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Moments of Upper Record Values from Marshall-Olkin Exponential Distribution

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Abstract: Marshall-Olkin exponential distribution has been established and studied by Salah, et, al. [10] and Salah [7]. In this paper, we study the upper record values from Marshall-Olkin exponential distribution. We derive several relations for the moments of the upper record values from Marshall-Olkin exponential distribution, these relations may then be used to compute all means, variances and covariances of the upper record values based on Marshall-Olkin exponential distribution.

Keywords: Order statistics; Marshall-Olkin exponential distribution; record values; upper record; single moments; product moments.
AMS Subject Classification: 62G30,62G99, 62E15.

1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. Assume that

$$Y_n = \max\{X_1, X_2, ..., X_n\}, for \quad n \geq 1. \tag{1}$$

Define X_j to be an upper record value of Eq.(1), if $X_j > Y_{j-1}$, for $j > 1$. Similarly one can define the lower record. Hence X_1 is an upper record as well as a lower record value. The indices at which the upper record values occur are given by the record times $\{U(n), n \geq 1\}$, where $U(n) = \min\{j : j > U(n-1), X_j > X_{U(n-1)}, n \geq 2\}$, with $U(1) = 1$, we assume that all upper record values $X_{U(i)}, i \geq 1$ occur at the sequence $\{X_n : n \geq 1\}$ of i.i.d. random variables.

The statistical study of record values of a sequence of i.i.d. continuos random variables was first carried out by Chandler [4]. For a survey on important result developed in this area one may refer to Nagaraja [3], Ahsanullah [5] and Arnold et al. [2]. Many statistical applications deals with record values such as sports and athletic events, oil and mining surveys, industrial stress testing, and bioscience. In the context of bioscience , suppose we are interested in the behavior of human organs as in kidneys or lungs. Similarly in the assessment of glucose level among diabetic patients, the researcher may be interested to study the behavior of the ordered records of glocagine. Knowledge of certain distributional function of record values sequence is adequate to determine the common distribution of the underlying observation. For example, the sequence of moment generating functions (MGF) is adequate for such a characterization. Moreover, obtain recurrence relations of the MGF of record values allow us to generate the MGF and then the moments of the entire sequence of record values. As a sequence, the estimation of the parameters involved in the parent distribution and prediction of the future record in the variational series can be derived based on moments. For more details and references, see Balakrishnan and Ahsanullah [11,12], Nagaraja [3], Ahsanullah [5], and Raqab [8,9].

Marshall and Olkin [1] introduced an interesting method for adding a new parameter to an existing distribution. The resulting new distribution, known as the Marshall-Olkin extended distribution, includes the original distribution as a special case and gives more flexibility to model various types of distributions.

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Suppose we have a given distribution with cdf say $H(x)$, the Marshall-Olkin extended distribution is defined by

$$F(x) = \frac{H(x)}{\alpha + (1-\alpha)H(x)}, \quad -\infty < x < \infty, \ \alpha > 0, \tag{2}$$

If we let,

$$H(x) = 1 - e^{-x}, \ \ 0 \le x < \infty, \tag{3}$$

where $H(x)$ is the cdf of the exponential distribution. Then substituting Eq. (3) into Eq. (2), we get the so called Marshall-Olkin exponential distribution, and for simplicity, let us denote it by MOE distribution. The cdf and the pdf of the MOE distribution is given respectively by

$$F(x) = 1 - \frac{\alpha e^{-x}}{1-(1-\alpha)e^{-x}}, \ 0 \le x < \infty, \alpha > 0. \tag{4}$$

$$f(x) = \frac{\alpha e^{-x}}{[1-(1-\alpha)e^{-x}]^2}, \ 0 \le x < \infty, \alpha > 0. \tag{5}$$

Note that When $\alpha = 1$, Eqs. (4) and (5) reduce to the standard exponential distribution which is the original distribution. Also when $\alpha = 2$, we have the half-logistic distribution, for more details see Salah (2010).

From Eqs.(4) and (5) one can be readily seen the following three relations

$$f(x) = [1-F(x)] + \frac{1-\alpha}{\alpha}[1-F(x)]^2, \ \alpha > 0, \tag{6}$$

$$f(x) = F(x)[1-F(x)] + \frac{1}{\alpha}[1-F(x)]^2, \ \alpha > 0, \tag{7}$$

$$f(x) = \frac{1}{\alpha}\left[1 + (\alpha-2)F(x) + (1-\alpha)F^2(x)\right], \ \alpha > 0. \tag{8}$$

The MOE distribution was introduced and studied by Salah et. al. [10] and Salah [6, 7]. They studied the order statistics and associated inferences from MOE distribution, they derived many recurrence relations in order to find the means, variances and covariances of MOE distribution based on order statistics.

In this paper, we derived recurrence relations for the single and product moments of upper record values.

2 Relations for Single Moments

In this section, we derive the single moments of upper record values from MOE distribution. To this end, let $\{X_j, \ \ j \ge 1\}$ be i.i.d. continuous random variable with cdf $F(x)$ and pdf $f(x)$. Let $T_0 = 1$, and $T_n = \min\{i : i > T_{n-1}, X_i > X_{T_{n-1}}\}$, then the sequence of upper record values $\{X_{U(n)}, \ n > 1\}$ is defined by $X_{U(n)} = X_{T_{n-1}}, n = 1, 2, \ldots$ and its pdf of the nth upper record $X_{U(n)}$ is given by

$$f_n(x) = \frac{1}{(n-1)!}\{-\log[1-F(x)]\}^{n-1} f(x), \quad -\infty < x < \infty, \ n = 1, 2, \ldots, \tag{9}$$

Suppose $\{X_j, \ \ j \ge 1\}$ be a sequence of i.i.d. random variable from MOE distribution with cdf and pdf as described in Eqs. (4) and (5) respectively, let us denote $E(X_{U(n)})$ by μ_n, $E(X^k_{U(n)})$ by $\mu_n^{(k)}$, $Var(X_{U(n)})$ by $\beta_{n,n}$, $E(X_{U(n)}X_{U(m)})$ by $\mu_{m,n}$ and $Cov(X_{U(n)}, X_{U(m)})$ by $\beta_{m,n}$. Then we have the following theorems satisfied by MOE distribution.

Theorem 1. *Let $\{X_{U(n)}, \ n > 1\}$ be the upper record values from MOE distribution, we have*

$$\mu_n = n + \log\alpha - \sum_{k=1}^{\infty}\left(\frac{\alpha-1}{\alpha}\right)^k \frac{1}{k(k+1)^n}, \ \ \alpha > 0.5. \tag{10}$$

*Proof.*from Eq.(9) we may write

$$\begin{aligned}\mu_n &= E(X_{U(n)}) \\ &= \iint_0^\infty x f_n(x)dx \\ &= \frac{1}{(n-1)!}\iint_0^1 F^{-1}(u)\{-\log[1-u]\}^{n-1}du,\end{aligned}$$

since from the cdf of MOE distribution we have

$$F^{-1}(u) = \log(1-(1-\alpha)u) - \log(1-u),$$

and

$$\begin{aligned}\mu_n = & \frac{1}{(n-1)!}\iint_0^1 \log(1-(1-\alpha)u)\{-\log[1-u]\}^{n-1}du \\ & + \frac{1}{(n-1)!}\iint_0^1 \{-\log[1-u]\}^{n-1}du,\end{aligned}$$

by taking the transformation $w = -\log[1-u]$, we obtain

$$\mu_n = \frac{1}{(n-1)!}\{\iint_0^\infty w^n e^{-w}dw + \iint_0^\infty w^{n-1}e^{-w}\log\left(1-\frac{1-\alpha}{\alpha}e^{-w}\right)dw + \log\alpha \iint_0^\infty w^{n-1}e^{-w}dw\}. \tag{11}$$

Upon using the following facts

$$\Gamma(n) = \iint_0^\infty t^{n-1}e^{-t}dt, \tag{12}$$

$$\log(1-e^{-y}) = -\sum_{k=1}^\infty \frac{e^{-ky}}{k}. \tag{13}$$

By applying Eqs. (12) and (13) in Eq. (11), we get

$$\mu_n = \frac{\Gamma(n+1)}{(n-1)!} + \frac{\Gamma(n)}{(n-1)!}\log\alpha + \frac{1}{(n-1)!}\sum_{k=1}^\infty \frac{1}{k}\left(\frac{\alpha-1}{\alpha}\right)^k \iint_0^\infty w^{n-1}e^{-(k+1)w}dw. \tag{14}$$

Then the result follow easily from Eq. (14).

Theorem 2.*The second moment for the upper record values from MOE distribution is given by*

$$\mu_n^{(2)} = n(n-1) + 2\mu_n - \sum_{r=2}^\infty \frac{1}{(r+1)^2}\sum_{i=1}^{r-1}\frac{1}{i}\frac{1}{r-i}\left(\frac{\alpha-1}{\alpha}\right)^{i+2}, \quad n \geq 1, \quad \alpha > 0.5. \tag{15}$$

*Proof.*From Eq. (9), we have

$$\begin{aligned}
\mu_n^{(2)} &= \iint_0^\infty x^2 f_n(x)dx \\
&= \frac{1}{(n-1)!}\iint_0^1 \left[F^{-1}(u)\right]^2 \{-\log[1-u]\}^{n-1} du \\
&= \frac{1}{(n-1)!}\iint_0^1 [\log(1-(1-\alpha)u)]^2 \{-\log[1-u]\}^{n-1} du \\
&+ \frac{2}{(n-1)!}\iint_0^1 \log(1-(1-\alpha)u)\{-\log[1-u]\}^n du \\
&+ \frac{1}{(n-1)!}\iint_0^1 \{-\log[1-u]\}^{n+1} du. \qquad (16)
\end{aligned}$$

By taking the transformation $w = -\log[1-u]$, and upon using Eqs. (12) and (13) then Eq.(16) will be

$$\begin{aligned}
\mu_n^{(2)} &= \frac{\Gamma(n+2)}{(n-1)!} + \frac{2}{(n-1)!}\iint_0^\infty \log\left(1-(1-\alpha)\left(1-e^{-w}\right)\right) w^{n-1}e^{-w}dw \\
&+ \frac{1}{(n-1)!}\iint_0^\infty \left\{\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\alpha-1}{\alpha}\right)^k e^{-kw}\right\}^2 w^{n-1}e^{-w}dw. \qquad (17)
\end{aligned}$$

If we suppose that $\pi_r =$ the coefficient of e^{-rw} in $\left\{\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\alpha-1}{\alpha}\right)^k e^{-kw}\right\}^2$ and with the help of Eq.(12), then we may write Eq.(17) as

$$\begin{aligned}
\mu_n^{(2)} &= n(n+1) + 2(\mu_n - n) + \frac{1}{(n-1)!}\sum_{r=2}^{\infty}\pi_r \iint_0^\infty w^{n-1}e^{-(k+1)w}dw \\
&= n(n-1) + 2\mu_n - \sum_{r=2}^{\infty}\pi_r \frac{1}{(r+1)^n}. \qquad (18)
\end{aligned}$$

Note that

$$\begin{aligned}
\pi_r &= the\ coefficient\ of\ \ e^{-rw}\ in\ \left\{\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\alpha-1}{\alpha}\right)^k e^{-kw}\right\}^2 \\
&= \left\{\frac{\alpha-1}{\alpha}e^{-w} + \left(\frac{\alpha-1}{\alpha}\right)^2\frac{e^{-2w}}{2} + \left(\frac{\alpha-1}{\alpha}\right)^3\frac{e^{-3w}}{3} + \ldots\right\} \\
&\times \left\{\frac{\alpha-1}{\alpha}e^{-w} + \left(\frac{\alpha-1}{\alpha}\right)^2\frac{e^{-2w}}{2} + \left(\frac{\alpha-1}{\alpha}\right)^3\frac{e^{-3w}}{3} + \ldots\right\} \\
&= \sum_{i=1}^{r-1}\frac{1}{i}\frac{1}{r-i}\left(\frac{\alpha-1}{\alpha}\right)^{i+2}, \quad \alpha > 0.5. \qquad (19)
\end{aligned}$$

The result follows from Eqs. (18) and (19).

Remark. The variances for the upper record values from MOE distribution follows by substituting Eqs.(10) and (15) in the following equation

$$Var(X_{U(n)}) = \beta_{n,n} = \mu_n^{(2)} - (\mu_n)^2 .$$

3 Relations for Product Moments

Consider the joint density function of $X_{U(n)}$ and $X_{U(m)}$ to be

$$\begin{aligned} f_{m,n}(x,y) = c_{m,n}\{-\log[1-F(x)]\}^{m-1}\frac{f(x)}{1-F(x)} \\ \times\{-\log[1-F(y)]+\log[1-F(x)]\}^{n-m-1}f(y), \\ -\infty < x < y < \infty; \quad m,n = 1,2,..., \; m < n, \end{aligned} \tag{20}$$

where

$$c_{m,n} = \frac{1}{(m-1)!\,(n-m-1)!}.$$

In this section, we derive the product moments of upper record values from MOE distribution.

Theorem 3.*For the upper record values from MOE distribution, we have*

$$\begin{aligned} \mu_{m,n} = (n-m+\log\alpha)+m\mu_m+\sum_{k=1}^{\infty}\left(\frac{\alpha-1}{\alpha}\right)^k\frac{1}{k\,(k+1)^n}\left(\log\alpha+\frac{m}{k+1}\right) \\ +\sum_{k=1}^{\infty}\left(\frac{\alpha-1}{\alpha}\right)^k\frac{1}{k\,(k+1)^{n-m}}\sum_{j=1}^{\infty}\left(\frac{\alpha-1}{\alpha}\right)^j\frac{1}{j\,(k+j+1)^m}, \end{aligned} \tag{21}$$

where $\alpha > 0.5, \; 1 \le m < n.$

*Proof.*From Eq.(20) we have

$$\begin{aligned} \mu_{m,n} &= \iint_0^{\infty}\iint_x^{\infty} xy\,f_{m,n}(x,y)dydx \\ &= c_{m,n}\iint_0^{1}\iint_u^{1} F^{-1}(u)\{-\log[1-u]\}^{m-1}\frac{1}{1-u}I(u)du. \end{aligned} \tag{22}$$

where

$$I(u) = \iint_u^1 F^{-1}(w)\{-\log[1-w]+\log[1-u]\}^{n-m-1}dw. \tag{23}$$

Upon using Eq.(12), then Eq. (23) will be

$$I(u) = \iint_u^1 \{\log[1-(1-\alpha)w]-\log[1-w]\}\{-\log[1-w]+\log[1-u]\}^{n-m-1}dw.$$

By taking the transformation $t = -\log[1-w]+\log[1-u]$, we have for $I(u)$ after simplification

$$I(u)=(1-u)\{\iint\limits_0^\infty \log\left[1-\frac{1-\alpha}{\alpha}(1-u)e^{-t}\right]t^{n-m-1}e^{-t}dt+\log\alpha\iint\limits_0^\infty t^{n-m-1}e^{-t}dt$$
$$+\iint\limits_0^\infty t^{n-m}e^{-t}dt-\log(1-u)\iint\limits_0^\infty t^{n-m-1}e^{-t}dt\}.$$

Using Eqs.(12) and (13) in the last equation to get

$$I(u)=(1-u)(n-m-1)!\{\sum_{k=1}^\infty\left(\frac{\alpha-1}{\alpha}\right)^k\frac{(1-u)^k}{k(k+1)^{n-m}}+(n-m)$$
$$-\log(1-u)+\log\alpha\}. \tag{24}$$

Substituting Eq.(24) into Eq.(22) to get

$$\mu_{m,n}=\frac{1}{(m-1)!}\{(n-m+\log\alpha)\iint\limits_0^1\log[1-(1-\alpha)u]\{-\log(1-u)\}^{m-1}du$$
$$+(n-m+\log\alpha)\iint\limits_0^1\{-\log(1-u)\}^m du+\iint\limits_0^1\{-\log(1-u)\}^{m+1}du$$
$$+\iint\limits_0^1\log[1-(1-\alpha)u]\{-\log(1-u)\}^m du$$
$$+\sum_{k=1}^\infty\left(\frac{\alpha-1}{\alpha}\right)^k\frac{1}{k(k+1)^{n-m}}[\iint\limits_0^1(1-u)^k\log[1-(1-\alpha)u]\{-\log(1-u)\}^{m-1}du$$
$$+\iint\limits_0^1(1-u)^k\{-\log(1-u)\}^m du]\}. \tag{25}$$

Simplifying Eq. (25) to get

$$\mu_{m,n}=(n-m+\log\alpha)\mu_m+m\mu_{m+1}+\log\alpha\sum_{k=1}^\infty\left(\frac{\alpha-1}{\alpha}\right)^k\frac{1}{k(k+1)^n}$$
$$+m\sum_{k=1}^\infty\left(\frac{\alpha-1}{\alpha}\right)^k\frac{1}{k(k+1)^{n+1}}+A, \tag{26}$$

where

$$A=\frac{1}{(m-1)!}\sum_{k=1}^\infty\left(\frac{\alpha-1}{\alpha}\right)^k\frac{1}{k(k+1)^{n-m}}\iint\limits_0^\infty\log\left[1-\frac{1-\alpha}{\alpha}e^{-t}\right]t^{m-1}e^{-(k+1)t}dt.$$

Upon using Eq.(13) we have for A

$$A=\frac{1}{(m-1)!}\sum_{k=1}^\infty\left(\frac{\alpha-1}{\alpha}\right)^k\frac{1}{k(k+1)^{n-m}}\sum_{j=1}^\infty\left(\frac{\alpha-1}{\alpha}\right)^j\frac{1}{j}\iint\limits_0^\infty t^{m-1}e^{-(k+j+1)t}dt$$
$$=\sum_{k=1}^\infty\left(\frac{\alpha-1}{\alpha}\right)^k\frac{1}{k(k+1)^{n-m}}\sum_{j=1}^\infty\left(\frac{\alpha-1}{\alpha}\right)^j\frac{1}{j(k+j+1)^m}. \tag{27}$$

The result follow directly from Eqs. (26) and (27).

Remark. The covariance $\beta_{m,n}$ of the upper record from MOE distribution follows directly from Theorems (2.1) and (3.1) as

$$\beta_{m,n} = \mu_{m,n} - \mu_n \mu_m, \quad 1 \leq m < n.$$

4 Conclusion

In this paper, the moments of the upper record values from Marshall-Olkin exponential distribution are derived. Means, variances and covariances of the upper record values from MOE distribution can be computed directly from Remark (2.1) and (3.1). Records from MOE distribution are still an open problem and need more study and research dealing with it. One of the open problem is studying the Fisher information matrix for the observed and complete samples and the statistical inferences for the record values from the MOE distribution which is our next research paper.

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