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An Efficient Approach for Time–Fractional Damped Burger and Time–Sharma–Tasso–Olver Equations Using the FRDTM

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Abstract: The objective of this paper is to use the fractional reduced differential transform method (FRDTM) to find approximate analytical solutions to the time-fractional Sharma–Tasso–Olver equation and the time-fractional damped Burger equation. The fractional derivatives are described in the Caputo sense. We compare our results with those from existing methods such as the homotopy analysis method (HAM), variation iteration method (VIM) and the Adomian decomposition method (ADM). Also, the results we obtained in this paper are in a good agreement with the exact solutions; hence, this technique is powerful and efficient as an alternative method for finding approximate and exact solutions for nonlinear fractional PDEs.

Keywords: Reduced Differential Transform Method (RDTM), Sharma Tasso Olver (STO) equation, Schrodinger equation, Telegraph, equation, Approximate solutions, Analytical solutions.

1 Introduction

In recent years, there is an increase of the number of mathematical modeling in physical applications that arises in diverse fields of physics and engineering which usually result in nonlinear fractional partial and ordinary differential equations. So, a huge interest in them has been aroused recently due to their widespread applications. Finding analytic numerical solutions for these fractional differential equations is very important in applied mathematics. It is worth mentioning here that there exists no method, in general, that gives an exact solution for fractional differential equation, and so finding approximate solutions is valuable in science. Two of the fractional differential equations arising in science and engineering are the fractional Sharma–Tasso–Olver and the fractional damped Burger equation with time-fractional derivatives.

Many authors used numerical and analytic methods to solve linear and non-linear fractional equations. A few of these methods to name: the Differential Transform Method (DTM) [24, 25, 26], the Adomian Decomposition Method (ADM) [11, 17, 30, 31], the Variational Iteration Method (VIM) [11, 36] and the Homotopy Perturbation Method (HPM) [28, 37]. The RDTM was first introduced by Y. Keskin in his Ph.D. [14, 15, 16] which is presented to overcome very complicated calculations. This method unlike the traditional DTM techniques it provides us with approximate solution and in some cases an exact solution, in a rapidly convergent power series. Usually, a few number of iterations needed of the series solution for numerical purposes with high accuracy.

Recently, Keskin and Oturanc [15] used the FRDTM to solve fractional partial differential equations. Finally, Esen A, Tasbozan O and Yagmurlu M [7, 8], used the HAM to obtained approximate solution of the fractional Sharma–Tasso–Olver equation and the fractional damped burger and Cahn–Allen equations.

First, we consider the nonlinear time-fractional Damped Burger equation of the form:

$$D_t^{\alpha} u(x,t) + u(x,t) Du(x,t) - D_x^2 u(x,t) + \lambda u(x,t) = 0, t > 0,\ 0 < \alpha \leq 1 \tag{1}$$

subject to the initial condition

$$u(x,0) = \lambda x, \tag{2}$$

where α is a parameter describing the order of the fractional derivative and $u(x,t)$ is a function of x and t. Note that we are using the fractional derivative in the Caputo sense. Moreover, the exact solution of the Damped Burger equation with $\alpha = 1$ is given by [30]:

$$u(x,t) = \frac{\lambda x}{2e^{\lambda t} - 1}, \tag{3}$$

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where λ is a constant.

Second, we consider the nonlinear fractional Sharma–Tasso–Olver equation of the form:

$$D_t^{\alpha}u(x,t)+aD_xu^3(x,t)+\frac{3}{2}aD_x^2u^2(x,t)+aD_x^3u(x,t)=0,\ t>0,0<\alpha\leq 1 \tag{4}$$

subject to the initial condition

$$u(x,0)=\sqrt{\frac{1}{a}}\tanh\left(\sqrt{\frac{1}{a}}x\right), \tag{5}$$

where a is a constant and α is a parameter describing the order of the fractional derivative and $u(x,t)$ is a function of x and t. Moreover, the exact solution of the Sharma–Tasso–Olver equation with $\alpha=1$ is given by:

$$u(x,t)=\sqrt{\frac{1}{a}}\tanh\left(\sqrt{\frac{1}{a}}(x-t)\right). \tag{6}$$

The rest of this paper is organized as follows: In Section 2, we give some important facts and definitions related to fractional calculus. In section 3, the fractional reduced differential transform method is introduced. Sections 4 and 5 are devoted to apply the method to a test problems and present graphs to show the effectiveness of the FRDTM for some values of x and t. Section 6 we present tables of numerical calculations. Finally, section 7 discussion and conclusion of this paper.

2 Preliminary of Fractional Calculus

In this section, we present some of the main definitions and facts that we will use in this research paper. Some of these basic definitions are due to Liouville see, [12, 13]:

Definition 1. *A real function $f(x)$, $x>0$ is said to be in the space C_μ, $\mu\in\mathbb{R}$ if there exists a real number $q(>\mu)$, such that $f(x)=x^q g(x)$, where $g(x)\in C[0,\infty)$, and it is said to be in the space C_μ^m if $f^{(m)}\in C_\mu$, $m\in\mathbb{N}$.*

Definition 2. *For a function f, the Riemann–Liouville fractional integral operator of order $\alpha\geq 0$, is defined as*

$$\left\{\begin{array}{l} J^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)\,dt,\ \alpha>0,\ x>0 \\ J^0 f(x)=f(x) \end{array}\right\}. \tag{7}$$

Caputo and Mainardi [13] presented a modified fractional differentiation operator D^α in their work on the theory of viscoelasticity to overcome the disadvantages of the Riemann–Liouville derivative when someone tries to model real world problems.

Definition 3. *The fractional derivative of f in the Caputo sense can be defined as*

$$D^{\alpha}f(x)=J^{m-\alpha}D^m f(x)$$
$$=\frac{1}{\Gamma(m-\alpha)}\int_0^x(x-t)^{m-\alpha-1}f^{(m)}(t)dt,m-1<\alpha\leq m,\ m\in\mathbb{N},$$
$$x>0,\ f\in C_{-1}^m.$$

Lemma 4. If $m-1<\alpha\leq m$, $m\in\mathbb{N}$and$f\in C_\mu^m,\mu\geq -1$, then

$$\left\{\begin{array}{l} D^{\alpha}J^{\alpha}f(x)=f(x),\ x>0 \\ J^{\alpha}D^{\alpha}f(x)=f(x)-\sum_{k=0}^{m-1}f^{(k)}(0^+)\frac{x^k}{k!},\ m-1<\alpha<m \end{array}\right\}. \tag{8}$$

We would like to mention here, the Caputo fractional derivative is used because it allows traditional initial and boundary conditions to be included in the formulation of our problem.

3 Methodology of the FRDTM

In this section, we will give the methodology of the FRDTM. So let's start with a function of two variables $u(x,t)$ which is analytic and $k-$times continuously differentiable with respect to time t and space x in the domain of our interest. Assume we can represent this function as a product of two single-variable functions $u(x,t)=f(x).g(t)$. From the definitions of the DTM, the function can be represented as follows:

$$u(x,t)=\left(\sum_{i=0}^{\infty}F(i)x^i\right)\left(\sum_{j=0}^{\infty}G(j)t^j\right)=\sum_{k=0}^{\infty}U_k(x)t^k \tag{9}$$

where $U(i,j)=F(i).G(j)$ is called the spectrum of $u(x,t)$. Some basic operations of the reduced differential transformation can be obtained as follows [15, 16]:

Definition 4. *If $u(x,t)$ is analytic and continuously differentiable with respect to space variable x and time t in the domain of interest, then the t-dimensional spectrum function*

$$U_k(x)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{\alpha k}}{\partial t^{\alpha k}}u(x,t)\right]_{t=t_0} \tag{10}$$

is the reduced transformed function of $u(x,t)$, where α is a parameter which describes the order of time-fractional derivative.

Throughout this paper, $u(x,t)$ represents the original function and $U_k(x)$ represents the reduced transformed function. The differential inverse transform of $U_k(x)$ is given by

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)(t-t_0)^{k\alpha}. \tag{11}$$

From equations (11) and (10) one can deduce

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{\alpha k}}{\partial t^{\alpha k}}u(x,t)\right]_{t=t_0}(t-t_0)^{\alpha k} \tag{12}$$

Note that when $t=0$, Eq.(12) becomes

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{\alpha k}}{\partial x^{\alpha k}}u(x,t)\right]_{t=0}t^{k\alpha}. \tag{13}$$

Note that from the above discussion, one can realize that the RDTM is derived from the power series expansion of a function. Some basic operations of the reduced differential transformation obtained from equations (10) and (11) are given in the table below:

Table 1. Basic operations of the FRDTM [13, 14, 15]

Functional Form	Transformed form
$u(x,t)$	$\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=0}$
$\gamma u(x,t)\pm\beta v(x,t)$	$\gamma U_k(x)\pm\beta V_k(x)$, γ and β are constant.
$u(x,t).v(x,t)$	$\Sigma_{i=0}^{k}U_i(x)\,V_{k-i}(x)$
$u(x,t).v(x,t).w(x,t)$	$\Sigma_{i=0}^{k}\Sigma_{j=0}^{i}U_j(x)\,V_{i-j}(x)\,W_{k-i}(x)$
$\frac{\partial^{n\alpha}}{\partial t^{n\alpha}}u(x,t)$	$\frac{\Gamma(k\alpha+n\alpha+1)}{\Gamma(k\alpha+1)}U_{k+n}(x)$
$\frac{\partial^{n}}{\partial x^{n}}u(x,t)$	$\frac{\partial^{n}}{\partial x^{n}}U_k(x)$
$x^m t^n u(x,t)$	$x^m U_{k-n}(x)$
$x^m t^n$	$x^m\delta(k\alpha-n)$, where $\delta(k\alpha-n)=\left\{\begin{matrix}1, & \alpha k=n\\ 0, & \alpha k\neq n\end{matrix}\right\}$

Remark. In Table 1, Γ represents the Gamma function, which is defined by

$$\Gamma(z):=\int_0^\infty e^{-t}t^{z-1}dt,\ \ z\in\mathbb{C}. \tag{14}$$

Notice that the Gamma function is the continuous extension to the fractional function. Throughout this paper, we will be using the recursive relation $\Gamma(z+1)=z\Gamma(z), z>0$ to calculate the value of the Gamma function of all real numbers by knowing only the value of the Gamma function between 1 and 2.

Now, we illustrate the basic idea of the FRDTM by considering a general fractional nonlinear nonhomogeneous partial differential equation with initial condition of the form

$$D_t^\alpha U(x,t)+R\left(U(x,t)\right)+N\left(U(x,t)\right)=h(x,t), \tag{15}$$

subject to the initial conditions

$$U(x,0)=f(x),\ U_t(x,0)=g(x), \tag{16}$$

where $D_t^\alpha U(x,t)$ is the Caputo fractional derivative of the function $U(x,t)$, R is the linear differential operator, N represents the general nonlinear operator and $h(x,t)$ is the source term.

Applying the FRDTM to both sides of Eq. (15), we obtain

$$L\left(u(x,t)\right)+R\left(u(x,t)\right)+N\left(u(x,t)\right)=L\left(h(x,t)\right). \tag{17}$$

Using the FRDTM formulas in Table 1, we can find:

$$L\left(U(x,t)\right)=f(x)+u^\alpha L\left(h(x,t)\right)-u^\alpha L\left(R\left(U(x,t)\right)-N\left(U(x,t)\right)\right). \tag{18}$$

Using the FRDTM inverse transform on both sides of Eq. (19)to get

$$U(x,t)=H(x,t)-L^{-1}\left(u^\alpha L\left(R\left(U(x,t)\right)+N\left(U(x,t)\right)\right)\right). \tag{19}$$

where $H(x,t)$ represents the term coming from the source term and the prescribed initial conditions. Now from equation (17), we can write the initial conditions as:

$$U_0(x)=f(x);\ U_1(x)=g(x) \tag{20}$$

To find all other iterations, we first substitute equation (21) into equation (20) and then we find all the values of $U_k(x)$. Finally, we apply the inverse transformation to all the values $\{U_k(x)\}_{k=0}^n$ to obtain the approximate solution:

$$\widehat{u}\,(x,t)=\sum_{k=0}^{n}U_k(x)t^{k\alpha}, \tag{21}$$

where n is the number of iterations we need to find the intended approximate solution.

Hence, the exact solution of our problem is given by $u(x,t)=\lim_{n\to\infty}\widehat{u}\,(x,t)$.

4 Solution of Time–Fractional Damped Burger Equation by the FRDTM

In this section, we apply the RDTM to the nonlinear time-fractional damped burger equation.

4.1 Time–fractional damped burger equation

First, consider the time–fractional damped burger equation:

$$D_t^\alpha u(x,t)+u(x,t)Du(x,t)-D_x^2u(x,t)+\lambda u(x,t)=0,\ t>0,\ 0<\alpha\le 1 \tag{22}$$

subject to the initial condition

$$u(x,0)=\lambda x, \tag{23}$$

where the exact solution of the non-fractional Damped Burger equation with $\alpha=1$ [30] is

$$u(x,t)=\frac{\lambda x}{2e^{\lambda t}-1}, \tag{24}$$

where λ is a constant.
Applying the FRDTM to Eq. (22) and Eq. (23) we get

$$U_{k+1}(x)=\frac{\Gamma(k\alpha+\alpha)}{\Gamma(k\alpha+\alpha+1)}\left(\frac{d^2}{dx^2}U_k(x)-\lambda U_k(x)-\sum_{i=0}^{k}U_i(x)\frac{d}{dx}U_{k-i}(x)\right), \tag{25}$$

where the initial condition

$$U_0(x)=\lambda x. \tag{26}$$

Now, substitute Eq. (26) into Eq. (25) to obtain the following:

$$U_1(x)=-\frac{2x\lambda^2\Gamma(\alpha)}{\Gamma(\alpha+1)}=-\frac{2x\lambda^2}{\alpha},U_2(x)=\frac{6x\lambda^3\Gamma(\alpha)\Gamma(2\alpha)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}=\frac{3x\lambda^3}{\alpha^2}$$

$$U_3(x)=-\frac{11x\lambda^4}{3\alpha^3},U_4(x)=\frac{25x\lambda^5}{4\alpha^4},\ldots$$

We continue in this manner and after a few iterations, the differential inverse transform of $\{U_k(x)\}_{k=0}^{\infty}$ will provide us with the following approximate solution:

$$\begin{aligned}\widehat{u}(x,t) &= \Sigma_{k=0}^{\infty} U_k(x)t^{\alpha k} = U_0(x) + U_1(x)t^{\alpha} + U_2(x)t^{2\alpha} + \ldots \\ &= \lambda x - \frac{2x\lambda^2}{\alpha}t^{\alpha} + \frac{3x\lambda^3}{\alpha^2}t^{2\alpha} - \frac{11x\lambda^4}{3\alpha^3}t^{3\alpha} + \frac{25x\lambda^5}{4\alpha^4}t^{4\alpha} + \ldots \\ &= \lambda x\left(1 - \frac{2\lambda}{\alpha}t^{\alpha} + \frac{3\lambda^2}{\alpha^2}t^{2\alpha} - \frac{11\lambda^3}{3\alpha^3}t^{3\alpha} + \frac{25\lambda^4}{4\alpha^4}t^{4\alpha} + \ldots\right)\end{aligned} \tag{27}$$

Hence, the approximate solution is convergent rapidly to the exact solution. Also, it only takes few terms to get analytic function. Now, we calculate numerical results of the approximate solution $u(x,t)$ for different values of $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 0.75$, $\alpha = 0.9$ and different values of x and t.

The numerical results of the approximate solution obtained by FRDTM and exact solution given by Esen [7] are shown in figures 1(a)-1(d) when $\lambda = 1$ for different values of x, t and α.

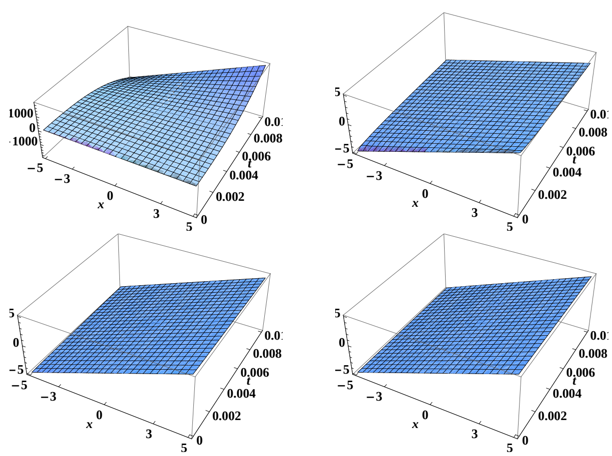

Fig. 1: The approximate solution for example 4.1 when $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 0.75$ and $\alpha = 0.90$ respectively

From figure 1 above one can observe that the values of the approximate solution at different grid points and different values of αobtained by FRDTM are close to the values of the exact solution with high accuracy and the accuracy increases as the order of approximation increases.

4.2 Numerical Examples

To show the efficiency of the present method, we compare the FRDTM solutions of time-fractional damped burger equation for $\alpha = 1$, $\lambda = 1$ with the exact solution given by $u(x,t) = \frac{x}{2e^t - 1}$, see [30].

Consider the nonlinear damped PDE when $\alpha = 1,\ \lambda = 1$ given by:

$$u_t + u u_x - u_{xx} + u = 0, \tag{28}$$

subject to the initial condition

$$u(x,0) = x. \tag{29}$$

Applying the FRDTM to Eq. (28) and Eq. (29) we get

$$U_{k+1}(x) = \frac{1}{k+1}\left(\frac{d^2}{dx^2}U_k(x) - U_k(x) - \sum_{i=0}^{k} U_i(x)\frac{d}{dx}U_{k-i}(x)\right), \tag{30}$$

where the initial condition

$$U_0(x) = x, \tag{31}$$

where the $U_k(x)$, is the transform function of the $t-$dimensional spectrum.

Now substitute Eq. (31) into Eq. (32) and for $k \geq 1$ we obtain

$$U_1(x) = -2x,\ U_2(x) = 3x,\ U_3(x) = -\frac{13x}{3},\ U_4(x) = \frac{25x}{4}, \ldots \tag{32}$$

So after a few iterations, the differential inverse transform of $\{U_k(x)\}_{k=0}^{\infty}$ will give the following approximate solution:

$$\begin{aligned}\widehat{u}(x,t) &= \sum_{k=0}^{\infty} U_k(x)t^k \\ &= U_0(x) + U_1(x)t + U_2(x)t^2 + U_3(x)t^3 + \ldots \\ &= x - 2xt + 3xt^2 - \frac{13x}{3}t^3 + \frac{25x}{4}t^4 - \frac{541x}{60}t^5 + \ldots \\ &= x\left(1 - 2t + 3t^2 - \frac{13t^3}{3} + \frac{25t^4}{4} - \frac{541t^5}{60} + \ldots\right) \\ &= \frac{x}{2e^t - 1}.\end{aligned}$$

This is the exact solution of the standard damped burger equation in Eq.(28).

5 Solution of Time–Fractional Sharma–Tasso–Olver Equation by the FRDTM

In this section, we apply the RDTM to the time-fractional Sharma–Tasso–Olver equation.

5.1 Time–Fractional Sharma–Tasso–Olver Equation

Consider the nonlinear time-fractional Sharma–Tasso–Olver equation which is given by:

$$D_t^{\alpha}u(x,t) + aD_x u^3(x,t) + \frac{3}{2}aD_x^2 u^2(x,t) + aD_x^3 u(x,t) = 0,\ t > 0, 0 < \alpha \leq 1, \tag{33}$$

subject to the initial condition

$$u(x,0) = \sqrt{\frac{1}{a}}\tanh\left(\sqrt{\frac{1}{a}}x\right), \tag{34}$$

where the exact solution of the Sharma–Tasso–Olver equation with $\alpha = 1$ is given by

$$u(x,t) = \sqrt{\frac{1}{a}} \tanh\left(\sqrt{\frac{1}{a}}(x-t)\right). \tag{35}$$

Applying the FRDTM to Eq. (33) and Eq. (34) we get

$$\begin{aligned} U_{k+1}(x) &= \frac{-\Gamma(k\alpha+\alpha)}{\Gamma(k\alpha+\alpha+1)}\left(a\frac{d^3}{dx^3}U_k(x) + \frac{3}{2}a\frac{d^3}{dx^3}\left(\sum_{i=0}^{k} U_i(x)U_{k-i}(x)\right)\right) \\ &+ \frac{-a\Gamma(k\alpha+\alpha)}{\Gamma(k\alpha+\alpha+1)}\left(\frac{d}{dx}\left(\sum_{i=0}^{k}\sum_{j=0}^{i} U_{k-i}(x)U_{i-j}(x)U_j(x)\right)\right). \end{aligned} \tag{36}$$

where the initial condition

$$U_0(x) = \sqrt{\frac{1}{a}} \tanh\left(\sqrt{\frac{1}{a}}x\right) \tag{37}$$

Now, substitute Eq. (37) into Eq. (36) to obtain the following:

$$U_1(x) = \frac{-sech^2\left(\sqrt{\frac{1}{a}}x\right)}{a\alpha}$$

$$U_2(x) = \frac{-\left(\frac{1}{a}\right)^{3/2} sech^2\left(\sqrt{\frac{1}{a}}x\right) tanh\left(\sqrt{\frac{1}{a}}x\right)}{\alpha^2}$$

$$U_3(x) = \frac{sech^6\left(\sqrt{\frac{1}{a}}x\right)\left(\left(3+2cosh\left(2\sqrt{\frac{1}{a}}x\right)-cosh\left(4\sqrt{\frac{1}{a}}x\right)\right)\right)}{12a^2\alpha^3}.$$

We continue in this manner and after a few iterations, the differential inverse transform of $\{U_k(x)\}_{k=0}^{\infty}$ will provide us with the following approximate solution:

$$\begin{aligned} \widehat{u}(x,t) &= \sum_{k=0}^{\infty} U_k(x) t^k \\ &= U_0(x) + U_1(x)t^{\alpha} + U_2(x)t^{2\alpha} + \ldots. \end{aligned}$$

Hence, the approximate solution is convergent rapidly to the exact solution. Now, we calculate numerical results of the approximate solution $u(x,t)$ for different values of $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 0.75$, $\alpha = 0.90$ and different values of x and t.

The numerical results for the approximate solution obtained by FRDTM and the exact solution given in Eq. (35) are shown in figures below for a constant value of $a = 4$ and for different values of x, t and α.
From figure 2 below one can observe that the values of the approximate solution at different grid points and different values of α obtained by FRDTM are close to the values of the exact solution with high accuracy and the accuracy increases as the order of approximation increases.

5.2 Application of the RDTM

In this section, we describe the method explained in section 2 by considering a numerical example to show the efficiency and the accuracy of the RDTM. This example was done by the author, see [32].

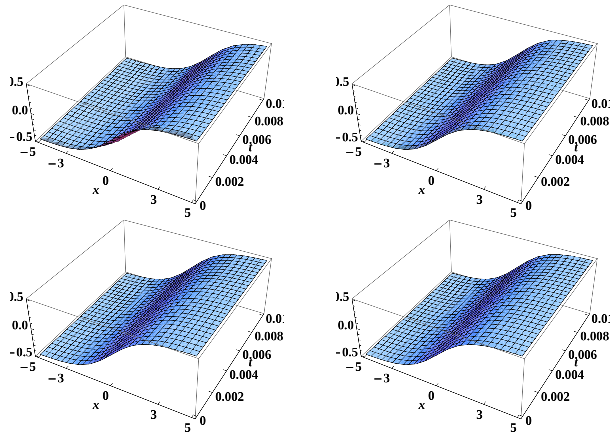

Fig. 2: The approximate solution for example 5.1 when $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 0.75$ and $\alpha = 0.90$ respectively

$$u_t + \alpha\left(u^3\right)_x + \frac{3}{2}\alpha(u^2)_{xx} + \alpha u_{xxx} = 0, \tag{38}$$

where α is a constant.

In the case when $\alpha = 4$, the STO becomes [32]:

$$u_t + 4\left(u^3\right)_x + 6(u^2)_{xx} + 4u_{xxx} = 0, \tag{39}$$

subject to the initial conditions

$$u(x,0) = \frac{1}{2}tanh\left(\frac{x}{2}\right); \quad u_t(x,0) = \frac{-1}{4}sech^2\left(\frac{x}{2}\right), \tag{40}$$

where the exact solution is

$$u(x,t) = \frac{1}{2}tanh\left(\frac{x-t}{2}\right). \tag{41}$$

Applying the FRDTM to Eq. (18) and Eq. (17), we obtain the recursive relation

$$\begin{aligned} U_{k+1}(x) &= \frac{-1}{k+1}\left(4\frac{d^3U_k(x)}{dx^3} + 4\frac{d}{dx}\left(\sum_{i=0}^{k}\sum_{j=0}^{i} U_{i-j}(x)U_j(x)U_{k-i}(x)\right)\right) \\ &+ \frac{-6}{k+1}\left(\frac{d^2}{dx^2}\left(\sum_{i=0}^{k} U_i(x)U_{k-i}(x)\right)\right). \end{aligned} \tag{42}$$

where the $U_k(x)$, is the transform function of the $t-$dimensional spectrum. Note that

$$U_0(x) = \frac{1}{2}tanh\left(\frac{x}{2}\right); \quad U_1(x) = \frac{-1}{4}sech^2\left(\frac{x}{2}\right). \tag{43}$$

Now, substitute Eq. (21) into Eq. (20) to obtain the following:

$$U_2(x) = \frac{sinh(x)}{4(1+cosh(x))^2}$$

$$U_3(x) = -\frac{1}{48}(cosh(x)-2)sech^4\left(\frac{x}{2}\right)$$

$$U_4(x) = -\frac{(cosh(x)-5)tanh\left(\frac{x}{2}\right)}{48(1+cosh(x))^2}.$$

We continue in this manner and after a few iterations, the differential inverse transform of $\{U_k(x)\}_{k=0}^{\infty}$ will provide us with the following approximate solution:

$$\begin{aligned}\widehat{u}\,(x,t) &= \sum_{k=0}^{\infty} U_k(x)\,t^k \\ &= U_0(x)+U_1(x)\,t+U_2(x)\,t^2+U_3(x)\,t^3+\ldots. \\ &= \frac{1}{2}tanh\left(\frac{x}{2}\right)-\frac{1}{4}sech^2\left(\frac{x}{2}\right)t-\frac{sinh(x)}{4(1+cosh(x))^2}t^2 \\ &- \frac{1}{48}(cosh(x)-2)sech^4\left(\frac{x}{2}\right)t^3+\ldots.\end{aligned}$$

Hence, the approximate solution converges rapidly to the exact solution and the exact solution of the problem is given by $u(x,t)=\lim_{n\to 0}\widehat{u}_n\,(x,t)$.

From figure 3 below one can observe that the values of the approximate solution at different grid points obtained by FRDTM are very close to the values of the exact solution with high accuracy with only five iterations and the accuracy increases as the order of approximation increases.

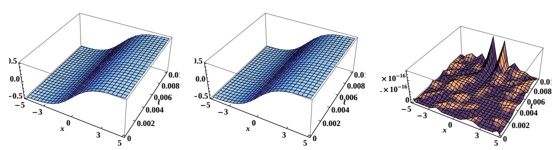

Fig. 3: The approximate, exact solutions and absolute error, respectively for example 3.1 when $-5<x<5$ and $0<t<0.01$

Also figure 4 below shows the exact solution, approximate solution of $u(x,t)$ for the values of $x=-5,-3,3,5$ and $t=0.02,0.04,0.06,0.08,0.1$.

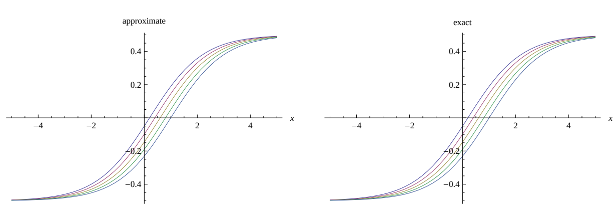

Fig. 4: The approximate and exact solutions for example 3.1 when $-5<x<5$ and $0<t<0.1$

6 Tables of Numerical Calculations

The comparison of the results of the FRDTM and the exact solution for $\alpha=1$ is given in table 2 and table 3 for different values of x and t. We present in table 2 the results obtained by the FRDTM for different values of α with only the 5th order approximate solution $u(x,t)$ and the exact solution given in Eq. (24) with $\lambda=1$. Also, in table 3 we present the results obtained by the FRDTM with 5th order approximate solution $u(x,t)$ and the exact solution given in Eq. (35) with $a=4$.

Table 2 The results obtained by the FRDTM for different values of α and $\lambda=1$ for example 4.1 with $n=5$

x	t	$\alpha=0.5$ Numerical	$\alpha=0.75$ Numerical	$\alpha=1$ Exact	Numerical
−5	0.002	−4.211841	−4.876244	−4.980060	−4.980060
	0.004	−3.938004	−4.794473	−4.960239	−4.960239
	0.006	−3.744961	−4.724456	−4.940535	−4.940535
	0.008	−3.592556	−4.661485	−4.920949	−4.920949
−3	0.002	−2.527105	−2.925746	−2.988036	−2.988036
	0.004	−2.362803	−2.876684	−2.976143	−2.976143
	0.006	−2.246977	−2.834674	−2.964321	−2.964321
	0.008	−2.155534	−2.796891	−2.952569	−2.952569
3	0.002	2.527105	2.925746	2.988036	2.988036
	0.004	2.362803	2.876684	2.976143	2.976143
	0.006	2.246977	2.834674	2.964321	2.964321
	0.008	2.155534	2.796891	2.952569	2.952569
5	0.002	4.211841	4.876244	4.980060	4.980060
	0.004	3.938004	4.794473	4.960239	4.960239
	0.006	3.744961	4.724456	4.940535	4.940535
	0.008	3.592556	4.661485	4.920949	4.920949

Table 3 The results obtained by the FRDTM for different values of α and $a=4$ for example 5.1 with $n=5$

x	t	$\alpha=0.5$ Numerical	$\alpha=0.75$ Numerical	$\alpha=1$ Exact	Numerical
−5	0.002	−0.493876	−0.49339	−0.493320	−0.493320
	0.004	−0.494098	−0.493447	−0.493334	−0.493334
	0.006	−0.494262	−0.493496	−0.493347	−0.493347
	0.008	−0.494397	−0.49354	−0.493360	−0.493360
−3	0.002	−0.456455	−0.453141	−0.452664	−0.452664
	0.004	−0.457972	−0.453523	−0.452755	−0.452755
	0.006	−0.459102	−0.453856	−0.452844	−0.452844
	0.008	−0.460032	−0.45416	−0.452934	−0.452934
3	0.002	0.448366	0.452001	0.452484	0.452484
	0.004	0.446521	0.451607	0.452393	0.452393
	0.006	0.445064	0.451259	0.452302	0.452302
	0.008	0.443806	0.450937	0.452211	0.452211
5	0.002	0.492686	0.493223	0.493294	0.493294
	0.004	0.492412	0.493165	0.493281	0.493281
	0.006	0.492194	0.493113	0.493267	0.493267
	0.008	0.492007	0.493066	0.493254	0.493254

7 Conclusion

In this paper, we successfully applied the FRDTM to find approximate analytical solution of the fractional Sharma–Tasso–Olver equation and the fractional damped Burger equation for different values of $\alpha$and the results we obtained in example 4.1 and example 5.1 were in excellent agreement with the exact solutions for $\alpha=1$, $\lambda=1$ and $\alpha=1$, $a=4$, respectively. The FRDTM introduces a significant improvement in the fields over existing techniques because it takes less calculations and it takes less work compared by other methods. Our goal in the future is to apply this method to other nonlinear fractional PDEs which arise in other areas of science such as Biology, Medicine and Engineering. Computations of the paper have been carried out using the computer package of Mathematica 7.

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