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Derivations on *MA*-Semirings

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Abstract: The main purpose of this paper is study and investigate some results concerning a derivation$D$ on aMA- semiring R , when R admits to satisfy some conditions, where$R$ acts as semiprime MA-semiring and prime MA-semiring.

Keywords: Semirings, MA-semirings, Dependent elements,Derivation, Orthogonal derivation

1 Introduction

The theory of commutators plays an important role in the study of Lie algebras [2], prime rings [8], and[14] and $C*$-algebras[7]. It has tremendous applications in the theory of derivations of rings and modules as well.Dependent elements play an important role in solving the functional equations which exist in different disciplines e.g. Quantum Mechanics, Computer science (see [13]. A mapping $F : R \to R$ is a free action if the only dependent element of F is zero.The notion of free action was introduced by Murray and Von Neumann on von Neumann algebras and explored in [15] on commutative von Neumann algebras. The concept was further generalized by Kallman [9] in the context of automorphisms on von Neumann algebras where commutativity is dropped. Choda [3],and[4] studied the dependent elements in $C*$-algebras. Dependent elements are also studied by several authors in the context of operator algebras (see [3] and reference there in). We can find a brief account of dependent elements in $W*$-algebras in the book of Stratila[20]and [15]. Laradji and Thaheem [13] initiated the study of dependent elements of endomorphisms. Recently, Vukman and Kosi-Ulble [23] have explore the study of dependent elements of certain mappings on prime and semiprime rings.Here we are able to introduce the concept of dependent element in the certain class of semirings. As semirings, ordered semirings have some application to the theory of automata (see [11],[16]and [21] and formal languages (see [12], therefore, we believe that this paper has a lot of potential to solve functional equations in computerscience. Vukman [22],and [23] in the setting of MA-semirings.By semiring we mean a non empty set R with two binary operations$'+'$ and $':'$ such that $(R;+)$; $(R;:)$ form semigroup,$'+'$ commutative and $a.(b+c) = a.b+a.c$;$(b+c).a = b.a+c.a$ hold for all $a,b,c \in R$. If there exists $0 \in R$ such that $a+0 = 0+a = a$ and$a0 = 0a = 0$ for all$a \in R$; then$R$ is said to be a semiring with $'0'$(see [6]. Anon empty subset I of R is said to be ideal if for$u,v \in I$; $r \in R$ imply$u+v \in I$ and $ur,ru \in I$. An ideal I is said to be k -ideal if $a+b \in I$;$b \in I$ implies that a I (for more detail see [6].Following the terminology of [6] an element $a \in R$ is said to be additively regular if there exists unique $b \in R$ such that

$$a+b+a = a \tag{i}$$

$$b+a+b = b \tag{ii}$$

(see [6]; the element b is said to be pseudo inverse of a which is, indeed, unique (see[6]. The uniqueness allows us to denote pseudo inverse of a by $\acute{a}$. Note that the semirings satisfying i are also referred as regular semirings by some authors (see [10],and [19] . However, the class of semirings satisfying i ii is also referred as inverse semiring introduced by Karvellas [10]. In fact, the class of MA-semirings is same as the class of additively commutative inverse semirings with 0(see Karvellas [10], satisfying the condition A2 of Bandlet, Petrich [1]. More explicitly, Bandlet studied the semirings satisfying ii, which further satisfy the conditions

$$x(x+\acute{x}) = x+\acute{x} \text{ for all } x \in R \tag{A1}$$

$$y(x+x') = (x+x')y, \text{ for all } x,y \in R \tag{A2}$$

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$$x+(x+x)y=1 \text{for all} x,y\in R \tag{A3}$$

and proved some remarkable results in the theory of regular semirings. The theory of semirings (satisfying the conditioni, A1,A2,A3 is further enhanced by several authors (see [19],and[5]. In 1982, H.J Bandlet and Mario Petrich [1] considered additively commutative semirings satisfying i, such semirings have been used by several authors (see V.N.Salli [17]. Let R be anMA-semiring.R is said to be prime if $aRb=0$ implies that $a=0$ or $b=0$. And R is called be a semiprime if $aRa=0$ implies that $a=0$. A prime ring is semiprime but the converse is not true in general .Let R be an MA-semiring, an element $a\in R$ is said to be dependent element of a mapping $F:R\rightarrow R$, if $F(x)a+ax=0$ for all $x\in R$. An MA-semiring R is 2-torsion free in case $2x=o$ implies that $x=o$ for any $x\in R$. An additive mapping $D:R\rightarrow R$ is said to be derivation on R if $D(xy)=D(x)y+xD(y)$ for all$x,y\in R$. Let$R$ be anMA-semiring, : $R\rightarrow R$ is an automorphism, an additive mapping $D:R\rightarrow R$ is said to -derivation on R if $D(xy)=D(x)(y)+xD(y)$ for all $x,y\in R$. Then : $R\rightarrow R$, : $R\rightarrow R$ are two automorphism, an additive mapping D, : $R\rightarrow R$ is said to (α,β)-derivation on$R$ if $D,(xy)=D,(x)(y)+(x)D,(y)$for all $x,y\in R$. A mapping D is called centralizing if $[D(x),x]\in Z(R)$ for all $x\in R$, in particular,if $[D(x),x]=0$ for all$x\in R$, then it is called commuting. Let R be a prime MA semiring. The derivations$D$ and G of$R$ are called orthogonal if $D(x)RG(y)=0=G(y)RD(x)$.In this paper we study and investigate some results concerning derivation d onMA-semiring,we give some results about that. The following lemmas are necessary for the paper.

Lemma 1[18 : Lemma 2.11] Let$R$ be a semiprimeMA-semiring, then there is no non-zero nilpotent dependent element in$R$.

Lemma 2[18 : Lemma 2.12] Let R be a 2-torsion free semiprime MA-semiring and let $a,b\in R$.If $axb+bxa=0$ for all $x\in R$; then $axb=bxa=0$ for all $x\in R$.

Lemma 3 [18:Lemma 2.3] Let R be a semiprime MA-semiring and an element $a\in R$ such that $[x,a]a=0$ or $a[x,a]=0$ for all $x\in R$, then $a\in Z(R)$.

2 The Main Results

Theorem 3.1: Let$R$ be a 2-torsion free prime MA-semiring and D is derivation of R .If $D(x^2)=0$for all $x\in R$,the$D$ is commuting on$R$.

Proof: By hypothesis $D(x^2)=0$for all $x\in R$. The linearization gives

$$D(xy+yx)=0 \quad \text{for all } x\in R \tag{3.1}$$

Replacing $y$by yx in (3.1) (using (3.1) again) and replacing y by $D(x)y$ in the equation, so obtained, we have

$$xD(x)yD(x)+D(x)yxD(x)=0 \quad \text{for all } x\in R \tag{3.2}$$

In view of Lemma 2, from the last equation, we have $xD(x)=0$ for all $x\in R$ or$D(x)=0$ for all $x\in R$ If$D(x)=0$ for all $x\in R$; then it easy to get that D is commuting on$R$. Now assume that $xD(x)=0$ for all $x\in R$.The replacement of x by $x+y$ in this expression and using it again, we get

$$xD(y)+yD(x)=0 \quad \text{for all } x,y\in R \tag{3.3}$$

Replacing y by y^2 in the last relation and using hypothesis, we get

$$y^2D(x)=0 \quad \text{for all } x,y\in R \tag{3.4}$$

Left-multiplying (3.3) by y with using (3.4),we obtain

$$yxD(y)=0 \quad \text{for all } x,y\in R \tag{3.5}$$

Putting (3.5) in (3.2),we get $xD(x)yD(x)=0$ for all $x\in R$. Since R is primeMA- semiring, therefore,$D(x)=0$ for all $x\in R$, This shows that$D=0$, This completes the proof.

Theorem3.2: Let R be a 2-torsion free semiprime MA-semiring and $D_{\alpha,\beta}$ is derivation of R .If $D_{\alpha,\beta}(x^2)=0$for all $x\in R$,the$D_{\alpha,\beta}$ is commuting on R, where $\alpha,\beta\in Aut(R)$.

Proof: By hypothesis $D_{\alpha,\beta}(x^2)=0$ for all $x\in R$,on linearizing , we have $D_{\alpha,\beta}(xy+yx)=0$ for all $x\in R$ .Replacing y by yx in the last equation and using it again, we get $(xy+yx)D_{,}(x)=0$ implies that$(\beta(x)\beta(y)+\beta(y)\beta(x))D_{\alpha,\beta}(x)=0$.As β is an automorphism, z is an arbitrary element, therefore, $\beta(y)=z$ for some $y\in R$, $(\beta(x)z+z\beta(x))D_{\alpha,\beta}(x)=0$. Replacing z by $D_{\alpha,\beta}(x)y$ in the last relation, we have $\beta(x)D_{\alpha,\beta}(x)yD_{\alpha,\beta}(x)+D_{\alpha,\beta}(x)y\beta(x)D_{\alpha,\beta}(x)=0$ for all$x\in R$. In view of Lemma 2, the last relation implies that $\beta(x)D_{\alpha,\beta}(x)=0$ or $D_{\alpha,\beta}(x)=0$ for all $x\in R$. If $D_{\alpha,\beta}(x)=0$ for all $x\in R$; then nothing is left to show so assume that $\beta(x)D_{\alpha,\beta}(x)=0$ Linearizing last equation, we have$\beta(x)D_{\alpha,\beta}(y)+\beta(y)D_{\alpha,\beta}(x)=0$ for all $x,y\in R$. Replacing y by y^2 in the last relation and using hypothesis, we have $\beta(y)^2D_{\alpha,\beta}(x)=0$ for all $x,y\in R$. As β is an automorphism, therefore $\beta(y)$ can be replaced by an arbitrary element $z\in R$, we get $D_{\alpha,\beta}(x)z^2D_{\alpha,\beta}(x)=0$ for all $x,y\in R$. As R is prime, so is semiprime, therefore, by Lemma1, we have $D_{\alpha,\beta}(x)=0$ for all $x\in R$. This shows that $D_{\alpha,\beta}=0$. This completes the proof.

Theorem3.3: Let R be a 2-torsion free semiprime MA-semiring and let D,G and H be derivation of$R$. Then the mapping $x(D(G(x))+H(x)$ is a free action.

Proof: If$D=0$ or$G=0$; then the result is trivial. Let a is a dependent element of the mapping F; then by

$$F(x)a+ax=0 \quad \text{for all } x\in R \tag{3.6}$$

If$F(x)$ stands for $D(G(x))+H(x)$; then routine calculation gives

$$F(xy)=F(x)y+G(x)D(y)+D(x)G(y)+xF(y) \quad \text{for all } x,y\in R \tag{3.7}$$

. Replacing x by xa in (3.6) and using the relation (3.7), we get $F(x)aa + G(x)D(a)a + D(x)G(a)a + xF(a)a + axa = 0$ for all $x,y \in R$. In view of (3.6), the last equation can be rewritten as

$$G(x)D(a)a + D(x)G(a)a + xF(a)a = 0 \qquad (3.8)$$

Replacing x by yx in (3.8) and using (3.8) again, we get

$$G(y)xD(a)a + D(y)xG(a)a = 0 \qquad (3.9)$$

Replacing y by$a$; x by ax in (3.9) and applying Lemma 2, we get

$$D(a)axG(a)a = 0 \quad \text{for all} x \in R. \qquad (3.10)$$

By post multiplication of (3.9) by $zG(a)a$ and then using (3.10), we have $D(y)xG(a)azG(a)a = 0$ for all $x \in R$. As $D \neq 0$; therefore by primeness of R; we have $G(a)azG(a)a = 0$ for all $z \in R$; and again applying primeness of$R$; we get $G(a)a = 0$: Now the relation (3.9) reduces to $G(y)xD(a)a = 0$ for all $x,y \in R$. As $G \neq 0$; therefore we can conclude that $D(a)a = 0$: Now $G(a)a = D(a)a = 0$; therefore, the equation (3.8), becomes $xF(a)a = 0$. By replacing x by a in (3.6) and then by pre multiplication by x, we get $xF(a)a + xaa = 0$ By using (3.10) it reduces to $0 = xaa+ = xa = a^2xa^2$ for all $x \in R$. Semiprimeness of R implies that $a^2 = 0$; then by Lemma 1, we have $a = 0$ and hence the given mapping is a free action. This completes the proof.

Theorem3.4: Let R be a non commutative semiprime MA-semiring and let D be an (α,β)-derivation of$R$ such that $D([x,y]) = 0$ for all $x,y \in R$; then D is commuting on R.

Proof: By hypothesis

$$D([x,y]) = 0 \quad \text{for all } x,y \in R. \qquad (3.11)$$

Replacing y by yx in (11) and using it, we get $0 = D([x,y]) = D(xyx + yxx) = D((xy + yx)x) = D([x,y]x) = \beta([x,y])D(x) = \beta(xy + yx)x)D(x) = \beta(x)\beta(y) + \beta(y)\beta(x))D(x) = (\beta(x)\beta(y) + \beta(y)(\beta(x)))D(x) = [\beta(x),\beta(y)]D(x)$ As β is an automorphism, therefore, $\beta(y)$ can be replaced by an arbitrary element w,then

$$[\beta(x),w]D(x) = 0 \quad \text{for all } x,y \in R. \qquad (3.12)$$

Replacing x by $x+z$ in (12) and using (12) again, we get

$$[\beta(z),w]D(x) + [\beta(x),w]D(z) = 0 \quad \text{for all } x,y \in R. \qquad (3.13)$$

Replacing w byyw in (13), by [18:Theorem A] and using (13) again, we get $[\beta(z),y]wD(x) + [\beta(x),y]wD(z) = 0$ for all $x,y,z \in R$. Replacing w by $D(x)w[\beta(z),y]$ in the last equation and by (12), we get $[\beta(z),y]D(x)w[\beta(z),y]D(x) = 0$ for all $x,y,z \in R$. By semiprimeness of R from the last expression, we get $[\beta(z),y]D(x) = 0$ for all $x,y,z \in R$. $[t,y]D(x) = 0$ for all $t,y,x \in R$: (sinceβ is automorphism) By Lemma 3,we get $D(x) \in Z(R)$ for all $x \in R$.This completes the proof.

Proposition 3.5: Let R be a 2-torsion free semiprime MA-semiring and let $a,b \in R$.If $axb + bxa = 0$ for all $x \in R$,then $ab = ba = 0$.

Proof: By hypothesis $axb + bxa = 0$ for all $x \in R$. For all $s,t \in R$; replacing x by sat and adding $asbta + asbta$ to both sides of the last equation, we get $asatb + bsata + asbta + asbta = asbta + asbta$, By using hypothesis, we have $asbta + asbta = 0$; that is $2asbta = 0$ for all $s,t \in R$. This implies that $2aRbRa = 0$.Since R is 2-torsion free semiprime , implies that $aRbRa = 0$. Pre multiplying the last equation bybR and using the semiprimeness of$R$, we have$bRa = 0$. Left-multipying by$a$ and right multiplying by$b$,we get $abRab = 0$. By using the semiprimeness of R, we get $ab = 0 = ba$.This completes the proof.

Proposition 3.6: Let R be a semiprime MA-semiring and let there exists $a \in R$ such that $a[x,y]a = 0$ for all $x,y \in R$, then $[[x,a],a] = 0$ for all $x \in R$.

Proof: By supposition $a[x,y]a = 0$ for all $x,y \in R$. Replacing$y$ by ya in the above relation, we get $a[x,ya]a = a[x,y]aa + ay[x,a]a = ay[x,a]a = 0$ for all $x,y \in R$ [18:Theorem$A$].This implies that $[x,a]ay[x,a]a = 0$.By using the semiprimeness of R implies that

$$[x,a]a = 0 \quad \text{for all } x \in R. \qquad (3.14)$$

. According to the Lemma3, we get $a \in Z(R)$, So by using this result we obtain from the relation (14),

$$a[x,a] = 0 \quad \text{for all } x \in R. \qquad (3.15)$$

. Subtracting (14) and(15),we completes the proof.

Theorem 3.7: Let R be a prime MA-semiring and let D be an (α,β)-derivation of R such that $D([x,y]) = 0$ for all $x,y \in R$; their $R$is commutative or $D = 0$.

Proof: By our hypothesis we have the relation

$$D([x,y]) = 0 \quad \text{for all } x,y \in R. \qquad (3.16)$$

Replacing y by yx in (16) and using it, we get $0 = D([x,y]) = D(xyx + yxx) = D((xy + yx)x) = D([x,y]x) = (\beta(x)\beta(y) + \beta(y)(\beta(x)))D(x) = [\beta(x),\beta(y)]D(x)$ Since acts as an automorphism of R, therefore, the $\beta(y)$ in above relation can be replaced by an arbitrary element w, then

$$[\beta(x),w]D(x) = 0 \quad \text{for all } x,w \in R. \qquad (3.17)$$

Replacing w by wr with using R be a prime MA-semiring and β is an automorphism, then relation (17) leads to Either $[\beta(x),w] = [x,w] = 0$ for all $x,w \in R$. Or $D = 0$.This completes the proof.

Theorem 3.8 Let R be a 2-torsion free prime MA-semiring and D,G are derivations such that $D(x)G(x) = G(x)d(x)$ for all $x \in R$, then $D(x)$ and $G(x)$ orthogonal on R.

Proof: Since D,G are derivations and$R$ is MA-semiring.

Then, we have $DG(xy) = D(G(xy)) = D(G(x)y + xG(y)) = DG(x)y + G(x)D(y) + D(x)G(y) + xDG(y)$. So, we obtain DG is also a derivation, therefore ,we have $DG(xy) = DG(x)y + xDG(y)$. Thus ,we have $G(x)D(y) + D(x)G(y) = 0$ for all $x, y \in R$. Replacing y by x in above relation, we get $G(x)D(x) + D(x)G(x) = 0$ for all $x, y \in R$. Since D and G are commuting on R, we get $D(x)G(x) + D(x)G(x) = 0$. $2D(x)G(x) = 0$. Since R be a 2-torsion free prime MA-semiring, then $D(x)G(x) = 0$ for all $x \in R$. We set $a = D(x)$, then $aG(x) = 0$ for all $x \in R$, replace x by xy. Then $0 = aG(xy) = a(G(x)y + xG(y)) = aG(x)y + axG(y) = axG(y)$ for all $x, y \in R$. Since R is a prime MA-semiring , if $G(y) \neq 0$ for some $y \in R$, then, $a = 0 = D$. By same method we can prove $G = 0$. However, we obtain either $D = 0$ or $G = 0$,which leads to

$D(x)RG(y) = 0 = G(y)RD(x)$ for all $x, y \in R$.This completes the proof.

Proposition 3.9 Let R be a semiprime MA-semiring. Suppose that D and G are derivations of R. If D and G are orthogonal there exists $a, b \in R$ such that $DG(x) = ax + xb$ for all $x \in R$. Then $a = -b$.

Proof: Suppose that $DG(x) = ax + xb$ for all $x \in R$. Replacing x by xy, we get $DG(xy) = a(xy) + (xy)b$ Alternatively, $D(G(x)y + xG(y)) = DG(x)y + G(x)D(y) + D(x)G(y) + xDG(y) = axy + xyb$ $xby + xay + D(x)G(y) + G(x)D(y) = 0$ for all $x, y \in R$ Since D and G are orthogonal, then $xby + xay = 0$ for all $x, y \in R$. $x(b + a)y = 0$ for all $x, y \in R$. By using the semiprineness of R, we get $a = -b$.This completes the proof.

3 Conclusion

The theory of commutators plays an important role in the study of Lie algebras , prime rings and $C*$-algebras. It has tremendous applications in the theory of derivations of rings and modules as well.Dependent elements play an important role in solving the functional equations which exist in different disciplines e.g. Quantum Mechanics, Computer science.This paper is studying and it investigate some results concerning a derivation D on a MA- semiring R , when R admits to satisfy some conditions, where R acts as a 2-torsion free semiprime MA-semiring , prime MA-semiring and suppose that d and g are orthogonal derivations.

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