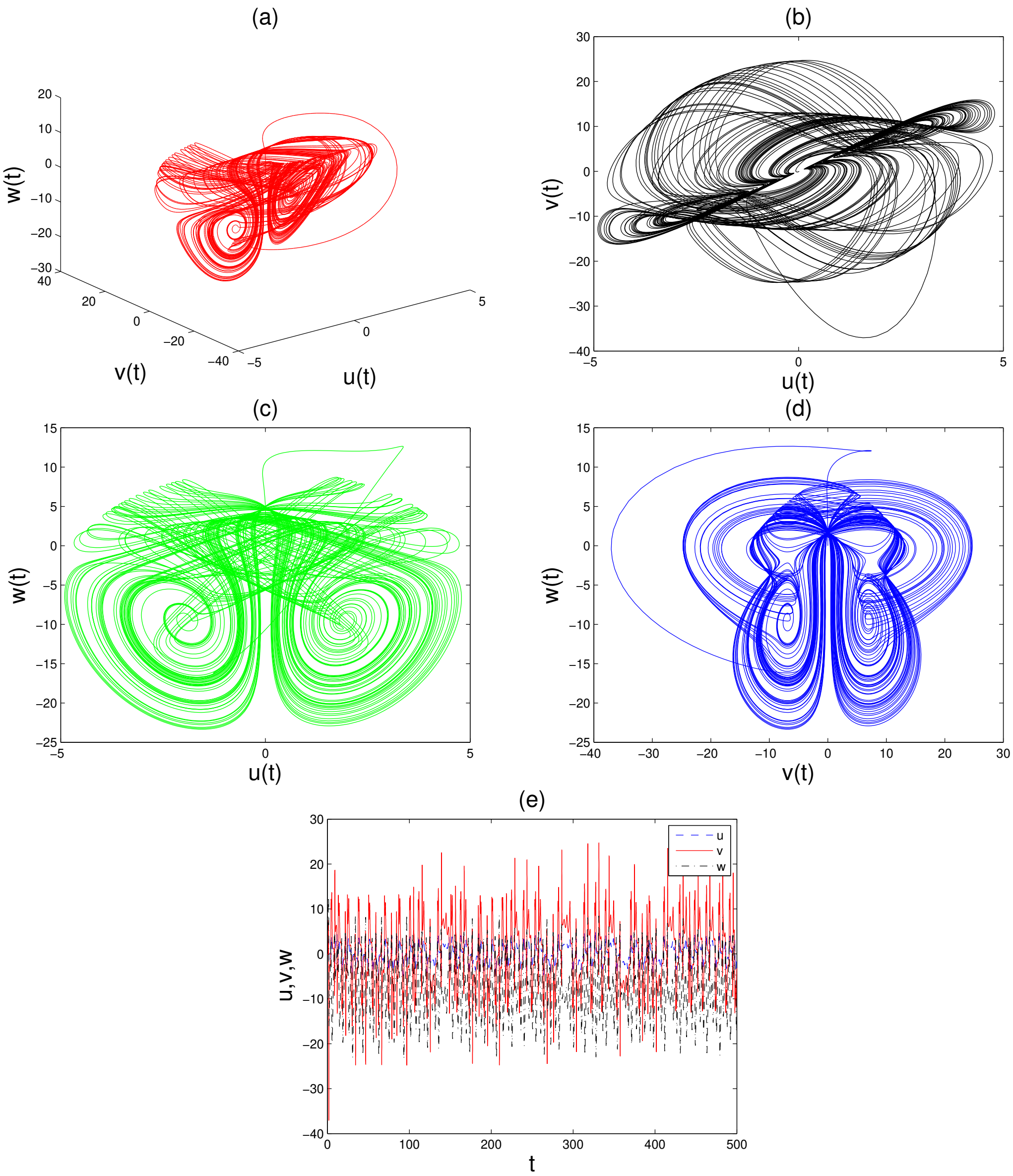

Fig. 5: Fractional attractors of system (21) with $\gamma = 0.94$.

Theorem 33 Let F denotes a function not necessarily differentiable on $[a,T]$. Then the fractional derivative of f of order γ in the Riemann-Liouville sense is defined by

$$
\begin{aligned}
{}^{RL}\mathscr{D}_{0,t}^{\gamma}\{y(t)\} = \frac{1}{\Delta t\Gamma(2-\gamma)} & \left\{ \sum_{s=0}^{j} \frac{F(t_{s+1})+F(t_s)}{2}\left[(t_{j+1}-t_{s+1})^{1-\gamma}-(t_{j+1}-t_s)^{1-\gamma}\right] \right. \\
& \left. -\sum_{s=1}^{j} \frac{F(t_s)+F(t_{s-1})}{2}\left[(t_j-t_{s-1})^{1-\gamma}-(t_j-t_s)^{1-\gamma}\right] \right\} \\
& +\mathbb{E}_{\gamma,j},
\end{aligned}
$$

where

$$|\mathbb{E}_{\gamma,j}| \leq C\left(t_{j+1}^{1-\gamma}-t_j^{1-\gamma}\right).$$

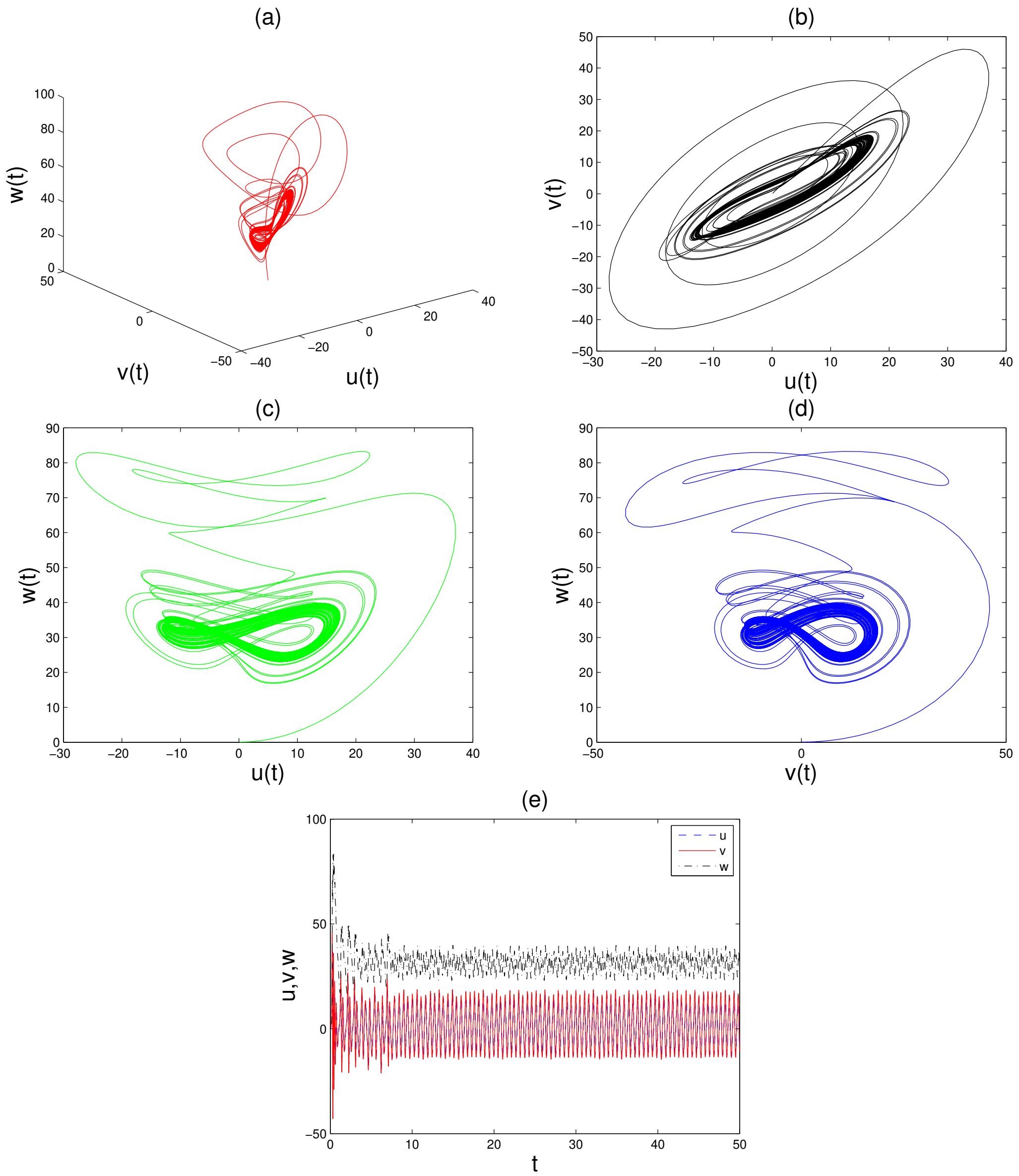

Fig. 6: Fractional attractors of system (22) with $\gamma = 0.59$.

*Proof.*Considering the error term $|\mathbb{E}_{\gamma,j}|$, we have

$$
\begin{aligned}
&\frac{1}{\Gamma(1-\gamma)}\sum_{s=0}^{j}\int_{t_s}^{t_{s+1}}\frac{F(\tau)-F(t_{s+1})}{(t_{j+1}-\tau)^{\gamma}}d\tau\\
&\qquad=\frac{1}{\Gamma(1-\gamma)}\sum_{s=0}^{j}\int_{t_s}^{t_{s+1}}\frac{(F(\tau)-F(t_{s+1}))(r-t_{s+1})}{(\tau-t_{s+1})(t_{j+1}-\tau)^{\gamma}}d\tau\\
&\qquad=\frac{1}{\Gamma(1-\gamma)}\sum_{s=0}^{j}\int_{t_s}^{t_{s+1}}\frac{F(\lambda_s)(\tau-t_{s+1})}{(t_{j+1}-\tau)}d\tau,\quad \tau<t\le t_{j+1}.
\end{aligned}
\tag{16}
$$

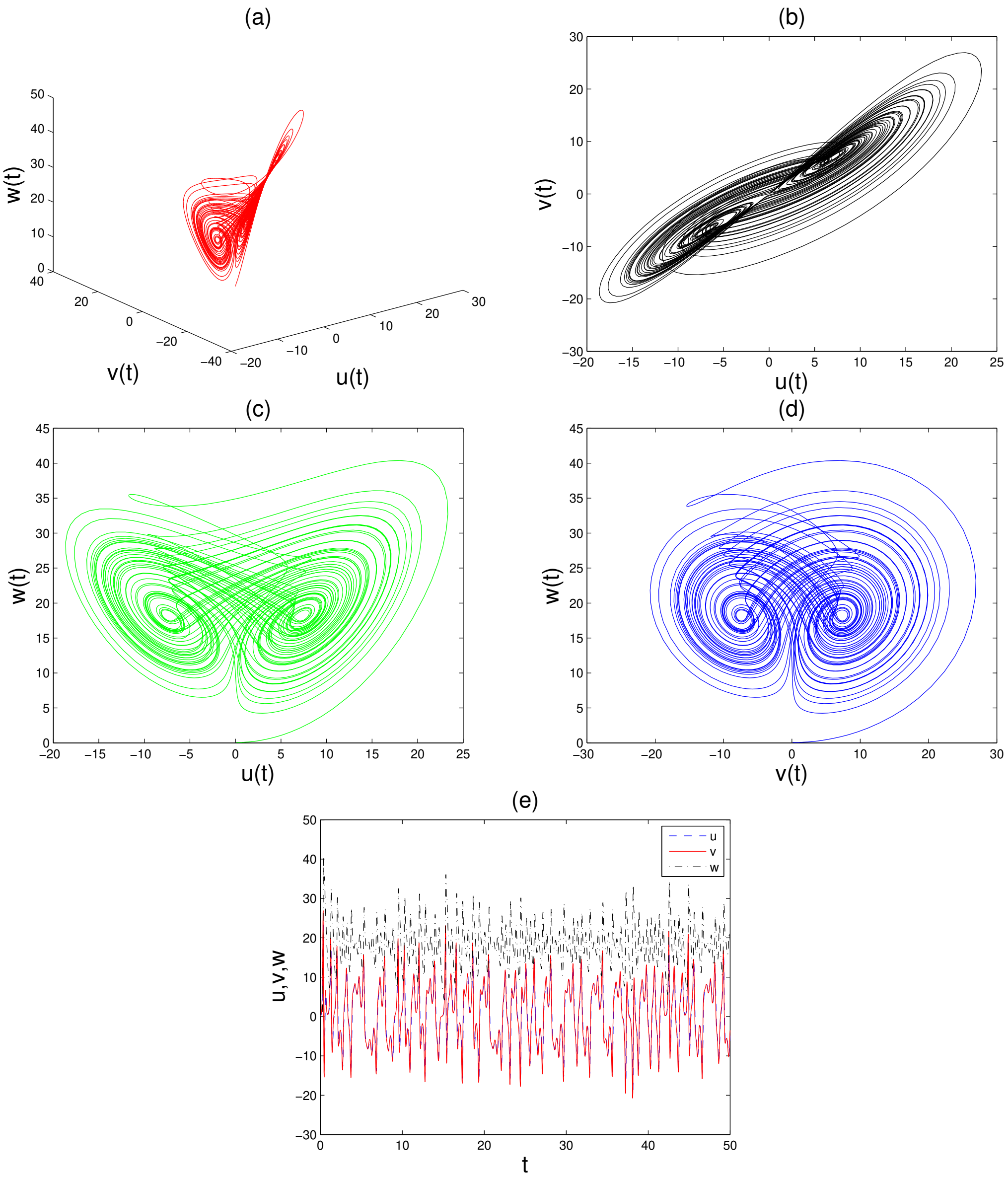

Fig. 7: Fractional attractors of system (22) with $\gamma = 0.74$.

Therefore,

$$\left|\frac{1}{\Gamma(1-\gamma)}\sum_{s=0}^{j}\int_{t_s}^{t_{s+1}}\frac{F(\lambda_s)(\tau-t_{s+1})}{(t_{j+1}-\tau)}d\tau\right|$$
$$\leq \frac{t_{j+1}^{-\gamma}}{\Gamma(1-\gamma)}\max_{0\leq t\leq t_{j+1}}|F(t)|\sum_{s=0}^{j}\int_{t_s}^{t_{s+1}}\frac{1}{(t_{j+1}-\tau)^{\gamma}}d\tau$$
$$\leq \frac{\Delta t}{\Gamma(2-\gamma)}\max_{0\leq t\leq t_{j+1}}|F(t)|t_{j+1}^{1-\gamma}. \tag{17}$$

Similarly,

$$\left|\frac{1}{\Gamma(1-\gamma)}\sum_{s=0}^{j-1}\int_{t_s}^{t_{s+1}}\frac{F(\tau)-F(t_{s+1})}{(t_j-\tau)^{\gamma}}d\tau\right| \leq \frac{\Delta t}{\Gamma(2-\gamma)}\max_{0\leq t\leq t_{j+1}}|F(t)|t_j^{1-\gamma}. \tag{18}$$

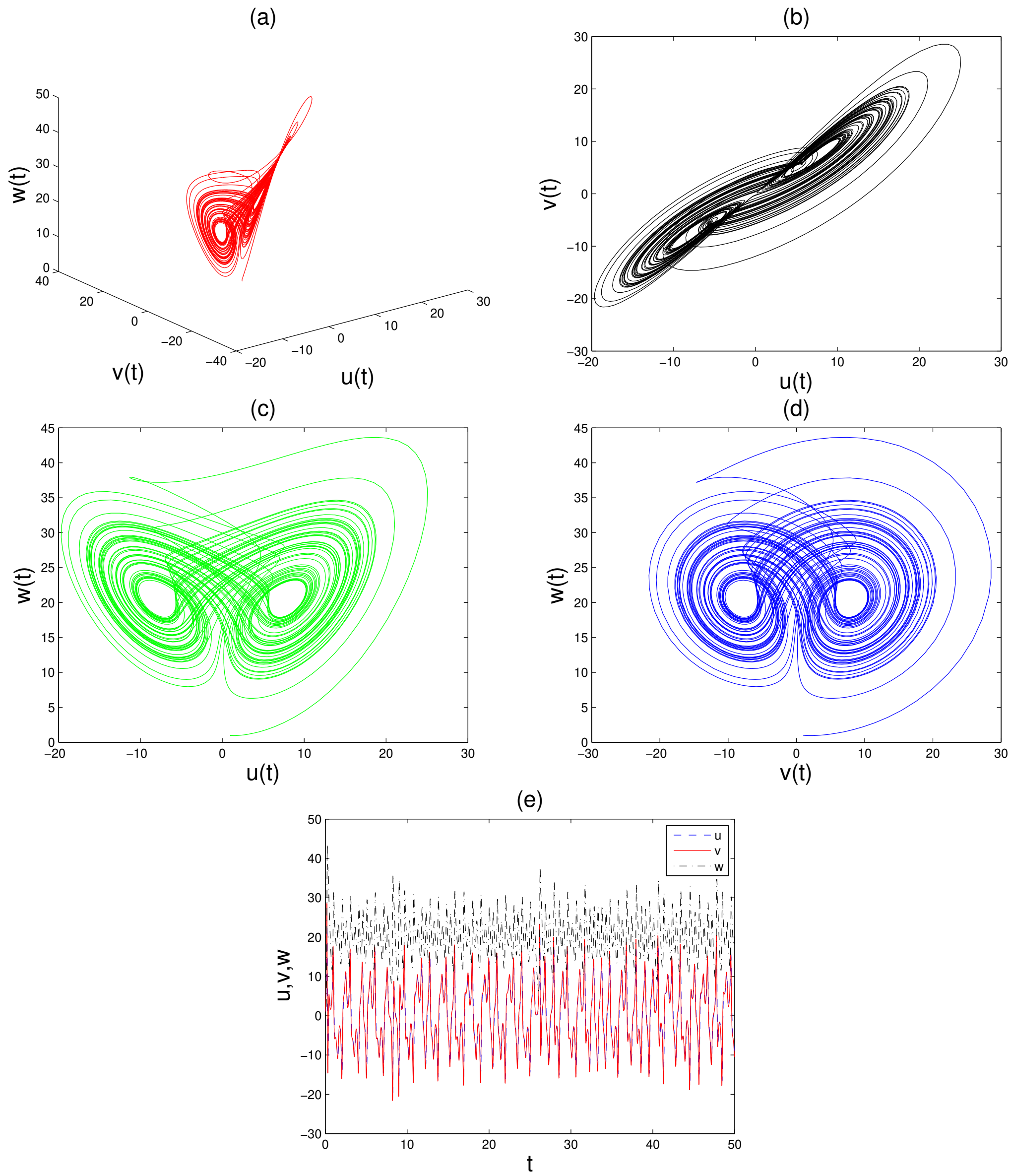

Fig. 8: Fractional attractors of system (22) with $\gamma = 1.00$.

Finally, we have the required result as

$$\frac{1}{\Gamma(2-\gamma)} \max_{0 \le t \le t_{j+1}} |F(t)|$$

which completes the proof.

4 Experimental results

Some examples of chaotic models drawn across the literature are addressed in this section. The classical time derivatives in such models will be replaced using the Riemann-Liouville fractional operator of order $0 < \gamma \le 1$. We also intend to compare the integer order result obtained when $\gamma = 1$ with fractional order cases at $\gamma \in (0,1)$.

4.1 Example 1

Consider the following fractional nonlinear autonomous chaotic system

$$\begin{aligned} {}^{RL}\mathscr{D}_0^{\gamma}u(t) &= v-u, \\ {}^{RL}\mathscr{D}_0^{\gamma}v(t) &= \alpha v-uw, \\ {}^{RL}\mathscr{D}_0^{\gamma}w(t) &= uv-\beta, \end{aligned} \tag{19}$$

which has two equilibrium points obtained by setting ${}^{RL}\mathscr{D}_0^{\gamma}=0$ in (19), and solve for u,v,w. That is

$$v^*-u^*=0,\ \ \alpha v^*-u^*w^*=0,\ \ u^*v^*-\beta=0.$$

Simple calculation shows that the interior points $E^*=(u^*,v^*,w^*)$ corresponds to $(\pm\sqrt{\beta},\pm\sqrt{\beta},\alpha)$. The corresponding Jacobian matrix is given by

$$\mathbf{A}=\begin{pmatrix} -1 & 1 & 0 \\ -w & \alpha & -u \\ v & u & 0 \end{pmatrix}.$$

At point E^*, the Jacobian matrix becomes

$$\mathbf{A}=\begin{pmatrix} -1 & 1 & 0 \\ -\alpha & \alpha & -\sqrt{\beta} \\ \sqrt{\beta} & \sqrt{\beta} & 0 \end{pmatrix}_{E^*}$$

which has the characteristic equation

$$\lambda^3-(\alpha+1)\lambda^2+\beta(\lambda+2)=0 \tag{20}$$

with resulting eigenvalues calculated as $\lambda_1=-1$, $\lambda_2=0.25-0.968245j$ and $\lambda_3=0.25+0.968245j$ for parameter values $\alpha=0.5$ and $\beta=0.5$. It should be noted that the real parts of this eigenvalues are nonnegative. This implies that the equilibrium states are unstable. Consequently, emergence of chaos is evident. In the experiment, we simulate with initial values $u_0=0.01, v_0=0.01, w_0=0$ to obtain numerical results as shown in Figures 1 and 2 for $\gamma=0.77$ and $\gamma=0.85$, respectively. Figure 3 corresponds to integer order results obtained for $\gamma=1.00$.

4.2 Example 2

As an extension, we consider a new four-scroll chaotic attractor introduced to the following fractional differential system [23]

$$\begin{aligned} {}^{RL}\mathscr{D}_0^{\gamma}u(t) &= a_1v-a_2u+a_3uw \\ {}^{RL}\mathscr{D}_0^{\gamma}v(t) &= -a_4uw-a_5u+a_6vw+a_7u \\ {}^{RL}\mathscr{D}_0^{\gamma}w(t) &= a_8-a_9v^2. \end{aligned} \tag{21}$$

The choice of parameters $a_1=1, a_2=0.7, a_3=0.3, a_4=4, a_5=4.4, a_6=1, a_7=0.1, a_8=10, a_9=1$ to obtain chaotic attractors in Figures 4 and 5.

4.3 Example 3

Finally, we consider three-dimensional autonomous Lu system [24]

$$\begin{aligned} {}^{RL}\mathscr{D}_0^{\gamma}u(t) &= a(v-u), \\ {}^{RL}\mathscr{D}_0^{\gamma}v(t) &= -uw+cv \\ {}^{RL}\mathscr{D}_0^{\gamma}w(t) &= uv-bw. \end{aligned} \tag{22}$$

For the experiments, we utilize the Lu system parameters $a=35, b=3$ and $c=28$ to obtain results in Figures 6, 7 and 8 for different instances of fractional power $\gamma\in(0,1]$.

5 Conclusion

In this paper, some classical order chaotic systems have been modelled using the Riemann-Liouville fractional derivative of order $\gamma \in (0,1)$. Mathematical analysis of the general system is presented. Numerical approximation technique is also introduced. Some numerical experiments with applications in physics, engineering are considered in some instances of γ with comparison to classical order case. We deduce that both classical and fractional order systems almost display a similar distribution in phase.

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