Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/060202

Maximum Principles for Nonlinear Fractional Differential Equations in Reliable Space

Mohammed Al-Refai

Department of Mathematics, Faculty of Sciences, Yarmouk University, Irbid, Jordan

Received: 2 Apr. 2019, Revised: 20 Jun. 2019, Accepted: 2 Jul. 2019
Published online: 1 Apr. 2020

Abstract: In the present paper, we formulate and prove weak and strong maximum principles for non-linear fractional differential equations with Riemann-Liouville fractional derivative of order $0 < \alpha < 1$. Compared to the previous studies, our results are obtained in a wider space $C_{1-\alpha}[0,T]$, and are extendable to multi-term fractional differential equations.

Keywords: Maximum principle, fractional differential equations, Riemann-Liuoville derivative.

1 Introduction

We consider the following nonlinear fractional boundary value problem

$$(D_0^\alpha u)(t) = f(t,u),\ 0 < t < T,\ 0 < \alpha < 1, \tag{1}$$

$$\lim_{t\to 0^+} t^{1-\alpha} u(t) = u_0, \tag{2}$$

where $(D_0^\alpha u)(t)$ is the left Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$, defined by

$$(D_0^\alpha u)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} u(s) ds.$$

Theory of Eq. (1) was discussed in [1] in the space $C[0,T]$ and for the initial condition $u(0) = u_0$. The problem was transformed to an equivalent Volterra fractional integral equation. We mention here that the space $C[0,T]$ is restrictive for the Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$, and it is more proper to consider a boundary condition given in (2). The linear case of the problem (1)-(2) was discussed in [2], in a much wider space $C_{1-\alpha}[0,T]$, where two maximum principles were obtained. Maximum principle is commonly used to analyze the solutions of fractional differential equations. Recently, several maximum principles have been derived to fractional differential equations with several types of fractional derivatives. The applications of maximum principles in exploring fractional differential equations were indicated in [1,2,3,4,5,6,7,8,9,10,11,12].
The present paper addresses a weak and a strong maximum principles for the nonlinear fractional boundary value problem (1)-(2) in the space $C_{1-\alpha}[0,T]$. The results extend to nonlinear multi-term fractional differential equations.
For $\mu \geq 0$, the space $C_\mu[0,T]$ is the space of all functions f such that $t^\mu f(t) \in C[0,T]$. It is known that $C[0,T] = C_0[0,T] \subset C\mu[0,T] \subset L^1(0,T)$.

The organization of the manuscript is given below. Section 2 presents the main results. Conclusions are depicted in section 3.

2 Maximum principles

We first prove the following comparison principle, which is a crucial result in our analysis. We have

* Corresponding author e-mail: m_alrefai@yu.edu.jo

Lemma 1. *Let function $h \in C_{1-\alpha}[0,T]$ satisfy the following:*

$$h(t) \geq 0, t \in (0,t_1], \ h(t_1) = 0, \ 0 < t_1 < T.$$

Then it holds that

$$(D_0^\alpha h)(t_1) \leq 0, \textit{ for all } \ 0 < \alpha < 1.$$

Proof. Let $r(t) = \int_0^t k(t-s)h(s)ds$; where $k(s) = s^{-\alpha}$. We have

$$\begin{aligned} r(t_1+\Delta t) - r(t_1) &= \int_0^{t_1+\Delta t} k(t_1+\Delta t - s)h(s)ds - \int_0^{t_1} k(t_1 - s)h(s)ds, \\ &= \int_0^{t_1} k(t_1+\Delta t - s)h(s)ds + \int_{t_1}^{t_1+\Delta t} k(t_1+\Delta t - s)h(s) - \int_0^{t_1} k(t_1 - s)h(s)ds, \\ &= \int_0^{t_1} \Big(k(t_1+\Delta t - s) - k(t_1 - s)\Big)h(s)ds + \int_{t_1}^{t_1+\Delta t} k(t_1+\Delta t - s)h(s)ds, \\ &= I_1 + I_2. \end{aligned}$$

Since $(D_0^\alpha h)(t_1)$ exists, see [13], we prove that

$$\lim_{\Delta t \to 0^+} \frac{r(t_1+\Delta t) - r(t_1)}{\Delta t} \leq 0, \tag{3}$$

and the proof of

$$\lim_{\Delta t \to 0^-} \frac{r(t_1+\Delta t) - r(t_1)}{\Delta t} \leq 0, \tag{4}$$

will follow. Since $\Delta t > 0$, we have $t_1 + \Delta t - s > t_1 - s$, which implies $k(t_1+\Delta t - s) < k(t_1 - s)$, as $k(s)$ is decreasing. Because $h(t) \geq 0, t \in (0,t_1]$ we have

$$I_1 = \int_0^{t_1} \Big(k(t_1+\Delta t - s) - k(t_1 - s)\Big)h(s)ds \leq 0. \tag{5}$$

Since $t_1 > 0$, $h(t)$ continues on $[t_1, t_1+\Delta t]$. Because $h(t_1) = 0$, there exists $\Delta t > 0$, for every $\varepsilon > 0$, such that

$$|h(t)| \leq \varepsilon(1-\alpha), \ \text{ for all } |t - t_1| < \Delta t.$$

Thus,

$$\begin{aligned} I_2 &= \int_{t_1}^{t_1+\Delta t} k(t_1+\Delta t - s)h(s)ds \leq \varepsilon(1-\alpha)\int_{t_1}^{t_1+\Delta t} k(t_1+\Delta t - s)ds \\ &= \varepsilon(1-\alpha)\int_{t_1}^{t_1+\Delta t} (t_1+\Delta t - s)^{-\alpha}ds = \varepsilon(1-\alpha)\frac{1}{1-\alpha}(\Delta t)^{1-\alpha} = \varepsilon(\Delta t)^{1-\alpha} < \varepsilon. \end{aligned}$$

The lase equation yields

$$I_2 \leq 0. \tag{6}$$

Combining Eqs. (5) and (6) with $\Delta t \to 0^+$ to prove the result in Eq. (3), which completes the proof.

We start with the following weak maximum principle.

Theorem 1. *(The weak maximum principle) Let $u, v \in C_{1-\alpha}[0,T]$ satisfy the following inequalities,*

$$\lim_{t \to 0^+} t^{1-\alpha}u(t) = u_0 < \lim_{t \to 0^+} t^{1-\alpha}v(t) = v_0, \tag{7}$$

$$D_0^\alpha u - f(t,u) < D_0^\alpha v - f(t,v), \ 0 < t < T. \tag{8}$$

Then $u < v$ on $(0,T)$ and $u \leq v$ on $(0,T]$.

*Proof.*We first show that $u < v$ on $(0,T)$. Assume the result is untrue, then there exists $t_1 \in (0,T)$ such that

$$u(t_1) = v(t_1),\ u(t) < v(t), t \in (0,t_1).$$

Let $h(t) = v(t) - u(t), t \in (0,t_1]$, then it holds that

$$h(t) \geq 0,\ t \in (0,t_1],\ h(t_1) = 0.$$

Thus,

$$(D_0^\alpha h)(t_1) \leq 0, \text{ or } (D_0^\alpha v)(t_1) \leq (D_0^\alpha u)(t_1),$$

based on the result in Lemma 1. So, at $t = t_1$, we have $f(t_1,u(t_1)) = f(t_1,v(t_1))$, which together with $(D_0^\alpha v)(t_1) \leq (D_0^\alpha u)(t_1)$, contradict (10) and complete the proof of $u < v$ on $(0,T)$.
Since $u,v \in C_{1-\alpha}[0,T]$, then u and v are continuous on $(0,T]$, which together with $u < v$ on $(0,T)$, imply $u \leq v$ on $(0,T]$.

For the following strong maximum principle, we assume that $f(t,u)$ is k-Lipschitz in $C_{1-\alpha}[0,T]$. That is,

$$|f(t,u) - f(t,v)| \leq k|u-v|, \text{ for some } k > 0, \text{and for all, } u,v \in C_{1-\alpha}[0,T].$$

Theorem 2.*(The strong-maximum principle) Let $u,v \in C_{1-\alpha}[0,T]$ satisfy the following inequalities:*

$$\lim_{t\to 0^+} t^{1-\alpha}u(t) = u_0 < \lim_{t\to 0^+} t^{1-\alpha}v(t) = v_0, \tag{9}$$

$$D_0^\alpha u - f(t,u) \leq D_0^\alpha v - f(t,v),\ 0 < t < T, \tag{10}$$

where $f(t,u)$ is k-Lipschitz in $C_{1-\alpha}[0,T]$. Then, $u < v$ on $(0,T)$, and $u \leq v$ on $(0,T]$.

*Proof.*We define the auxiliary function $w = u + z,\ t \in (0,T]$, where z is the solution of

$$\left(D_0^\alpha z\right)(t) = -2kz, \quad 0 < t < T,\ 0 < \alpha < 1, \tag{11}$$

$$\lim_{t\to 0^+} t^{1-\alpha}z(t) = z_0 = \frac{v_0 - u_0}{2} > 0. \tag{12}$$

The unique solution of Eqs. (11)-(12) is, see [?]

$$z(t) = z_0\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(-2kt^\alpha),\ t \in (0,T].$$

Since $E_{\alpha,\alpha}(t) > 0$, we have $z(t) = w(t) - u(t) > 0, \quad t \in (0,T]$. We have

$$w_0 = \lim_{t\to 0^+} t^{\alpha-1}w(t) = u_0 + z_0 = u_0 + \frac{v_0 - u_0}{2} = \frac{v_0 + u_0}{2} < v_0,$$

and

$$\begin{aligned}
\left(D_0^\alpha w\right)(t) - f(t,w) &= \left(D_0^\alpha u\right)(t) + \left(D_0^\alpha z\right)(t) - f(t,w), \\
&= \left(D_0^\alpha u\right)(t) - 2kz - [f(t,w) - f(t,u)] - f(t,u), \\
&= \left(D_0^\alpha u\right)(t) - 2k(w-u) - [f(t,w) - f(t,u)] - f(t,u), \\
&\leq \left(D_0^\alpha u\right)(t) - f(t,u) - 2k(w-u) + k(w-u), \\
&\leq \left(D_0^\alpha v\right)(t) - f(t,v) - k(w-u) = \left(D_0^\alpha v\right)(t) - f(t,v) - kz, \\
&< \left(D_0^\alpha v\right)(t) - f(t,v).
\end{aligned}$$

Thus, by the weak maximum principle we have $w < v$ on $(0,T)$. Since $z > 0$, we have $w > u$ on $(0,T)$ and thereby $u < v$ on $(0,T)$. The result $u \leq v$ on $(0,T]$ will follow as $u,v \in C(0,T]$.

In the following, we extend the weak and strong maximum principles for nonlinear multi-term fractional differential equations. We have

Theorem 3.*(The weak maximum principle) Let $u,v \in C_{1-\alpha}[0,T]$ satisfy the following inequalities:*

$$\lim_{t\to 0^+} t^{1-\alpha}u(t) = u_0 < \lim_{t\to 0^+} t^{1-\alpha}v(t) = v_0,$$

$$\left(D_0^{\alpha_m} + \sum_{i=1}^{m-1} c_i D_0^{\alpha_i}\right)u(t) - f(t,u) < \left(D_0^{\alpha_m} + \sum_{i=1}^{m-1} c_i D_0^{\alpha_i}\right)v(t) - f(t,v),\ 0 < t < T, \tag{13}$$

where $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < 1$, and $c_i \geq 0, i = 1,\cdots,m-1$. Then $u < v$ on $(0,T)$, and $u \leq v$ on $(0,T]$.

*Proof.*We follow analogous steps to the proof of Theorem 1. We first show that $u < v$ on $(0,T)$. Assume the result is untrue, then there exists $t_1 \in (0,T]$ such that

$$u(t_1) = v(t_1),\; u(t) < v(t), t \in (0,t_1).$$

Let $h(t) = v(t) - u(t), t \in (0,t_1]$, then it holds that

$$h(t) \geq 0,\; t \in (0,t_1],\; h(t_1) = 0.$$

Thus,

$$(D_0^{\alpha_i} h)(t_1) \leq 0, \text{ or } (D_0^{\alpha_i} v)(t_1) \leq (D_0^{\alpha_i} u)(t_1), i = 1,\cdots,m,$$

based on the result in Lemma 1. Since $c_i \geq 0, i = 1,\cdots,m-1$, we have

$$\left(D_0^{\alpha_m} + \sum_{i=1}^{m-1} c_i D_0^{\alpha_i}\right) v(t_1) \leq \left(D_0^{\alpha_m} + \sum_{i=1}^{m-1} c_i D_0^{\alpha_i}\right) u(t_1). \tag{14}$$

So, at $t = t_1$, we have $f(t_1,u(t_1)) = f(t_1,v(t_1))$, which together with the result in Eq. (14) contradict (13) and complete the proof of $u < v$ on $(0,T)$.
Since u and v are continuous on $(0,T]$, and $u < v$ on $(0,T)$, then $u \leq v$ on $(0,T]$.

Theorem 4.*(The strong maximum principle) Let $u,v \in C_{1-\alpha}[0,T]$ satisfy the following inequalities,*

$$\lim_{t \to 0^+} t^{1-\alpha} u(t) = u_0 < \lim_{t \to 0^+} t^{1-\alpha} v(t) = v_0, \tag{15}$$

$$\left(D_0^{\alpha_m} + \sum_{i=1}^{m-1} c_i D_0^{\alpha_i}\right) u(t) - f(t,u) \leq \left(D_0^{\alpha_m} + \sum_{i=1}^{m-1} c_i D_0^{\alpha_i}\right) v(t) - f(t,v),\; 0 < t < T,$$

where $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < 1$, $c_i \geq 0, i = 1,\cdots,m-1$, and $f(t,u)$ is k-Lipschitz in $C_{1-\alpha}[0,T]$. Then $u < v$ on $(0,)$, and $u \leq v$ on $(0,T]$.

*Proof.*The proof is obtained by applying analogous statements in the proof of Theorem 2, and by considering the minor changes in the proof of Theorem 3.

3 Conclusion

We have developed maximum principles for nonlinear fractional differential equations involving the Riemann-Liouville fractional derivative of order $0 < \alpha < 1$. The obtained maximum principles can be implemented to investigate various types of fractional differential equations [3,4,14]. We highlight below the importance of the new results compared with the previous ones:

1. The space $C_{1-\alpha}[0,T]$ is wider than $C[0,T]$, so it is more proper for the Riemann-Liouville fractional derivative of order $0 < \alpha < 1$.
2. The results are extendable to multi-term nonlinear fractional differential equations, but it is not the case with the previous approaches. Also, it is difficult to transform a multi-term fractional differential equation to an equivalent integral equation.
3. The type of the boundary conditions is more proper than the Dirichlet boundary conditions in the case of the Riemann-Liouville fractional derivative.

References

[1] V. Lakshmikantham and A. S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Anal.* **69**, 2677–2682 (2008).
[2] J. J. Nieto, Maximum principles for fractional differential equations derived from Mittag-Leffler functions, *Appl. Math. Lett.* **23**, 1248-1251 (2010).
[3] M. Al-Refai and Y. Luchko, Maximum principles for the fractional diffusion equations with the Riemann-Liouville fractional derivative and their applications, *Fract. Calc. Appl. Anal.* **17**, 483-498 (2014).
[4] M. Al-Refai and Y. Luchko, Maximum principle for the multi-term time-fractional diffusion equations with the Riemann-Liouville fractional derivatives, *J. Appl. Math. Comput.* **257**, 40–51 (2015).

[5] M. Al-Refai and Y. Luchko, Analysis of fractional diffusion equations of distributed order: maximum principles and its applications, *Analysis*, DOI: 10.1515/anly-2015-5011 (2015).
[6] M. Al-Refai, Fractional differential equations involving Caputo fractional derivative with Mittag-Leffler non-singular kernel: comparison principles and applications, *Electr. J. Differ. Equ.* **36**, 1-10 (2018).
[7] M. Al-Refai and K. Pal, A maximum principle for a fractional boundary value problem with convection term and applications, *Math. Model. Anal.* **24**(1), 62-71 (2019).
[8] D. Baleanu and O. G. Mustafaa, On the global existence of solutions to a class of fractional differential equations, *Comput. Math. Appl.* **59**(5), 1835-1841 (2010).
[9] Y. Luchko, Maximum principle for the generalized time-fractional diffusion equation, *J. Math. Anal. Appl.* **351**, 218-223 (2019).
[10] Y. Luchko, Maximum principle and its application for the time-fractional diffusion equations, *Fract. Calc. Appl. Anal.* **14**(1), 110-124 (2011).
[11] G. Wang, D. Baleanu and L. Zhang, Monotone iterative method for a class of nonlinear fractional differential equations, *Fract. Calc. Appl. Anal.* **15**(2), 244-252 (2012).
[12] S. Owyed, M. A. Abdou, A. Abdel-Aty, A. A. Ibraheem, R. Nekhili, D. Baleanu, New optical soliton solutions of space-time fractional nonlinear dynamics of microtubules via three integration schemes, *Journal of Intelligent and Fuzzy systems* **38**, 2859–2866 (2020)
[13] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science B.V., Amsterdam, 2006.
[14] J. Singh, D. Kumar and D. Baleanu, On the analysis of chemical kinetics system pertaining to a fractional derivative with Mittag-Leffler type kernel, *Chaos* **27**(10), 103-113 (2017).