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Transmuted Weighted Exponential Distribution and Its Application

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Abstract: In this paper, we have introduced Transmuted Weighted Exponential Distribution (TWED) and investigated its different characteristic as well as structural properties. Expressions for the Renyi entropy, Bonferroni curve and Lorenz curve have also been derived. In addition to it, we have also derived expressions for the PDF of first order, n^{th} order and r^{th} order statistics. Two types of data sets are considered for making the comparison between special cases of TWED in terms of fitting.

Nomenclature:

TWED:	Transmuted Weighted Exponential distribution
TED:	Transmuted Exponential distribution
TLBED:	Transmuted Length Biased Exponential distribution
TABED:	Transmuted Area Biased Exponential distribution
WED:	Weighted Exponential distribution
ED:	Exponential distribution
LBED:	Length Biased Exponential distribution
ABED:	Area Biased Exponential distribution
MRL:	Mean Residual Life
QRTM:	Quadratic Rank Transmutation Map
AIC	Akaike information criterion
AICC	Akaike information criterion corrected
BIC	Bayesian information criterion

Keywords: Transmuted Weighted Exponential Distribution, Descriptive Statistics, Reliability, Entropy, Bonferroni and Lorenz Curve, Order Statistics, AIC, AICC, BIC.

1 Introduction

Statistics is the science of drawing inferences about random phenomena in which chances play a major role. Being a statistician, the first and foremost concern is to predict some future events with much higher accuracy and the accuracy of which can be guaranteed only by using the flexible and suitable models for modelling. In day-to-day life, the applicability of statistics, particularly statistical modelling is so vast that there is merely any field where statistics can't be used. Use of statistics as a whole and statistical modeling in particular in different disciplines is meaningful only if it can be used the right way. Since for carrying out any type of statistical analysis, sampling techniques plays the fundamental role. Quite often people don't contemplate on the type of sampling technique used in collecting the data due to which, most of the assumptions get violated and accordingly the results drawn do not seem to be valid anymore in real life. Thus, it is very imperative to look into the ways of sampling techniques used in extracting the part (sample) from the whole (population), before we go for statistical modelling. Sometimes, situations arise in real life where observations get selected with probabilities proportional to some function, usually known as weight function e.g. PPS (probability proportional to size)

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and size biased sampling. It will not be genuine to carry out statistical modelling without taking into consideration the consequences of such situations. We sometimes require different types of model for fitting purpose. These models already do not exist. So to overcome such requirements, we use to develop some new models. These newly developed classes of distributions provide greater flexibility in modeling complex data and the results drawn from them seems quite sound and genuine. Thus our main concern becomes, to give importance especially to model specification and the data interpretation. Some of the unifying approaches for the development of new class of distributions are Transmutation, Truncation, Kumaraswamy-G, Marshall and Olkin-G, exponentiated-G, construction of weighted model etc. Weighted models take into consideration the method of ascertainment, by adjusting the probabilities of actual occurrence of events to arrive at a specification of the probabilities of those events as observed and recorded. We may arrive at the wrong conclusions, while failing to make such adjustment. The concept of weighted distributions can be traced to the work of Fisher [5], in connection with his studies, on how methods of ascertainment can influence the form of distribution of recorded observations. Later it was introduced and formulated in general terms by Rao[14]. In Rao's paper [14], he identified various situations that can be modeled by weighted distributions. The concept of weighted distribution is also evident from the study of the effect of methods of ascertainment upon estimation of frequencies by Fisher [5]. L.L. Macdonald [12] showed the need for teaching weighted distribution theory. It is necessary to use the concept of weighted distribution while dealing with a stochastic process in which probability of generating an observation varies from observation to observation.

The concept of weighted distributions attracted a lot of researchers to contemplate on and to carry out research on same. Van Deusen [19] arrived at size biased distribution theory independently and applied it to fitting distributions of diameter at breast height (DBH), data arising from horizontal point sampling (HPS) (Grosenbaugh) inventories. Subsequently, Lappi and Bailey [10] used weighted distributions to analyze HPS diameter increment data. In fisheries, Taillie*et al.*[17] modeled populations of fish stocks using weights. In ecology, Dennis and Patil [4] used stochastic differential equations to arrive at a weighted gamma distribution as the stationary probability density function for the stochastic population model with predation effects. Warren [20] was first to apply size biased distributions in connection with sampling wood cells. Jing [8] introduced the weighted inverse Weibull distribution and beta-inverse Weibull distribution. Gove [6] reviewed some of the more recent results on size-biased distributions pertaining to parameter estimation in forestry. Kvam, P [9] studied Length bias in the measurements of Carbon Nanotubes. Ahmed *et al.* [2] worked on new moment method of estimation of parameters of size-biased classical gamma distribution and its characterization. Reshi *et al.* [15] worked on new moment method of estimation of parameters of size-biased classical gamma distribution and its characterization.

There exist a lot of model construction techniques in statistical literature. Herein, we are going to use the technique of transmutation and the concept of weighted distributions for introducing the transmuted weighted Exponential distribution (TWED). The motive behind the construction of TWED is to assess its potentiality and flexibility in modelling a particular data set. Quadratic Rank Transmutation Map (QRTM) has been used for transmutation. Researchers have studied transmuted versions of so many already existing distributions. Ahmad *et al.*[1] studied the Transmuted model of the Inverse Raleigh Distribution. Aryal*et al.*[3] studied the Transmuted Extreme Value Distribution. Hussain*et al.*[7] studied the Transmuted Exponentiated Gamma Distribution. Merovci*et al.* [13] studied the Transmuted Lindley Distribution. Now we are going to investigate different characteristic as well as structural properties of TWED.

2 Derivation of Transmuted Weighted Exponential Distribution

In this section, we have introduced TWED with the help of QRTM. QRTM was introduced by Shaw and Buckley [16], for developing the transmuted version of a particular distribution. QRTM is given by:

$$F_T(x) = (1+\beta)F(x) - \beta\{F(x)\}^2 \tag{2.1}$$

Where $F_T(x)$ and $F(x)$ are respectively the distribution functions of Transmuted random variable and original random variable.

The PDF of weighted Exponential distribution is given by:

$$f_\omega(x) = \frac{\lambda^{(\omega+1)} x^{\omega} e^{-\lambda x}}{\omega!};\ x \geq 0, \lambda > 0, \omega > 0 \tag{2.2}$$

And the corresponding CDF is given by:

$$F_\omega(x) = 1 - e^{-\lambda x} \sum_{j=0}^{\omega} \frac{(\lambda x)^j}{j!} = 1 - \Gamma(\omega+1, x)/\Gamma(\omega+1) \tag{2.3}$$

Substituting (2.3) as the base line distribution in (2.1), we will obtain the CDF of TWED as:

$$F_T(x) = 1 + e^{-\lambda x}\sum_{j=0}^{\omega}\frac{(\lambda x)^j}{j!}\left\{\beta - \beta e^{-\lambda x}\sum_{j=0}^{\omega}\frac{(\lambda x)^j}{j!} - 1\right\} = \frac{[\Gamma(\omega+1) - \Gamma(\omega+1,\lambda x)][\Gamma(\omega+1) + \beta\Gamma(\omega+1,\lambda x)]}{[\Gamma(\omega+1)]^2} \quad (2.4)$$

PDF of TWED will be given by differentiating (2.4) as:

$$f_T(x) = \lambda e^{-\lambda x}\frac{(\lambda x)^{\omega}}{\omega!}\left[1 + \beta\left\{2e^{-\lambda x}\sum_{j=0}^{\omega}\frac{(\lambda x)^j}{j!} - 1\right\}\right] = \frac{\lambda e^{-\lambda x}(\lambda x)^{\omega}[(1-\beta)\Gamma(\omega+1) + 2\beta\Gamma(\omega+1,\lambda x)]}{[\Gamma(\omega+1)]^2}$$

$$; \; x \ge 0, \lambda > 0, \omega \ge 0, |\beta| \le 1 \quad (2.5)$$

Table 1: Some special cases of TWED.

		Distn.	$f_T(x)$	$F_T(x)$
$\lvert\beta\rvert \le 1$	$\omega \ge 0$	TWED	$\frac{\lambda e^{-\lambda x}(\lambda x)^{\omega}[(1-\beta)\Gamma(\omega+1)+2\beta\Gamma(\omega+1,\lambda x)]}{[\Gamma(\omega+1)]^2}$	$\frac{[\Gamma(\omega+1)-\Gamma(\omega+1,\lambda x)][\Gamma(\omega+1)+\beta\Gamma(\omega+1,\lambda x)]}{[\Gamma(\omega+1)]^2}$
$\beta = 0$	$\omega = 0$	ED	$\lambda e^{-\lambda x}$	$1-\Gamma(1,\lambda x)$
	$\omega = 1$	LBED	$\lambda^2 x e^{-\lambda x}$	$[\Gamma(2)-\Gamma(2,\lambda x)][\Gamma(2)+\beta\Gamma(2,\lambda x)]/[\Gamma(2)]^2$
	$\omega = 2$	ABED	$\lambda^3 x^2 e^{-\lambda x}/2$	$[\Gamma(3)-\Gamma(3,\lambda x)][\Gamma(3)+\beta\Gamma(3,\lambda x)]/[\Gamma(3)]^2$

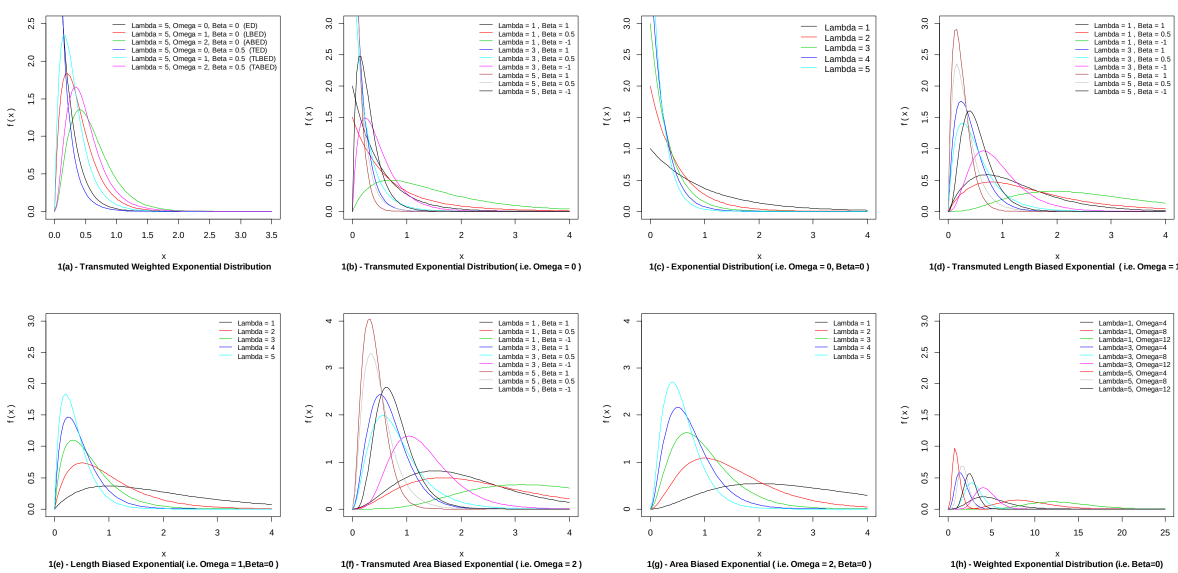


Figure-1

3 Structural Properties of TWED

In this section, various structural properties of TWED are discussed. Expressions for r^{th} non-central moment, mgf, Characteristic function, mean, variance, coefficient of variation, skewness and kurtosis are given.

Theorem 3.1: If a random variable X follows TWED with the following PDF.

$$f_T(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{\omega}[(1-\beta)\Gamma(\omega+1) + 2\beta\Gamma(\omega+1,\lambda x)]}{[\Gamma(\omega+1)]^2}$$

Then its r^{th} moment about zero i.e. μ_r' is given by:

$$\mu_r' = \frac{1}{\lambda^r}\left[\frac{(1-\beta)\Gamma(\omega+r+1)}{\Gamma(\omega+1)} + \frac{2\beta\,\Gamma(2\omega+r+2)\,{}_2F_1(\omega+r+1,2\omega+r+2;\omega+r+2;-1)}{(\omega+r+1)}\right]$$

Where ${}_2F_1(a,b;c;z)$ is known as hyper geometric function.

Proof:

$$\mu_r' = \int_0^\infty x^r \frac{\lambda e^{-\lambda x}(\lambda x)^\omega\left[(1-\beta)\Gamma(\omega+1)+2\beta\Gamma(\omega+1,\lambda x)\right]}{\left[\Gamma(\omega+1)\right]^2}dx$$

$$\mu_r' = \frac{\lambda^{\omega+1}}{\{\Gamma(\omega+1)\}^2}\left[(1-\beta)\Gamma(\omega+1)I_1 + 2\beta I_2\right] \tag{3.1}$$

Where,

$$I_1 = \int_0^\infty e^{-\lambda x}\, x^{\omega+r}\,dx = \frac{\Gamma(\omega+r+1)}{\lambda^{\omega+r+1}} \tag{3.2}$$

$$I_2 = \int_0^\infty e^{-\lambda x}\, x^{\omega+r}\,\Gamma(\omega+1,\lambda x)dx = \frac{\Gamma(2+r+2\omega)\,{}_2F_1(\omega+r+1,2\omega+r+2;\omega+r+2;-1)}{(\omega+r+1)\lambda^{\omega+r+1}} \tag{3.3}$$

Now, using (3.2) & (3.3) in (3.1) we get:

$$\mu_r' = \frac{1}{\lambda^r}\left[\frac{(1-\beta)\Gamma(\omega+r+1)}{\Gamma(\omega+1)} + \frac{2\beta\,\Gamma(2\omega+r+2)\,{}_2F_1(\omega+r+1,2\omega+r+2;\omega+r+2;-1)}{(\omega+r+1)}\right] \tag{3.4}$$

a. Descriptive Statistics:

a) <u>*Moments:*</u>

First four moments about origin can be respectively obtained by substituting $r = 1,2,3,4$ in equation (3.4) and are given as below:

$$\mu_1' = \frac{1}{\lambda}\left[\frac{(1-\beta)\Gamma(\omega+2)}{\Gamma(\omega+1)} + \frac{2\beta\,\Gamma(2\omega+3)\,{}_2F_1(\omega+2,2\omega+3;\omega+3;-1)}{(\omega+2)}\right] = \frac{1}{\lambda\omega!}\{(1-\beta)G_1+\beta S_1\} \tag{3.5}$$

$$\mu_2' = \frac{1}{\lambda^2}\left[\frac{(1-\beta)\Gamma(\omega+3)}{\Gamma(\omega+1)} + \frac{2\beta\,\Gamma(2\omega+4)\,{}_2F_1(\omega+3,2\omega+4;\omega+4;-1)}{(\omega+3)}\right] = \frac{1}{\lambda^2\omega!}\{(1-\beta)G_2+\beta S_2\} \tag{3.6}$$

$$\mu_3' = \frac{1}{\lambda^3}\left[\frac{(1-\beta)\Gamma(\omega+4)}{\Gamma(\omega+1)} + \frac{2\beta\,\Gamma(2\omega+5)\,{}_2F_1(\omega+4,2\omega+5;\omega+5;-1)}{(\omega+4)}\right] = \frac{1}{\lambda^3\omega!}\{(1-\beta)G_3+\beta S_3\} \tag{3.7}$$

$$\mu_4' = \frac{1}{\lambda^4}\left[\frac{(1-\beta)\Gamma(\omega+5)}{\Gamma(\omega+1)} + \frac{2\beta\,\Gamma(2\omega+6)\,{}_2F_1(\omega+5,2\omega+6;\omega+6;-1)}{(\omega+5)}\right] = \frac{1}{\lambda^4\omega!}\{(1-\beta)G_4+\beta S_4\} \tag{3.8}$$

b) <u>*Variance , coefficient of variation, Skewness and kurtosis:*</u>

Variance is given by:

$$\sigma^2 = \frac{\{(1-\beta)G_2+\beta S_2\}}{\lambda^2\omega!} - \frac{\{(1-\beta)G_1+\beta S_1\}^2}{\lambda^2(\omega!)^2} \tag{3.9}$$

Coefficient of variation is given by:

$$c.v.=\sqrt{\frac{(\omega!)^2\{(1-\beta)G_2+\beta S_2\}}{\{(1-\beta)G_1+\beta S_1\}^2}-1} \tag{3.10}$$

Table 2: Characteristics of TWED at different value of ω, λ and β.

Distn.	ω	β	λ	μ	σ^2	γ_1	γ_2	$c.v.$
TED	0	0.8	5	0.120	0.0176	2.713602	15.59504	1.1055420
			10	0.060	0.0044	2.713602	15.59504	1.1055420
		0.4	5	0.160	0.0304	2.360567	11.54294	1.0897250
			10	0.080	0.0076	2.360567	11.54294	1.0897250
		0 **(ED)**	5	0.200	0.0400	2.000000	9.000000	1.0000000
			10	0.100	0.0100	2.000000	9.000000	1.0000000
		-0.4	5	0.240	0.0464	1.757702	7.701546	0.8975275
			10	0.120	0.0116	1.757702	7.701546	0.8975275
		-0.8	5	0.280	0.0496	1.628035	7.145682	0.7953949
			10	0.140	0.0124	1.628035	7.145682	0.7953949
TLBED	1	0.8	5	0.280	0.0416	1.780131	8.649408	0.7284314
			10	0.140	0.0104	1.780131	8.649408	0.7284314
		0.4	5	0.340	0.0644	1.652584	7.242660	0.7463869
			10	0.170	0.0161	1.652584	7.242660	0.7463869
		0 **(LBED)**	5	0.400	0.0800	1.414214	6.000000	0.7071068
			10	0.200	0.0200	1.414214	6.000000	0.7071068
		-0.4	5	0.460	0.0884	1.243075	5.384349	0.6463508
			10	0.230	0.0221	1.243075	5.384349	0.6463508
		-0.8	5	0.520	0.0896	1.178813	5.250957	0.5756396
			10	0.260	0.0224	1.178813	5.250957	0.5756396
TABED	2	0.8	5	0.450	0.0675	1.411301	6.679012	1.2909940
			10	0.225	0.0169	1.411301	6.679012	1.2909940
		0.4	5	0.525	0.0994	1.349684	5.862941	1.3119030
			10	0.262	0.0248	1.349684	5.862941	1.3119030
		0 **(ABED)**	5	0.600	0.1200	1.154701	5.000000	1.2909940
			10	0.300	0.0300	1.154701	5.000000	1.2909940
		-0.4	5	0.675	0.1294	1.009331	4.595043	1.2521590
			10	0.337	0.0323	1.009331	4.595043	1.2521590
		-0.8	5	0.750	0.1275	0.971959	4.584775	1.2055430
			10	0.375	0.0319	0.971959	4.584775	1.2055430

Coefficient of skewness is given by:

$$\gamma_1=\frac{\{(1-\beta)G_3+\beta S_3\}/\omega!-3\{(1-\beta)G_1+\beta S_1\}\{(1-\beta)G_2+\beta S_2\}/(\omega!)^2+2\{(1-\beta)G_1+\beta S_1\}^3/(\omega!)^3}{\left[\{(1-\beta)G_2+\beta S_2\}/\omega!-\{((1-\beta)G_1+\beta S_1)/\omega!\}^2\right]^{3/2}} \tag{3.11}$$

Coefficient of kurtosis is given by:

$$\gamma_2=\frac{\left[\begin{array}{l}\{(1-\beta)G_4+\beta S_4\}/\omega!-4\{(1-\beta)G_3+\beta S_3\}\{(1-\beta)G_1+\beta S_1\}/(\omega!)^2+\\ 6\{(1-\beta)G_2+\beta S_2\}\{(1-\beta)G_1+\beta S_1\}^2/(\omega!)^3-3\{(1-\beta)G_1+\beta S_1\}^4/(\omega!)^4\end{array}\right]}{\left[\{(1-\beta)G_2+\beta S_2\}/\omega!-\{(1-\beta)G_1+\beta S_1\}^2/(\omega!)^2\right]^2} \tag{3.12}$$

Where, $G_r=\Gamma(\omega+r+1)$ & $S_r=\sum_{j=0}^{\omega}\frac{\Gamma(\omega+j+r+1)}{2^{(\omega+j+r)}\,j!}$

From table-2, it can be observed that:

a) With the increase in weight parameter, mean and variance increase whereas the coefficient of variation, skewness and kurtosis decrease.

b) With the increase in transmutation parameter, mean and variance decrease whereas the coefficient of variation, skewness and kurtosis increase.

c) With the increase in rate parameter, mean and variance decrease whereas the coefficient of variation, skewness and kurtosis remain constant.

Theorem 3.2: If a random variable X follows TWED with the following PDF.

$$f_T(x)=\frac{\lambda e^{-\lambda x}(\lambda x)^{\omega}\left[(1-\beta)\Gamma(\omega+1)+2\beta\,\Gamma(\omega+1,\lambda x)\right]}{\left[\Gamma(\omega+1)\right]^2}$$

Then its moment generating function and characteristic function are respectively given by:

$$M_X(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\frac{1}{\lambda^k}\left[\frac{(1-\beta)\Gamma(\omega+k+1)}{\Gamma(\omega+1)}+\frac{2\beta\,\Gamma(2\omega+k+2)\;{}_2F_1(\omega+k+1,2\omega+k+2;\omega+k+2;-1)}{(\omega+k+1)}\right]$$

$$\psi_X(t)=\sum_{k=0}^{\infty}\frac{(it)^k}{k!}\frac{1}{\lambda^k}\left[\frac{(1-\beta)\Gamma(\omega+k+1)}{\Gamma(\omega+1)}+\frac{2\beta\,\Gamma(2\omega+k+2)\;{}_2F_1(\omega+k+1,2\omega+k+2;\omega+k+2;-1)}{(\omega+k+1)}\right]$$

Proof:

$$M_X(t)=\int_0^{\infty}e^{tx}f_T(x)dx$$

$$M_X(t)=\int_0^{\infty}\sum_{k=0}^{\infty}\frac{(tx)^k}{k!}f_T(x)dx$$

$$M_X(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\int_0^{\infty}x^k f_T(x)dx$$

$$M_X(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\mu'_k$$

Now, by using equation (3.4) we get:

$$M_X(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\frac{1}{\lambda^k}\left[\frac{(1-\beta)\Gamma(\omega+k+1)}{\Gamma(\omega+1)}+\frac{2\beta\,\Gamma(2\omega+k+2)\;{}_2F_1(\omega+k+1,2\omega+k+2;\omega+k+2;-1)}{(\omega+k+1)}\right] \quad (3.13)$$

Also we have:

$\psi_X(t)=M_X(it)$

$$\psi_X(t)=\sum_{k=0}^{\infty}\frac{(it)^k}{k!}\frac{1}{\lambda^k}\left[\frac{(1-\beta)\Gamma(\omega+k+1)}{\Gamma(\omega+1)}+\frac{2\beta\,\Gamma(2\omega+k+2)\;{}_2F_1(\omega+k+1,2\omega+k+2;\omega+k+2;-1)}{(\omega+k+1)}\right] \quad (3.14)$$

4 Quartile and Random Number Generation

Inverse CDF Method is one of the methods used for generation of random numbers from a particular distribution. In this method, random numbers from a particular distribution are generated by solving the equation obtained on equating CDF of a distribution to a number p. The number p is itself being generated from $U(0,1)$. Thus following the same method for the generation of random numbers from TWED we will proceed as:

$F_T(x) = p \sim U(0,1)$

$$\frac{[\Gamma(\omega+1)-\Gamma(\omega+1,\lambda x)][\Gamma(\omega+1)+\beta\,\Gamma(\omega+1,\lambda x)]}{[\Gamma(\omega+1)]^2} = p \tag{4.1}$$

On solving equation (4.1) for x at fixed values of parameters ($\omega, \beta \,\&\, \lambda$), we will obtain the random number from the TWED. If $p = 0.25$, $p = 0.5$ and $p = 0.75$ the resulting solutions will be the first quartile (Q_1), Median (Q_2) and third quartile (Q_3) respectively. Equation (4.1) is complex and can't be solved manually. We used R software for solving it.

5 Reliability Analysis of TWED

a. Reliability Function

Reliability of a system is defined as the probability that it will survive beyond a specified time T based on a particular distribution. By definition, Reliability function (Survival function) is given by:

$R(t) = 1 - F(t)$

Thus, Reliability function of TWED is:

$$R_T(t) = 1 - \frac{[\Gamma(\omega+1)-\Gamma(\omega+1,\lambda x)][\Gamma(\omega+1)+\beta\,\Gamma(\omega+1,\lambda x)]}{[\Gamma(\omega+1)]^2} \tag{5.1}$$

b. Hazard rate is:

$$h_T(t) = \frac{\lambda e^{-\lambda x}(\lambda x)^{\omega}[(1-\beta)\Gamma(\omega+1)+2\beta\,\Gamma(\omega+1,\lambda x)]}{[\Gamma(\omega+1)]^2 - [\Gamma(\omega+1)-\Gamma(\omega+1,\lambda x)][\Gamma(\omega+1)+\beta\,\Gamma(\omega+1,\lambda x)]} \tag{5.2}$$

c. Mean Residual Life of TWED is:

The mean residual life function is widely used in the field of reliability. MRL of a product having age t is defined as the life to be expected for which a product survives after the age t. MRL is a conditional concept similar to that of failure rate and is conditioned on survival to time t. The main difference between MRL and failure rate function is that MRL provides information about the whole interval after t, whereas hazard rate provides information just after time t.

Let $F(t)$ be the distribution function of a life time T of a component (i.e. $F(t) = 0$, for $t < 0$) with finite first order moment, then mean residual life function is given by:

$$\mu(t) = E(T - t | T > t) = \begin{cases} \frac{1}{R(t)}\int_t^{\infty} R(x)dx \;\; ; & R(t) > 0 \\ 0\,; & R(t) = 0 \end{cases}$$

Thus, MRL for TWED will be given by:

$$\mu(t) = \left[\int_t^{\infty} e^{-\lambda x}\sum_{j=0}^{\omega}\frac{(\lambda x)^j}{j!}\left\{\beta e^{-\lambda x}\sum_{j=0}^{\omega}\frac{(\lambda x)^j}{j!} - \beta + 1\right\}dx\right] / \, e^{-\lambda t}\sum_{j=0}^{\omega}\frac{(\lambda t)^j}{j!}\left\{\beta e^{-\lambda t}\sum_{j=0}^{\omega}\frac{(\lambda t)^j}{j!} - \beta + 1\right\}$$

$$\mu(t) = \left\{\beta\sum_{j=0}^{\omega}\frac{\Gamma(2j+1, 2\lambda t)}{(j!)^2\lambda 2^{2j+1}} + \beta\sum_{i\neq j=0}^{\omega}\frac{\Gamma(i+j+1, 2\lambda t)}{i!\,j!\,\lambda 2^{i+j+1}} - (\beta-1)\sum_{j=0}^{\omega}\frac{\Gamma(j+1,\lambda t)}{j!\,\lambda}\right\} / \, e^{-\lambda t}\sum_{j=0}^{\omega}\frac{(\lambda t)^j}{j!}\left\{\beta e^{-\lambda t}\sum_{j=0}^{\omega}\frac{(\lambda t)^j}{j!} - \beta + 1\right\} \tag{5.3}$$

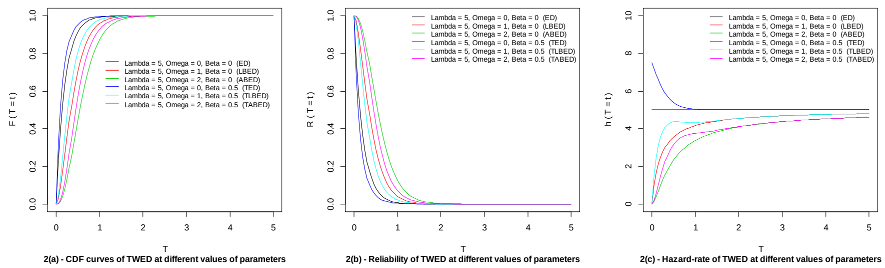


Figure 2

From fig. 2(c) it can be observed that, in case of TED, hazard rate is very high at the beginning then stars decreasing and becomes constant after some period of time. Whereas, for LBED, ABED, TLBED and TABED hazard rate is very low at the beginning, starts increasing and becomes asymptotically parallel to the hazard of Exponential distribution which is constant (i.e. $h(t)=\lambda$).

6 Information Measures of TWED

Renyi-Entropy

The Renyi entropy of a random variable X with PDF $f(x)$ is given by:

$$\gamma_R(\delta)=\frac{1}{1-\delta}\log\left[\int_0^\infty \{f(x)\}^\delta dx\right], \quad \text{Where } \delta>0 \ \& \ \delta\neq 1$$

Therefore, Renyi entropy for TWED will be given by:

$$\gamma_R(\delta)=\frac{1}{1-\delta}\log\left[\int_0^\infty\left\{\lambda e^{-\lambda x}\frac{(\lambda x)^\omega}{\omega!}\left\{1+\beta\left(2e^{-\lambda x}\sum_{j=0}^{\omega}\frac{(\lambda x)^j}{j!}-1\right)\right\}\right\}^\delta dx\right]$$

$$\gamma_R(\delta)=\frac{1}{1-\delta}\log\left[\frac{\lambda^{\delta(1+\omega)}}{(\omega!)^\delta}\int_0^\infty\left\{x^\omega e^{-\lambda x}+\beta x^\omega e^{-\lambda x}\left(2e^{-\lambda x}\sum_{j=0}^{\omega}\frac{(\lambda x)^j}{j!}-1\right)\right\}^\delta dx\right]$$

$$\gamma_R(\delta)=\frac{1}{1-\delta}\log\left[\frac{\lambda^{\delta(1+\omega)}}{(\omega!)^\delta}\int_0^\infty\left[\sum_{k=0}^{\delta}\binom{\delta}{k}\left(x^\omega e^{-\lambda x}\right)^{(\delta-k)}\left\{\beta x^\omega e^{-\lambda x}\left(2e^{-\lambda x}\sum_{j=0}^{\omega}\frac{(\lambda x)^j}{j!}-1\right)\right\}^k\right]dx\right]$$

$$\gamma_R(\delta)=\frac{1}{1-\delta}\log\left\{\frac{\lambda^{\delta(1+\omega)}}{(\omega!)^\delta}\sum_{k=0}^{\delta}\binom{\delta}{k}\beta^k\int_0^\infty e^{-\lambda\delta x}x^{\delta\omega}\left(2e^{-\lambda x}\sum_{j=0}^{\omega}\frac{(\lambda x)^j}{j!}-1\right)^k dx\right\}$$

$$\gamma_R(\delta)=\frac{1}{1-\delta}\log\left[\frac{\lambda^{\delta(1+\omega)}}{(\omega!)^\delta}\sum_{k=0}^{\delta}\binom{\delta}{k}\beta^k\int_0^\infty e^{-\lambda\delta x}x^{\delta\omega}\left\{\sum_{l=0}^{k}\binom{k}{l}2^{(k-l)}e^{-(k-l)\lambda x}\left(\sum_{j=0}^{\omega}\frac{(\lambda x)^j}{j!}\right)^{(k-l)}(-1)^{(l)}\right\}dx\right]$$

$$\gamma_R(\delta)=\frac{1}{1-\delta}\log\left\{\frac{\lambda^{\delta(1+\omega)}}{(\omega!)^\delta}\sum_{k=0}^{\delta}\sum_{l=0}^{k}\binom{k}{l}\binom{\delta}{k}(-1)^{(l)}2^{(k-l)}\beta^k\int_0^\infty e^{-(\delta+k-l)\lambda x}x^{\delta\omega}\left(\sum_{j=0}^{\omega}\frac{(\lambda x)^j}{j!}\right)^{(k-l)}dx\right\}$$

$$\gamma_R(\delta)=\frac{1}{1-\delta}\log\left[\frac{\lambda^{\delta(1+\omega)}}{(\omega!)^{\delta}}\sum_{k=0}^{\delta}\sum_{l=0}^{k}\sum_{i_1=0}^{k-l}\sum_{i_2=0}^{i_1}\cdots\sum_{i_\omega=0}^{i_{(\omega-1)}}\frac{(-1)^l\,2^{k-l}\,\delta!\,\beta^k\,\lambda^{-(\omega\delta+1)}\,\Gamma\left(\omega\delta+\sum_{r=1}^{\omega}i_r+1\right)(\delta+k-l)^{-(\omega\delta+\sum_{r=1}^{\omega}i_r+1)}}{(\delta-k)!\,l!\,(k-l-i_1)!\,i_\omega!\,(i_1-i_2)!\ldots(i_{(\omega-1)}-i_\omega)!\,\omega^{i_\omega}(\Gamma(\omega))^{i_{(\omega-1)}}}\right] \tag{6.1}$$

7 Statistical properties of TWED

a. Order Statistics

The formal investigation of order statistics dates back to 1925 when Tippet [18] derived the CDF of largest order statistics from standard normal distribution and found the mean of sample range. Order statistics has a lot of applications in the field of reliability and life testing. There is also an extensive role of order statistics in several aspects of statistical inference.

Let $x_{(1)},x_{(2)},\ldots x_{(n)}$ be an ordered sample of size n from TWED. Then the PDF of $X_{(1)},X_{(n)},X_{(r)}\,\&\,X_{(m+1)}$ are respectively given as below:

PDF of first order statistics (i.e. $X_{(1)}$) is given by:

$$f_{X_{1:n}}(x)=n\left[1-\frac{[\Gamma(\omega+1)-\Gamma(\omega+1,\lambda x)][\Gamma(\omega+1)+\beta\,\Gamma(\omega+1,\lambda x)]}{[\Gamma(\omega+1)]^2}\right]^{n-1}\frac{\lambda\,e^{-\lambda x}(\lambda x)^{\omega}[(1-\beta)\Gamma(\omega+1)+2\beta\,\Gamma(\omega+1,\lambda x)]}{[\Gamma(\omega+1)]^2} \tag{7.1}$$

PDF of largest order statistics (i.e. $X_{(n)}$) is given by:

$$f_{X_{n:n}}(x)=n\left[\frac{[\Gamma(\omega+1)-\Gamma(\omega+1,\lambda x)][\Gamma(\omega+1)+\beta\,\Gamma(\omega+1,\lambda x)]}{[\Gamma(\omega+1)]^2}\right]^{n-1}\frac{\lambda\,e^{-\lambda x}(\lambda x)^{\omega}[(1-\beta)\Gamma(\omega+1)+2\beta\,\Gamma(\omega+1,\lambda x)]}{[\Gamma(\omega+1)]^2} \tag{7.2}$$

PDF of $X_{(r)}$ (i.e. r^{th} order statistics) is given by:

$$f_{X_{r:n}}(x)=\begin{cases}\dfrac{n}{(r-1)!\,(n-r)!}\left[1-\dfrac{[\Gamma(\omega+1)-\Gamma(\omega+1,\lambda x)][\Gamma(\omega+1)+\beta\,\Gamma(\omega+1,\lambda x)]}{[\Gamma(\omega+1)]^2}\right]^{n-r}\times\\[2ex]\left[\dfrac{[\Gamma(\omega+1)-\Gamma(\omega+1,\lambda x)][\Gamma(\omega+1)+\beta\,\Gamma(\omega+1,\lambda x)]}{[\Gamma(\omega+1)]^2}\right]^{(r-1)}\times\dfrac{\lambda\,e^{-\lambda x}(\lambda x)^{\omega}[(1-\beta)\Gamma(\omega+1)+2\beta\,\Gamma(\omega+1,\lambda x)]}{[\Gamma(\omega+1)]^2}\end{cases} \tag{7.3}$$

And the PDF of Median $X_{(m+1)}$ order statistics is given by:

$$f_{X_{(m+1):n}}(x)=\frac{(2m+1)!}{m!\,m!}[1-F_T(x)]^m[F_T(x)]^m f_T(x)$$

$$f_{X_{(m+1):n}}(x)=\begin{cases}\dfrac{(2m+1)!}{m!\,m!}\left[1-\dfrac{[\Gamma(\omega+1)-\Gamma(\omega+1,\lambda x)][\Gamma(\omega+1)+\beta\,\Gamma(\omega+1,\lambda x)]}{[\Gamma(\omega+1)]^2}\right]^{m}\times\\[2ex]\left[\dfrac{[\Gamma(\omega+1)-\Gamma(\omega+1,\lambda x)][\Gamma(\omega+1)+\beta\,\Gamma(\omega+1,\lambda x)]}{[\Gamma(\omega+1)]^2}\right]^{m}\dfrac{\lambda\,e^{-\lambda x}(\lambda x)^{\omega}[(1-\beta)\Gamma(\omega+1)+2\beta\,\Gamma(\omega+1,\lambda x)]}{[\Gamma(\omega+1)]^2}\end{cases} \tag{7.4}$$

b. Bonferroni and Lorenz curves

Bonferroni and Lorenz curves are not only used in economics in order to study the relation between income and poverty, it is also being used in reliability, medicine, insurance and demography. The Bonferroni and Lorenz curve for the probability model with PDF $f(x)$ are respectively given by:

$$B(p)=\frac{1}{p\mu_1'}\int_0^q xf(x)dx\ \&\ L(p)=pB(p)=\frac{1}{\mu_1'}\int_0^q xf(x)dx\ \text{, where } q=F^{-1}(p).$$

Thus, the Bonferroni curve for TWED will be given by:

$$B(p)=\frac{1}{p\{(1-\beta)G_1+\beta S_1\}}\left[(1-\beta)\{1-\Gamma(\omega+2,\lambda q)\}+2\beta\sum_{j=0}^{\omega}\frac{1-\Gamma(\omega+j+2,2\lambda q)}{2^{(\omega+j+2)}\,j!}\right] \tag{7.5}$$

Lorenz curve for the same distribution will be given as below:

$$L(p)=pB(p)=\frac{1}{\{(1-\beta)G_1+\beta S_1\}}\left[(1-\beta)\{1-\Gamma(\omega+2,\lambda q)\}+2\beta\sum_{j=0}^{\omega}\frac{1-\Gamma(\omega+j+2,2\lambda q)}{2^{(\omega+j+2)}\,j!}\right] \tag{7.6}$$

8 Parameter Estimation

In this section we have estimated the parameters of TWED by applying the method of Moments and Maximum Likelihood technique.

a. Method of Moments

In the method of moments, parameters are estimated on solving the system of equations, obtained on equating sample moments to the corresponding population moments.

Let $x_1, x_2, \ldots x_n$ be a random sample of size n from the TWED. Thus on applying the method of moments we will have the following system of equations:

$$\bar{x}=\frac{1}{\lambda(\omega!)}\{(1-\beta)G_1+\beta S_1\}\Rightarrow\beta=\frac{1-\bar{x}\lambda(\omega!)}{(G_1-S_1)} \tag{8.1}$$

$$\bar{x}'=\frac{1}{n}\sum_{i=1}^{n}x_i^2=\frac{1}{\lambda^2(\omega!)}\{(1-\beta)G_2+\beta S_2\}\Rightarrow\beta=\frac{1-\bar{x}'\lambda^2(\omega!)}{(G_2-S_2)} \tag{8.2}$$

From (8.1) & (8.2) we will have:

$$\begin{aligned}&\bar{x}'\lambda^2(\omega!)(G_1-S_1)-\bar{x}\lambda(\omega!)(G_2-S_2)=(G_1-S_1)-(G_2-S_2)\\&p\lambda^2-q\lambda+s=0\end{aligned} \tag{8.3}$$

Where, $p=\bar{x}'(\omega!)(G_1-S_1)$, $q=\bar{x}(\omega!)(G_2-S_2)$ & $s=(G_2-S_2)-(G_1-S_1)$

Solving (8.3) for λ we get:

$$\hat{\lambda}_{mm}=\frac{q\pm\sqrt{q^2-4ps}}{2p}$$

$$\hat{\lambda}_{mm}=\frac{\bar{x}(\omega!)(G_2-S_2)\pm\sqrt{\left(\bar{x}(\omega!)(G_2-S_2)\right)^2-4\left\{\bar{x}'(\omega!)(G_1-S_1)\right\}\{(G_2-S_2)-(G_1-S_1)\}}}{2\bar{x}'(\omega!)(G_1-S_1)} \tag{8.4}$$

Substituting the value of $\hat{\lambda}_{mm}$ in (8.1) and solving for β we get:

$$\hat{\beta}_{mm}=\frac{1}{(G_1-S_1)}-\bar{x}\left[\frac{\bar{x}(\omega!)(G_2-S_2)\pm\sqrt{\left(\bar{x}(\omega!)(G_2-S_2)\right)^2-4\left\{\bar{x}'(\omega!)(G_1-S_1)\right\}\{(G_2-S_2)-(G_1-S_1)\}}}{2\bar{x}'(G_1-S_1)^2}\right] \tag{8.5}$$

b. Maximum Likelihood Method

Let $x_1, x_2, \ldots x_n$ be a random sample of size n from TWED. Therefore the log Likelihood function will be given by:

$$\log(l) = n(1+\omega)\log\lambda - \lambda\sum_{i=1}^{n} x_i + \omega\sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}\log\left\{(1-\beta)+2\beta\frac{\Gamma(\omega+1,\lambda x)}{\Gamma(\omega+1)}\right\} - n\log(\omega!)$$

On differentiating log likelihood function with respect to $\omega, \lambda \&, \beta$ and equating them to zero, we get the system of nonlinear equations. Manually solving the system of nonlinear equations is very tedious and cumbersome. Therefore, we used R and Wolfram Mathematica for estimating the required parameters.

9 Application

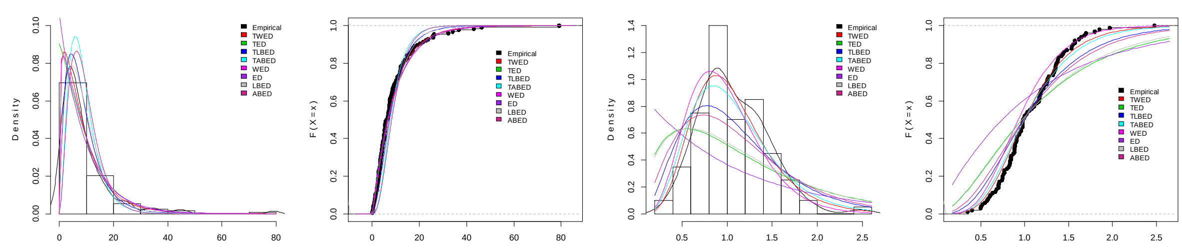
In this section, we considered both a real life and a simulated data set. The real life data set is regarding remission times (in months) of 128 Bladder cancer patients reported in Lee and Wang [11]. Another data set is of size 100, simulated from TWED with weight parameter = 5, rate parameter = 5 and transmutation parameter = 0.5. The data set is simulated by using the Inverse CDF method discussed in section 4. The two data sets are given in table-3.

Table 3

Remission times (in months) of 128 patients of Bladder cancer.															
00.08	02.09	03.48	04.87	06.94	08.66	13.11	23.63	00.20	02.23	03.52	04.98	06.97	09.02	13.29	00.40
02.26	03.57	05.06	07.09	09.22	13.80	25.74	00.50	02.46	03.64	05.09	07.26	09.47	14.24	25.82	00.51
02.54	03.70	05.17	07.28	09.74	14.76	26.31	00.81	02.62	03.82	05.32	07.32	10.06	14.77	32.15	02.64
03.88	05.32	07.39	10.34	14.83	34.26	00.90	02.69	04.18	05.34	07.59	10.66	15.96	36.66	01.05	02.69
04.23	05.41	07.62	10.75	16.62	43.01	01.19	02.75	04.26	05.41	07.63	17.12	46.12	01.26	02.83	04.33
07.66	11.25	17.14	79.05	01.35	02.87	05.62	07.87	11.64	17.36	01.40	03.02	04.34	05.71	07.93	11.79
18.10	01.46	04.40	05.85	08.26	11.98	19.13	01.76	03.25	04.50	06.25	08.37	12.02	02.02	03.31	04.51
06.54	08.53	12.03	20.28	02.02	03.36	06.76	12.07	21.73	02.07	03.36	06.93	08.65	12.63	22.69	05.49

Simulated Data with $\omega=5, \lambda=5\&, \beta=0.5$									
0.7660047	0.8714411	1.0745434	1.6769609	0.6980330	1.6408911	1.8529212	1.1779769	1.1389445	0.4992584
0.7028155	0.6691981	1.2122347	0.8830980	1.3368963	0.9952879	1.2548992	2.4773397	0.8791312	1.3500543
1.7959703	0.7095993	1.1664939	0.6044246	0.7677531	0.8850589	0.3484649	0.8814487	1.5512946	0.8404961
0.9794871	1.1044428	0.9910676	0.6804794	1.4473096	1.1877950	1.3804932	0.5791565	1.2638212	0.9095389
1.4335328	1.1607653	1.3597828	1.0530266	1.0283234	1.3714341	0.3947548	0.9746090	1.2766828	1.2199635
0.9749995	1.5282523	0.9357897	0.7445966	0.5168388	0.5662155	0.8168929	1.0167819	1.1795122	0.9052107
1.6953802	0.7944229	0.9565079	0.8327373	1.1654927	0.7583114	0.9759282	1.3309074	0.5413254	1.5673491
0.8392720	1.4744349	0.8467040	0.8340901	0.9737287	1.6197623	1.5366055	0.8888257	1.3498358	1.9680565
0.9324155	1.2475378	0.8985552	0.8258312	1.3156029	0.6991609	1.2455452	0.5990663	0.7453259	0.6281841
0.7391544	0.4932884	1.1548928	1.5701163	1.3526397	1.3862791	0.9527287	0.9083808	1.4127727	1.1105903

The motive behind considering these two data sets is to assess the potentiality and flexibility of TWED and its special cases in modelling. TWED and its special cases are fitted to the data sets given in table-3. MLE's and different comparison criteria are given in table-4, after the fitting of TWED and its special cases to the data sets considered.



3(a) - Emerical and fitted density curves 3(b) - Emperical and fitted CDF curves 3(c) - Emerical and fitted density curves 3(d) - Emperical and fitted CDF curves

Figure 3

Table 4: MLE's and different comparison criteria.

Data	Distn.	MLE's $\hat{\omega}$	$\hat{\beta}$	$\hat{\lambda}$	$-2\log(l)$	*BIC*	*AIC*	*AICC*
Remission Times in Months	TWED	0.270039	0.7111254	0.09362	822.282	833.1389	828.283	828.4764
	TED	0 *(known)*	-0.253742	0.12098	828.463	838.1673	832.463	832.5592
	TLBED	1 *(known)*	0.5811125	0.17429	843.465	853.1695	847.465	847.5614
	TABED	2 *(known)*	0.529122	0.28140	908.769	918.4730	912.769	912.8648
	WED	0.172346	0 *(known)*	0.12518	826.736	836.4390	830.736	830.8316
	ED	0 *(known)*	0 *(known)*	0.10677	828.684	833.5360	830.684	830.7155
	LBED	1 *(known)*	0 *(known)*	0.21357	853.593	858.4450	855.593	855.6247
	ABED	2 *(known)*	0 *(known)*	0.32031	921.996	926.8480	923.996	924.0281
Simulated Data	TWED	5.000380	0.499712	5.00070	82.4099	96.22547	88.4099	88.65996
	TED	0 *(known)*	-1.000000	1.26465	155.304	164.5152	159.305	159.4285
	TLBED	1 *(known)*	-0.999997	2.49925	106.107	115.3181	110.108	110.2315
	TABED	2 *(known)*	-0.999996	3.67172	87.0788	96.28916	91.0788	91.20253
	WED	5.168760	0 *(known)*	5.83644	87.4070	96.61740	91.4070	91.53077
	ED	0 *(known)*	0 *(known)*	0.93582	213.266	217.8717	215.267	215.3074
	LBED	1 *(known)*	0 *(known)*	1.72475	149.663	154.2685	151.663	151.7042
	ABED	2 *(known)*	0 *(known)*	2.71132	117.799	122.4045	119.799	119.8401

10 Conclusion

In this paper, we have introduced Transmuted Weighted Exponential distribution (TWED), which acts as a generalization to so many distributions viz. TED, TLBED, TABED, WED, LBED, ABED and classical ED. After introducing TWED, we investigated its different characteristic as well as structural properties. Two types of data sets have been considered in order to make comparison between special cases of TWED in terms of fitting. The data sets considered in this paper are both a real life and a simulated one. After the fitting of TWED and its special cases to the data sets considered, the results are given in Table-4. It is evident from the table-4 that, TWED possesses least values of AIC, AICC and BIC on its fitting, to both real life and simulated data set. Hence TWED will be treated as a best fitted distribution to the data sets given in table-3 as compared to its other special cases. Therefore, for the two data sets, distributions in the order of best fit are.

For Remission-times:

(Best) TWED→ED→WED→TED→TLBED→LBED→TABED→ABED (Good)

For Simulated Data:

(Best) TWED→TABED→WED→TLBED→ABED→LBED→TED→ED (Good)

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