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Generalized Upper Record Values from Modified Makeham Distribution and Related Inference

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Abstract: In this paper, we study the $k-$th upper record values from modified Makeham distribution and derive some simple recurrence relations satisfied by single and product moments. These relations are deduced for moments of record values. Further, conditional expectation and recurrence relation for single moments are used to characterize this distribution.

Keywords: Order statistics, $k-$th record values, upper records, single moments, product moments, recurrence relations, modified Makeham distribution, characterization.

1 Introduction

A random variable X is said to have a modified Makeham distribution if its probability density function (pdf) is of the form

$$f(x)=\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}exp\left(\left(\frac{x}{\alpha}\right)^{\beta}\right)exp\left(1-exp\left(\frac{x}{\alpha}\right)^{\beta}\right)$$

$$x\geq 0,\ \alpha,\ \beta>0 \tag{1}$$

and the distribution function (df) is of the form

$$\bar{F}(x)=exp\left(1-exp\left(\frac{x}{\alpha}\right)^{\beta}\right),\ \ x\geq 0,\ \ \alpha,\ \beta>0, \tag{2}$$

where $\bar{F}(x)=1-F(x)$.
It is easily observed that

$$f(x)=\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}[1+(-ln\bar{F}(x))]\bar{F}(x). \tag{3}$$

The relation in (3) will be used to derive some simple recurrence relations for the moments of $k-$th upper record values from the modified Makeham distribution.
This distribution has been widely used to describe human mortality and to fit actuarial data. For more details on this distribution one can find in Kosznik-Biernacka ([11,12,13]).
Let $\{X_n,\ n\geq 1\}$ be a sequence of independent and identically distributed (iid) random variables with df $F(x)$ and pdf $f(x)$. The $j-$ th order statistics of $X_1,X_2,\ldots,X_n$ is denoted by $X_{j:n}$. For a fixed $k\geq 1$ we define the sequence $\{U_n^{(k)},\ n\geq 1\}$ of $k-$th upper record times of $\{X_n,\ n\geq 1\}$ as follows:
$U_1^{(k)}=1$
$U_{n+1}^{(k)}=min\{j>U_n^{(k)}:X_{j:j+k-1}>X_{U_n^{(k)}:U_n^{(k)}+k-1}\}$.
For $k=1$ and $n=1,2,\ldots,$ we write $U_n^{(1)}=U_n$. Then $\{U_n,n\geq 1\}$ is the sequence of record times of $\{X_n,\ n\geq 1\}$. The sequence $\{Y_n^{(k)},\ n\geq 1\}$, where $Y_n^{(k)}=X_{U_n^{(k)}}$ is called the sequence of $k-$th upper record values of $\{X_n,\ n\geq 1\}$. For convenience, we shall also

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take $Y_0^{(k)} = 0$. Note that for $k = 1$ we have $Y_n^{(1)} = X_{U_n}, n \geq 1$, which are the record values of $\{X_n,\ n \geq 1\}$ (Ahsanullah [1]).
Then the pdf of $Y_n^{(k)}$ and the joint pdf of $Y_m^{(k)}$ and $Y_n^{(k)}$ are as follows:

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{\Gamma(n)} [-ln\bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x),\ \ n \geq 1, \tag{4}$$

$$f_{Y_m^{(k)},Y_n^{(k)}}(x,y) = \frac{k^n}{\Gamma n \Gamma(n-m)} [-ln\bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)}$$

$$\times [ln\bar{F}(x) - ln\bar{F}(y)]^{n-m-1} [\bar{F}(y)]^{k-1} f(y),\ x < y,\ 1 \leq m < n,\ n \geq 2. \tag{5}$$

(Dziubdziela and Kopociński [3], Grudzień [4].
The various developments on record values and related topics are extensively studied in the literature. See for example, Grudzień and Szynal [5], Pawlas and Szynal ([14,15]), Raqab and Ahsanullah [16], Khan and Zia [10], Khan et al. [9], Ahsanullah and Nevzorov [2] and Khan and Khan [8] among others.

2 Relations for single moments

Theorem 2.1. Fix a positive integer $k \geq 1$, for $n \geq 1$, $n \geq k$ and $j = 0, 1, \ldots,$

$$E(Y_n^{(k)})^j = \frac{\beta}{\alpha^\beta (j+\beta)} \{nE(Y_{n+1}^{(k)})^{j+\beta} - (n-k)E(Y_n^{(k)})^{j+\beta} - kE(Y_{n-1}^{(k)})^{j+\beta}\}. \tag{6}$$

Proof. For $n \geq 1$ and $j = 0, 1, \ldots,$ we have from (3) and (4)

$$E(Y_n^{(k)})^j = \frac{\beta k^n}{\alpha^\beta \Gamma n} \int_0^\infty x^{j+\beta-1} [-ln\bar{F}(x)]^{n-1} [\bar{F}(x)]^k dx$$

$$+ \frac{\beta k^n}{\alpha^\beta \Gamma n} \int_0^\infty x^{j+\beta-1} [-ln\bar{F}(x)]^n [\bar{F}(x)]^k dx.$$

Integrating by parts, taking $x^{j+\beta-1}$ as the part to be integrated and rest of the integrand for differentiation and then simplifying the resulting expression, we obtain the result given in Theorem 2.1.
Corollary 2.1. The recurrence relation for single moments of upper record values from the modified Makeham distribution has the form

$$E(X_{U_n}^j) = \frac{\beta}{\alpha^\beta (j+\beta)} \{nE(X_{U_{n+1}}^{j+\beta}) - (n-1)E(X_{U_n}^{j+\beta}) - E(X_{U_{n-1}}^{j+\beta})\}. \tag{7}$$

3 Relations for product moments

Theorem 3.1. For $m \geq 1$, $m \geq k$ and $i,\ j = 0, 1, 2, \ldots,$

$$E[(Y_m^{(k)})^i (Y_{m+1}^{(k)})^j] = \frac{\beta k}{\alpha^\beta (i+\beta)} \{E[(Y_m^{(k)})^{i+j+\beta}] - E[(Y_{m-1}^{(k)})^{i+\beta} (Y_m^{(k)})^j\}$$

$$+ \frac{m\beta}{\alpha^\beta (i+\beta)} \{E[(Y_{m+1}^{(k)})^{i+j+\beta}] - E[(Y_m^{(k)})^{i+\beta} (Y_{m+1}^{(k)})^j\}. \tag{8}$$

and for $1 \leq m \leq n-2$ and $i,\ j = 0, 1, 2, \ldots,$

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] = \frac{\beta k}{\alpha^\beta (i+\beta)} \{E[(Y_m^{(k)})^{i+\beta} (Y_{n-1}^{(k)})^j] - E[(Y_{m-1}^{(k)})^{i+\beta} (Y_{n-1}^{(k)})^j\}$$

$$+ \frac{m\beta}{\alpha^\beta (i+\beta)} \{E[(Y_{m+1}^{(k)})^{i+\beta} (Y_n^{(k)})^j] - E[(Y_m^{(k)})^{i+\beta} (Y_n^{(k)})^j\}. \tag{9}$$

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Proof. From (5) for $1 \leq m \leq n-1$ and $i,\ j = 0,1,2,...,$

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] = \frac{k^n}{\Gamma n \Gamma(n-m)} \int_0^\infty y^j f(y) I(y) dy, \tag{10}$$

where

$$I(y) = \int_0^y x^i [-ln\bar{F}(x)]^{m-1} [-ln\bar{F}(y) + ln\bar{F}(x)]^{n-m-1} \frac{f(x)}{\bar{F}(x)} dx.$$

Integrating $I(y)$ by parts and using (3), we obtain

$$\begin{aligned} I(y) = & \frac{\beta(n-m-1)}{\alpha^\beta(i+\beta)} \int_0^y x^{i+\beta} [-ln\bar{F}(x)]^{m-1} [-ln\bar{F}(y) + ln\bar{F}(x)]^{n-m-2} \frac{f(x)}{\bar{F}(x)} dx \\ & - \frac{\beta(m-1)}{\alpha^\beta(i+\beta)} \int_0^y x^{i+\beta} [-ln\bar{F}(x)]^{m-2} [-ln\bar{F}(y) + ln\bar{F}(x)]^{n-m-1} \frac{f(x)}{\bar{F}(x)} dx \\ & + \frac{\beta(n-m-1)}{\alpha^\beta(i+\beta)} \int_0^y x^{i+\beta} [-ln\bar{F}(x)]^{m} [-ln\bar{F}(y) + ln\bar{F}(x)]^{n-m-2} \frac{f(x)}{\bar{F}(x)} dx \\ & - \frac{m\beta}{\alpha^\beta(i+\beta)} \int_0^y x^{i+\beta} [-ln\bar{F}(x)]^{m-1} [-ln\bar{F}(y) + ln\bar{F}(x)]^{n-m-1} \frac{f(x)}{\bar{F}(x)} dx. \end{aligned}$$

Substituting this expression into (10) and simplifying, it leads to (9). Proceeding in a similar manner for the case $n = m+1$, the recurrence relation given in (8) can easily be established.

One can also note that Theorem 2.1 can be deduced from Theorem 3.1 by putting $j = 0$.

Corollary 3.1. The recurrence relation for single moments of upper record values from the modified Makeham distribution has the form

$$E(X_{U_m}^i X_{U_n}^j) = \frac{\beta}{\alpha^\beta(i+\beta)} \{E(X_{U_m}^{i+\beta} X_{U_{n-1}}^j) - E(X_{U_{m-1}}^{i+\beta} X_{U_{n-1}}^j)\} + \frac{m\beta}{\alpha^\beta(i+\beta)} \{E(X_{U_{m+1}}^{i+\beta} X_{U_n}^j) - E(X_{U_m}^{i+\beta} X_{U_n}^j)\} \tag{11}$$

4 Characterizations

Theorem 4.1. Fix a positive integer $k \geq 1$ and let j be a non-negative integer. A necessary and sufficient condition for a random variable X to be distributed with pdf given by (1) is that

$$E(Y_n^{(k)})^j = \frac{\beta}{\alpha^\beta(j+\beta)} \{nE(Y_{n+1}^{(k)})^{j+\beta} - (n-k)E(Y_n^{(k)})^{j+\beta} - kE(Y_{n-1}^{(k)})^{j+\beta}\} \tag{12}$$

for $n = 1,2,...,\ n \geq k$.

Proof. The necessary part follows from (6). On the other hand if the recurrence relation (12) is satisfied, then on rearranging the terms in (12) and using (4), we have

$$\begin{aligned} \frac{k^n}{\Gamma n} \int_0^\infty x^j [-ln\bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) dx = & \frac{\beta k^{n+1}}{\alpha^\beta(j+\beta)\Gamma n} \int_0^\infty x^{j+\beta} [-ln\bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) dx \\ & - \frac{\beta k^n}{\alpha^\beta(j+\beta)\Gamma(n-1)} \int_0^\infty x^{j+\beta} [-ln\bar{F}(x)]^{n-2} [\bar{F}(x)]^{k-1} f(x) dx \\ & + \frac{n\beta k^{n+1}}{\alpha^\beta(j+\beta)\Gamma(n+1)} \int_0^\infty x^{j+\beta} [-ln\bar{F}(x)]^{n} [\bar{F}(x)]^{k-1} f(x) dx \\ & - \frac{n\beta k^n}{\alpha^\beta(j+\beta)\Gamma n} \int_0^\infty x^{j+\beta} [-ln\bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) dx \end{aligned} \tag{13}$$

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Integrating the first and third integral on the right hand side in (13) by parts, we get

$$\frac{k^n}{\Gamma n}\int_0^\infty x^j[-ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)dx = \frac{\beta k^n}{\alpha^\beta \Gamma n}\int_0^\infty x^{j+\beta-1}[-ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^k dx$$

$$+\frac{\beta k^n}{\alpha^\beta \Gamma n}\int_0^\infty x^{j+\beta-1}[-ln\bar{F}(x)]^{n}[\bar{F}(x)]^k f(x)dx$$

which reduces to

$$\frac{k^n}{\Gamma n}\int_0^\infty x^j[-ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}\left\{f(x)-\frac{\beta}{\alpha^\beta}x^{\beta-1}[1+(-ln\bar{F}(x))]\bar{F}(x)\right\}=0 \tag{14}$$

Applying now a generalization of the Müntz-Szász Theorem (see for example Hwang and Lin [7]) to (14), we find that

$$f(x)=\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}[1+(-ln\bar{F}(x))]\bar{F}(x)$$

which proves that

$$\bar{F}(x)=exp\Big(1-exp\Big(\frac{x}{\alpha}\Big)^\beta\Big),\ \ x\geq 0,\ \ \alpha,\ \beta>0.$$

Remark 4.1. If $k=1$ we obtain the following characterization of the modified Makeham distribution

$$E(X^j_{U_n})=\frac{\beta}{\alpha^\beta(j+\beta)}\{nE(X^{j+\beta}_{U_{n+1}})-(n-1)E(X^{j+\beta}_{U_n})-E(X^{j+\beta}_{U_{n-1}})\},\ \ n=1,2,\ldots .$$

Corollary 4.1. Under the assumptions of Theorem 4.1 with $j=0$ the following relation

$$E(Y^{(k)}_{n+1})^\beta=\Big(\frac{n-k}{n}\Big)E(Y^{(k)}_n)^\beta+\frac{k}{n}E(Y^{(k)}_{n-1})^\beta+\frac{\alpha^\beta}{n},\quad n=1,2,\ldots$$

characterize the modified Makeham distribution.

Theorem 4.2. Let X be a non-negative random variable having an absolutely continuous $df\ F(x)$ with $F(0)=0$ and $0\leq F(x)\leq 1$ for all $x>0$, then

$$E[\xi(Y^{(k)}_n)\mid(Y^{(k)}_l)=x]=exp(1-e^{(x/\alpha)^\beta})\Big(\frac{k}{k+1}\Big)^{n-l},\quad n\geq k,\ \ l=m,\ \ m+1 \tag{15}$$

if and only if

$$\bar{F}(x)=exp\Big(1-exp\Big(\frac{x}{\alpha}\Big)^\beta\Big),\ \ x\geq 0,\ \ \alpha,\ \beta>0,$$

where

$$\xi(y)=exp(1-e^{(y/\alpha)^\beta}).$$

Proof. We have from (4) and (5)

$$E[\xi(Y^{(k)}_n)\mid(Y^{(k)}_m)=x]=\frac{k^{n-m}}{\Gamma(n-m)}\int_x^\infty exp(1-e^{(y/\alpha)^\beta})$$

$$\times[ln\bar{F}(x)-ln\bar{F}(y)]^{n-m-1}\Big(\frac{\bar{F}(y)}{\bar{F}(x)}\Big)^{k-1}\frac{f(y)}{\bar{F}(x)}dy. \tag{16}$$

By setting

$$u=\frac{\bar{F}(y)}{\bar{F}(x)}=\frac{exp(1-e^{(y/\alpha)^\beta})}{exp(1-e^{(x/\alpha)^\beta})}$$

from (2) in (16), we have

$$E[\xi(Y^{(k)}_n)\mid(Y^{(k)}_m)=x]=\frac{k^{n-m}}{\Gamma(n-m)}exp(1-e^{(x/\alpha)^\beta})\int_0^1 u^k(-lnu)^{n-m-1}du. \tag{17}$$

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We have Gradshteyn and Ryzhik [6]

$$\int_0^1 (-lnx)^{\mu-1} x^{\nu-1} dx = \frac{\Gamma\mu}{\nu^\mu}, \quad \mu > 0, \; \nu > 0. \tag{18}$$

On using (18) in (17), we have the result given in (15).
To prove sufficient part, we have

$$\frac{k^{n-m}}{\Gamma(n-m)} \int_x^\infty exp(1-e^{(y/\alpha)^\beta})[ln\bar{F}(x) - ln\bar{F}(y)]^{n-m-1}[\bar{F}(y)]^{k-1} f(y)dy = [\bar{F}(x)]^k g_{n|m}(x), \tag{19}$$

where

$$g_{n|m}(x) = exp(1-e^{(x/\alpha)^\beta})\Big(\frac{k}{k+1}\Big)^{n-m}.$$

Differentiating (19) both the sides with respect to x , we get

$$-\frac{k^{n-m} f(x)}{\bar{F}(x)\Gamma(n-m-1)} \int_x^\infty exp(1-e^{(y/\alpha)^\beta})[ln\bar{F}(x) - ln\bar{F}(y)]^{n-m-2}$$

$$\times[\bar{F}(y)]^{k-1} f(y)dy = g'_{n|m}(x)[\bar{F}(x)]^k - k\, g_{n|m}(x)[\bar{F}(x)]^{k-1} f(x)$$

or

$$-k\, g_{n|m+1}(x)[\bar{F}(x)]^{k-1} f(x) = g'_{n|m}(x)[\bar{F}(x)]^k - k\, g_{n|m}(x)[\bar{F}(x)]^{k-1} f(x).$$

Therefore,

$$\frac{f(x)}{\bar{F}(x)} = -\frac{g'_{n|m}(x)}{k[g_{n|m+1}(x) - g_{n|m}(x)]} = \frac{\beta x^{\beta-1} e^{(x|\alpha)^\beta}}{\alpha^\beta}, \tag{20}$$

where

$$g'_{n|m}(x) = -\frac{\beta x^{\beta-1} e^{(x|\alpha)^\beta}}{\alpha^\beta} exp(1-e^{(x/\alpha)^\beta})\Big(\frac{k}{k+1}\Big)^{n-m}$$

$$g_{n|m+1}(x) - g_{n|m}(x) = \frac{1}{k} exp(1-e^{(x/\alpha)^\beta})\Big(\frac{k}{k+1}\Big)^{n-m}.$$

Now integrating both the sides in (20) with respect to x between $(0,y)$, the sufficiency part is proved.
Remark 4.2. If $k = 1$, we get the following characterization of upper record values for modified Makeham distribution

$$E[\xi(X_{U_n}) \mid (X_{U_l}) = x] = exp(1-e^{(x/\alpha)^\beta})\Big(\frac{1}{2}\Big)^{n-l}, \quad l = m, \; m+1.$$

5 Conclusion

In this study some recurrence relations for single and product moments of $k-$th record values from the modified Makeham distribution have been established and some particular cases are also discussed. Further, conditional expectation and recurrence relation for single moments of $k-$th record values have been utilized to obtain the characterizing results of the modified Makeham distribution.

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