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A New Result on the Almost Increasing Sequences

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Abstract: In this paper, we generalize a known theorem dealing with $|C,1|_k$ summability factors to the $|\,C,\alpha,\beta\,|_k$ summability factors of infinite series. This theorem also includes some known and new results.

Keywords: Almost increasing sequences, Cesàro mean, absolute summability, infinite series, Hölder inequality, Minkowski inequality.

1 Introduction

A positive sequence (b_n) is said to be an almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Let $\sum a_n$ be a given infinite series. We denote by $t_n^{\alpha,\beta}$ the nth Cesàro mean of order (α,β), with $\alpha+\beta > -1$, of the sequence (na_n), that is (see [3])

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \tag{1}$$

where

$$A_n^{\alpha+\beta} = O(n^{\alpha+\beta}),\ A_0^{\alpha+\beta} = 1 \; and \; A_{-n}^{\alpha+\beta} = 0 \quad for\; n > 0. \tag{2}$$

The series $\sum a_n$ is said to be summable $|\,C,\alpha,\beta\,|_k$, $k \geq 1$, if (see [4])

$$\sum_{n=1}^{\infty} \frac{1}{n} \,|\, t_n^{\alpha,\beta} \,|^k < \infty. \tag{3}$$

If we take $\beta = 0$, then $|\,C,\alpha,\beta\,|_k$ summability reduces to $|\,C,\alpha\,|_k$ summability (see [5]).

2 The known result

Theorem A ([[7]). Let (φ_n) be a positive sequence and (X_n) be an almost increasing sequence. If the conditions

$$\sum_{n=1}^{\infty} n \left|\Delta^2 \lambda_n\right| X_n < \infty, \tag{4}$$

$$|\lambda_n| X_n = O(1) \; as \; n \to \infty, \tag{5}$$

$$\varphi_n = O(1) \; as \; n \to \infty, \tag{6}$$

$$n\Delta \varphi_n = O(1) \; as \; n \to \infty, \tag{7}$$

$$\sum_{v=1}^{n} \frac{|t_v|^k}{v X_v^{k-1}} = O(X_n) \; as \; n \to \infty \tag{8}$$

are satisfied , then the series $\sum a_n \lambda_n \varphi_n$ is summable $|C,1|_k$, $k \geq 1$.

3 The main result

The aim of this paper is to generalize Theorem A in the following form.
Theorem. Let (φ_n) be a positive sequence and (X_n) be an almost increasing sequence. If the conditions (4), (5), (6) and (7) are satisfied and the sequence $(w_n^{\alpha,\beta})$ defined by

$$w_n^{\alpha,\beta} = \begin{cases} \left|t_n^{\alpha,\beta}\right|, & \alpha = 1, \beta > -1 \\ \max_{1\leq v\leq n} \left|t_v^{\alpha,\beta}\right|, & 0 < \alpha < 1, \beta > -1 \end{cases} \tag{9}$$

satisfies the condition

$$\sum_{v=1}^{n} \frac{(w_v^{\alpha,\beta})^k}{v X_v^{k-1}} = O(X_n) \; as \; n \to \infty, \tag{10}$$

then the series $\sum a_n \lambda_n \varphi_n$ is summable $|\,C,\alpha,\beta\,|_k$, $0 < \alpha \leq 1$, $(\alpha+\beta-1) > 0$ and $k \geq 1$.
We need the following lemmas for the proof of our theorem.
Lemma 1 ([2]). If $0 < \alpha \leq 1$, $\beta > -1$ and $1 \leq v \leq n$, then

$$|\sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} a_p \,| \leq \max_{1\leq m\leq v} |\sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\beta} a_p \,|\,. \tag{11}$$

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Lemma 2 ([6]). Under the conditions (4) and (5), we have

$$nX_n|\Delta\lambda_n| = O(1) \text{ as } n\to\infty, \tag{12}$$

$$\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty. \tag{13}$$

3 Proof of the theorem

Let $(T_n^{\alpha,\beta})$ be the nth (C,α,β) mean, with $0<\alpha\le 1$ and $\beta>-1$, of the sequence $(na_n\lambda_n\varphi_n)$. Then, by (1), we have that

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n} A_{n-v}^{\alpha-1}A_v^{\beta} v a_v\lambda_v\varphi_n. \tag{14}$$

Thus, applying Abel's transformation first and then using Lemma 1, we have that

$$\begin{aligned}
T_n^{\alpha,\beta} &= \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}\Delta(\lambda_v\varphi_n)\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p \\
&\quad+\frac{\lambda_n\varphi_n}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v, \\
&= \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}(\lambda_v\Delta\varphi_v+\varphi_{v+1}\Delta\lambda_v)\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p \\
&\quad+\frac{\lambda_n\varphi_n}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v.
\end{aligned}$$

$$\begin{aligned}
|T_n^{\alpha+\beta}| &\le \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}|\lambda_v\Delta\varphi_v||\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}p\,a_p| \\
&\quad+\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}|\varphi_{v+1}\Delta\lambda_v||\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}p\,a_p| \\
&\quad+\frac{|\lambda_n\varphi_n|}{A_n^{\alpha+\beta}}|\sum_{v=1}^{v}A_{n-v}^{\alpha-1}A_v^{\beta}va_v| \\
&\le \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}A_v^{\alpha+\beta}w_v^{\alpha,\beta}|\lambda_v||\Delta\varphi_v| \\
&\quad+\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}A_v^{\alpha+\beta}w_v^{\alpha,\beta}|\varphi_{v+1}||\Delta\lambda_v| \\
&\quad+|\lambda_n||\varphi_n|w_n^{\alpha,\beta} \\
&= T_{n,1}^{\alpha,\beta}+T_{n,2}^{\alpha,\beta}+T_{n,3}^{\alpha,\beta}.
\end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-1}|T_{n,r}^{\alpha,\beta}|^k<\infty,\ \textit{for}\ r=1,2,3.$$

When $k>1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we get

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{-1}|T_{n,1}^{\alpha,\beta}|^k &\le \sum_{n=2}^{m+1} n^{-1}(A_n^{\alpha+\beta})^{-k}\times \\
&\quad\left\{\sum_{v=1}^{n-1}A_v^{\alpha+\beta}w_v^{\alpha,\beta}|\Delta\varphi_v||\lambda_v|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{1+(\alpha+\beta)k}}\sum_{v=1}^{n-1}(v^{\alpha+\beta})^k(w_v^{\alpha,\beta})^k|\Delta\varphi_v|^k|\lambda_v|^k\left\{\sum_{v=1}^{n-1}1\right\}^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{2+(\alpha+\beta-1)k}}\sum_{v=1}^{n-1}v^{(\alpha+\beta)k}(w_v^{\alpha,\beta})^k|\lambda_v|^k\frac{1}{v^k} \\
&= O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}(w_v^{\alpha,\beta})^kv^{-k}|\lambda_v|^k\sum_{n=v+1}^{m+1}\frac{1}{n^{2+(\alpha+\beta-1)k}} \\
&= O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}(w_v^{\alpha,\beta})^kv^{-k}|\lambda_v|^k\int_v^{\infty}\frac{dx}{x^{2+(\alpha+\beta-1)k}} \\
&= O(1)\sum_{v=1}^{m}(w_v^{\alpha,\beta})^k|\lambda_v||\lambda_v|^{k-1}\frac{1}{v} \\
&= O(1)\sum_{v=1}^{m}\frac{(w_v^{\alpha,\beta})^k|\lambda_v|}{v\,X_v^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\frac{(w_r^{\alpha,\beta})^k}{r\,X_r^{k-1}}+O(1)|\lambda_m|\sum_{v=1}^{m}\frac{(w_v^{\alpha,\beta})^k}{v\,X_v^{k-1}} \\
&= O(1)\sum_{v=1}^{m}|\Delta\lambda_v|X_v+O(1)|\lambda_m|X_m=O(1),\ m\to\infty
\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 2. Again, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{-1}|T_{n,2}^{\alpha,\beta}|^k &\le \sum_{n=2}^{m+1} n^{-1}(A_n^{\alpha+\beta})^{-k}\times \\
&\quad\left\{\sum_{v=1}^{n-1}A_v^{\alpha+\beta}w_v^{\alpha,\beta}|\varphi_{v+1}||\Delta\lambda_v|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{1+(\alpha+\beta)k}}\left\{\sum_{v=1}^{n-1}v^{\alpha+\beta}(w_v^{\alpha,\beta})|\Delta\lambda_v|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{1+(\alpha+\beta)k}}\sum_{v=1}^{n-1}\frac{v^{(\alpha+\beta)k}(w_v^{\alpha,\beta})^k|\Delta\lambda_v|}{X_v^{k-1}}\times \\
&\quad\left\{\sum_{v=1}^{n-1}X_v|\Delta\lambda_v|\right\}^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{1+(\alpha+\beta)k}}\sum_{v=1}^{n-1}\frac{v^{(\alpha+\beta)k}(w_v^{\alpha,\beta})^k|\Delta\lambda_v|}{X_v^{k-1}} \\
&= O(1)\sum_{v=1}^{m}\frac{v^{(\alpha+\beta)k}(w_v^{\alpha,\beta})^k|\Delta\lambda_v|}{X_v^{k-1}}\sum_{n=v+1}^{m+1}\frac{1}{n^{1+(\alpha+\beta)k}} \\
&= O(1)\sum_{v=1}^{m}\frac{v^{(\alpha+\beta)k}(w_v^{\alpha,\beta})^k|\Delta\lambda_v|}{X_v^{k-1}}\int_v^{\infty}\frac{dx}{x^{1+(\alpha+\beta)k}} \\
&= O(1)\sum_{v=1}^{m}v|\Delta\lambda_v|\frac{(w_v^{\alpha,\beta})^k}{vX_v^{k-1}}
\end{aligned}$$

$$= O(1)\textstyle\sum_{v=1}^{m}\Delta(v|\Delta\lambda_v|)\sum_{r=1}^{v}\frac{(w_r^{\alpha,\beta})^k}{rX_r^{k-1}}+O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}\frac{(w_v^{\alpha,\beta})^k}{vX_v^{k-1}}$$

$$= O(1)\textstyle\sum_{v=1}^{m-1}v|\Delta^2\lambda_v|X_v+O(1)\sum_{v=1}^{m-1}X_v|\Delta\lambda_v|+O(1)m|\Delta\lambda_m|X_m$$

$= O(1), \ as \ m \to \infty,$

by hypotheses of the theorem and Lemma 2. Finally, as in $T_{n,1}^{\alpha,\beta}$, we have that

$$\sum_{n=1}^{m} n^{-1} |T_{n,3}^{\alpha,\beta}|^k = \sum_{n=1}^{m} n^{-1} |\lambda_n \ \varphi_n \ w_n^{\alpha,\beta}|^k$$

$$= O(1) \sum_{n=1}^{m} \frac{(w_n^{\alpha,\beta})^k |\lambda_n|}{n X_n^{k-1}} = O(1), \ as \ m \to \infty.$$

by virtue of the hypotheses of the theorem and Lemma 2. This completes the proof of the theorem. It should be noted that, if we take β=0 and α=1, then we get Theorem A. If we take β=0, then we get a result concerning the $| C, \alpha |_k$ summability factors of infinite series. Also, if we take $k = 1$ and $\beta = 0$, then we get a new result dealing with the $| C, \alpha |$ summability factors of infinite series.

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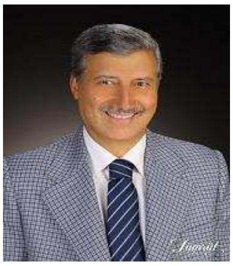

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