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A Hypergeometric Transformation for Gauss Function ${}_2F_1[a,b\,;\frac{1+a+b-2m}{2};\frac{1+z}{2}]$ and its Applications in Clausen Function

M. I. Qureshi[1] *and M. S. Baboo*[2,*]

[1] Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University), New Delhi -110025, India.
[2] School of Basic Sciences and Research, Sharda University, Greater Noida, Uttar Pradesh, 201306, India.

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Abstract: The object of this paper is to obtain a hypergeometric transformation for Gauss function ${}_2F_1[a,b\,;\frac{1+a+b-2m}{2};\frac{1+z}{2}]$ using a summation theorem of Fox. Some special cases of the hypergeometric transformation and summation theorems for Clausen hypergeometric function ${}_3F_2[a,b,\frac{2a+2b+3}{4}\,;1+a-b\pm m,\frac{2a+2b-1}{4};-1]$ with suitable convergence conditions, are also discussed.

Keywords: Clausen hypergeometric function; Gauss hypergeometric functions; Legendre duplication formula; Fox summation theorem.

1 Introduction, Definitions and Preliminaries

For the definitions of Pochhammer symbol, generalized hypergeometric function ${}_pF_q$, Gamma function, Legendre's duplication formula and other elementary results, we refer the literature [2], [4] and [7].

The following summation theorem plays a key role in the derivation of the main hypergeometric transformation (5).

Fox summation theorem [5, p.204, Equation (2.6)]

$$ {}_2F_1\left[\begin{matrix}A,B;\\ \frac{A+B+1-2M}{2};\end{matrix}\frac{1}{2}\right]=\frac{2^{A+B-2}\Gamma(\frac{A+B+1}{2})\Gamma(\frac{A+B+1-2M}{2})}{\sqrt{\pi}\,\Gamma(A)\,\Gamma(B)}\times\sum_{r=0}^{M}\binom{M}{r}\frac{2^r\Gamma(\frac{A+r}{2})\Gamma(\frac{B+r}{2})}{\Gamma(\frac{A+B+1+2r-2M}{2})}, \tag{1}$$

$$\left(A,B,\frac{1+A+B-2M}{2},\frac{1+A+B}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-\,;\,M\in\mathbb{N}_0\right).$$

The main object of this paper is to derive a generalization of the transformation formulas (2) and (3).

Kummer's transformation formula [6, p.82, Entry 72]

$$ {}_2F_1\left[\begin{matrix}a,b;\\ \frac{a+b+1}{2};\end{matrix}\frac{1+z}{2}\right]=\frac{\Gamma(\frac{a+b+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{a+1}{2})\,\Gamma(\frac{b+1}{2})}\,{}_2F_1\left[\begin{matrix}\frac{a}{2},\frac{b}{2};\\ \frac{1}{2};\end{matrix}z^2\right]+\frac{2z\,\Gamma(\frac{a+b+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{a}{2})\,\Gamma(\frac{b}{2})}\,{}_2F_1\left[\begin{matrix}\frac{a+1}{2},\frac{b+1}{2};\\ \frac{3}{2};\end{matrix}z^2\right], \tag{2}$$

$$\left(\frac{a+b+1}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-\,;\,|z|<1,\text{ or }z\in\{-1,1\},\,\Re(a+b)<1,\text{ or }z\in\{-i,i\},\Re(a+b)<3\,;\;i=\sqrt{(-1)}\right).$$

which was proved independently by Ramanujan [1, p.64, Entry 21] and is also recorded in [4, p.65, Equation 2.1.5(28), p.111, Equation 2.11(3)].

Recently the following transformation

$$ {}_2F_1\left[\begin{matrix}a,b;\\ \frac{a+b-1}{2};\end{matrix}\frac{1+z}{2}\right]$$

$$=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{a+b-1}{2})}{2}\left\{\frac{(b-1)}{\Gamma(\frac{b+1}{2})\Gamma(\frac{a+1}{2})}{}_3F_2\left[\begin{matrix}\frac{a}{2},\frac{b}{2},\frac{b+1}{2};\\ \frac{1}{2},\frac{b-1}{2};\end{matrix}z^2\right]+\right.$$

$$+\frac{4}{\Gamma(\frac{b}{2})\Gamma(\frac{a}{2})}{}_2F_1\left[\begin{matrix}\frac{a+1}{2},\frac{b+1}{2};\\ \frac{1}{2};\end{matrix}z^2\right]$$

$$+\frac{a}{\Gamma(\frac{b+1}{2})\Gamma(\frac{a+1}{2})}{}_2F_1\left[\begin{matrix}\frac{a+2}{2},\frac{b}{2};\\ \frac{1}{2};\end{matrix}z^2\right]+$$

* Corresponding author e-mail: mesub007@gmail.com

$$+\frac{2bz}{\Gamma(\frac{b}{2})\Gamma(\frac{a}{2})}{}_3F_2\left[\begin{array}{c}\frac{a+1}{2},\frac{b+1}{2},\frac{b+2}{2};\\ \frac{3}{2},\frac{b}{2};\end{array} z^2\right]$$
$$+\frac{2abz}{\Gamma(\frac{b+1}{2})\Gamma(\frac{a+1}{2})}{}_2F_1\left[\begin{array}{c}\frac{a+2}{2},\frac{b+2}{2};\\ \frac{3}{2};\end{array} z^2\right]+$$
$$+\frac{2(a+1)z}{\Gamma(\frac{b}{2})\Gamma(\frac{a}{2})}{}_2F_1\left[\begin{array}{c}\frac{a+3}{2},\frac{b+1}{2};\\ \frac{3}{2};\end{array} z^2\right]\Bigg\}, \quad (3)$$

$$\left(\frac{a+b-1}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-\ ;\ |z|<1,\text{or } z\in\{-1,1\}\ ,\ \Re(a+b)<-1,\text{or } z\in\{-i,i\}\ ,\ \Re(a+b)<1\right),$$

and reduction formula

$${}_3F_2\left[\begin{array}{c}a,b,b+\frac{1}{2};\\ d,b-\frac{1}{2};\end{array} -1\right]=$$
$$\frac{(2b-2a-1)}{(2b-1)}{}_2F_1\left[\begin{array}{c}a,b;\\ d;\end{array} -1\right]+\frac{(2a)}{(2b-1)}{}_2F_1\left[\begin{array}{c}a+1,b;\\ d;\end{array} -1\right], \quad (4)$$

$$\left(\Re(a+b-d)<0\ ;\ (b-\tfrac{1}{2}),d\in\mathbb{C}\backslash\mathbb{Z}_0^-\right),$$

were derived by the authors [9, Equations (3.3) and (4.3)].

2 The Main Transformation and its Demonstration

Our main transformation (presumably new) is stated here as the following theorem. The result derived in this section is interesting and potentially useful.

The following hypergeometric transformations holds true, when any values of parameters and variables leading to the results which do not make sense, are tacitly excluded.
Then

$${}_2F_1\left[\begin{array}{c}a,b;\\ \frac{1+a+b-2m}{2};\end{array} \frac{1+z}{2}\right]=\frac{2^{a+b-2}\Gamma(\frac{1+a+b}{2})}{\sqrt{\pi}\,\Gamma(a)\,\Gamma(b)}\times$$
$$\times\left\{\sum_{p=0}^{\infty}\frac{(\frac{1+a+b}{4})_p(\frac{3+a+b}{4})_p\,z^{2p}}{(\frac{1}{2})_p(\frac{1+a+b-2m}{4})_p(\frac{3+a+b-2m}{4})_p(p)!}\right.$$
$$\times\sum_{r=0}^{m}\binom{m}{r}\frac{2^r\,\Gamma(\frac{a+2p+r}{2})\Gamma(\frac{b+2p+r}{2})}{(\frac{1+a+b-2m+4p}{2})_r}+$$
$$+\frac{2(1+a+b)z}{(1+a+b-2m)}\sum_{p=0}^{\infty}\frac{(\frac{3+a+b}{4})_p(\frac{5+a+b}{4})_p\,z^{2p}}{(\frac{3}{2})_p(\frac{3+a+b-2m}{4})_p(\frac{5+a+b-2m}{4})_p(p)!}$$
$$\left.\times\sum_{r=0}^{m}\binom{m}{r}\frac{2^r\,\Gamma(\frac{a+2p+r+1}{2})\Gamma(\frac{b+2p+r+1}{2})}{(\frac{3+a+b-2m+4p}{2})_r}\right\}, \quad (5)$$

$$\left(a,b,\frac{1+a+b-2m}{2},\frac{1+a+b}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-\ ;\ |z|<1\ ;\ m\in\mathbb{N}_0\right),$$

provided that each of the series involved is absolutely convergent.

Proof. In order to derive the equation (5) we consider left hand side of the equation (5)

$${}_2F_1\left[\begin{array}{c}a,b;\\ \frac{1+a+b-2m}{2};\end{array} \frac{1+z}{2}\right]=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r(1+z)^r}{(\frac{1+a+b-2m}{2})_r r!\,2^r}$$
$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(\frac{1+a+b-2m}{2})_r r!\,2^r}\ {}_1F_0\left[\begin{array}{c}-r;\\ -;\end{array} -z\right]$$
$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(\frac{1+a+b-2m}{2})_r r!\,2^r}\ \sum_{p=0}^{r}\frac{(-r)_p(-z)^p}{p!}$$
$$=\sum_{r=0}^{\infty}\sum_{p=0}^{r}\frac{(a)_r(b)_r\,z^p}{(\frac{1+a+b-2m}{2})_r\,2^r p!(r-p)!}$$
$$=\sum_{r=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{r+p}(b)_{r+p}z^p}{(\frac{1+a+b-2m}{2})_{r+p}\,2^{r+p}(p)!(r)!}$$
$$=\sum_{r=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{r+2p}(b)_{r+2p}z^{2p}}{(\frac{1+a+b-2m}{2})_{r+2p}\,2^{r+2p}(2p)!(r)!}$$
$$+\sum_{r=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{r+2p+1}(b)_{r+2p+1}z^{2p+1}}{(\frac{1+a+b-2m}{2})_{r+2p+1}\,2^{r+2p+1}(2p+1)!(r)!}$$
$$=\sum_{p=0}^{\infty}\frac{(a)_{2p}(b)_{2p}z^{2p}}{(\frac{1+a+b-2m}{2})_{2p}\,2^{2p}(2p)!}\sum_{r=0}^{\infty}\frac{(a+2p)_r(b+2p)_r}{(\frac{1+a+b-2m+4p}{2})_r\,2^r\,(r)!}+$$
$$+\sum_{p=0}^{\infty}\frac{(a)_{2p+1}(b)_{2p+1}z^{2p+1}}{(\frac{1+a+b-2m}{2})_{2p+1}\,2^{2p+1}(2p+1)!}\sum_{r=0}^{\infty}\frac{(a+2p+1)_r(b+2p+1)_r}{(\frac{1+a+b-2m+4p+2}{2})_r\,2^r(r)!}$$

or

$${}_2F_1\left[\begin{array}{c}a,b;\\ \frac{1+a+b-2m}{2};\end{array} \frac{1+z}{2}\right]$$
$$=\sum_{p=0}^{\infty}\frac{(a)_{2p}(b)_{2p}z^{2p}}{(\frac{1+a+b-2m}{2})_{2p}\,2^{2p}(2p)!}\ {}_2F_1\left[\begin{array}{c}a+2p,b+2p;\\ \frac{1+a+b-2m+4p}{2};\end{array} \frac{1}{2}\right]+$$
$$+\sum_{p=0}^{\infty}\frac{(a)_{2p+1}(b)_{2p+1}z^{2p+1}}{(\frac{1+a+b-2m}{2})_{2p+1}\,2^{2p+1}(2p+1)!}\ {}_2F_1\left[\begin{array}{c}a+2p+1,b+2p+1;\\ \frac{1+a+b-2m+4p+2}{2};\end{array} \frac{1}{2}\right]. \quad (6)$$

Apply Fox summation theorem (1) in the right hand side of the equation (6), we get

$${}_2F_1\left[\begin{array}{c}a,b;\\ \frac{1+a+b-2m}{2};\end{array} \frac{1+z}{2}\right]=$$
$$=\sum_{p=0}^{\infty}\frac{(a)_{2p}(b)_{2p}2^{a+b-2}\,\Gamma(\frac{1+a+b+4p}{2})\Gamma(\frac{1+a+b-2m+4p}{2})z^{2p}}{(\frac{1+a+b-2m}{2})_{2p}\sqrt{\pi}\,\Gamma(a+2p)\,\Gamma(b+2p)(\frac{1}{2})_p(p)!}\times$$
$$\times\sum_{r=0}^{m}\binom{m}{r}\frac{2^r\,\Gamma(\frac{a+2p+r}{2})\Gamma(\frac{b+2p+r}{2})}{\Gamma(\frac{1+a+b+2r-2m+4p}{2})}+$$

$$+\sum_{p=0}^{\infty}\frac{(a)_{2p+1}(b)_{2p+1}z^{2p+1}2^{a+b-1}\,\Gamma(\frac{3+a+b+4p}{2})\Gamma(\frac{3+a+b-2m+4p}{2})}{(\frac{1+a+b-2m}{2})_{2p+1}\sqrt{\pi}\,\Gamma(a+2p+1)\,\Gamma(b+2p+1)(\frac{3}{2})_p(p)!}\times$$

$$\times\sum_{r=0}^{m}\binom{m}{r}\frac{2^r\,\Gamma(\frac{a+2p+r+1}{2})\Gamma(\frac{b+2p+r+1}{2})}{\Gamma(\frac{3+a+b+2r-2m+4p}{2})}. \qquad (7)$$

Using algebraic properties of Pochhammer symbols in equation (7), after simplification we obtain the right hand side of the hypergeometric transformation (5). This complete the proof of the theorem (5).

3 Special Cases

In this section, we obtain some special cases of our main transformation (5).

When $m=0$ in equation (5), we get Kummer's transformation formula (2). In Kummer's transformation formula (2) put $z=0$ we get Kummer's second summation theorem [6, p.134, Entry 2].

When $m=1$ in equation (5), we get

$${}_2F_1\left[\begin{matrix} a,b; \\ \frac{a+b-1}{2}; \end{matrix}\ \frac{1+z}{2}\right]$$

$$=\frac{\Gamma(\frac{a+b-1}{2})\Gamma(\frac{1}{2})}{2}\left\{\frac{(a+b-1)}{\Gamma(\frac{a+1}{2})\,\Gamma(\frac{b+1}{2})}{}_3F_2\left[\begin{matrix} \frac{a}{2},\frac{b}{2},\frac{a+b+3}{4}; \\ \frac{1}{2},\frac{a+b-1}{4}; \end{matrix}\ z^2\right]+\right.$$

$$+\frac{4}{\Gamma(\frac{a}{2})\,\Gamma(\frac{b}{2})}{}_2F_1\left[\begin{matrix} \frac{a+1}{2},\frac{b+1}{2}; \\ \frac{1}{2}; \end{matrix}\ z^2\right]$$

$$+\frac{2(a+b+1)z}{\Gamma(\frac{a}{2})\,\Gamma(\frac{b}{2})}{}_3F_2\left[\begin{matrix} \frac{a+1}{2},\frac{b+1}{2},\frac{a+b+5}{4}; \\ \frac{3}{2},\frac{a+b+1}{4}; \end{matrix}\ z^2\right]+$$

$$\left.+\frac{2abz}{\Gamma(\frac{a+1}{2})\,\Gamma(\frac{b+1}{2})}{}_2F_1\left[\begin{matrix} \frac{a+2}{2},\frac{b+2}{2}; \\ \frac{3}{2}; \end{matrix}\ z^2\right]\right\}, \qquad (8)$$

$$\left(\frac{a+b-1}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-\ ;\ |z|<1, \text{or } z\in\right.$$
$$\left.\{-1,1\}\ ,\ \Re(a+b)<-1, \text{or } z\in\{-i,i\}\ ,\ \Re(a+b)<1\right).$$

Now put $z=0$ in equation (8), we find

$${}_2F_1\left[\begin{matrix} a,b; \\ \frac{a+b-1}{2}; \end{matrix}\ \frac{1}{2}\right]=\frac{\Gamma(\frac{a+b+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{a+1}{2})\,\Gamma(\frac{b+1}{2})}+\frac{2\Gamma(\frac{a+b-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{a}{2})\,\Gamma(\frac{b}{2})}, \qquad (9)$$

$$\left(\frac{a+b-1}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

When $m=2$ in equation (5), we obtain

$${}_2F_1\left[\begin{matrix} a,b; \\ \frac{a+b-3}{2}; \end{matrix}\ \frac{1+z}{2}\right]$$

$$=\frac{\Gamma(\frac{a+b+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{a+1}{2})\,\Gamma(\frac{b+1}{2})}{}_4F_3\left[\begin{matrix} \frac{a}{2},\frac{b}{2},\frac{a+b+1}{4},\frac{a+b+3}{4}; \\ \frac{1}{2},\frac{a+b-1}{4},\frac{a+b-3}{4}; \end{matrix}\ z^2\right]+$$

$$+\frac{2(a+b-1)\Gamma(\frac{a+b-3}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{a}{2})\,\Gamma(\frac{b}{2})}{}_3F_2\left[\begin{matrix} \frac{a+1}{2},\frac{b+1}{2},\frac{a+b+3}{4}; \\ \frac{1}{2},\frac{a+b-1}{4}; \end{matrix}\ z^2\right]+$$

$$+\frac{ab\Gamma(\frac{a+b-3}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{a+1}{2})\,\Gamma(\frac{b+1}{2})}{}_2F_1\left[\begin{matrix} \frac{a+2}{2},\frac{b+2}{2}; \\ \frac{1}{2}; \end{matrix}\ z^2\right]+$$

$$+\frac{(a+b-1)(a+b+1)z\,\Gamma(\frac{a+b-3}{2})\Gamma(\frac{1}{2})}{2\,\Gamma(\frac{a}{2})\,\Gamma(\frac{b}{2})}\times$$

$$\times{}_4F_3\left[\begin{matrix} \frac{a+1}{2},\frac{b+1}{2},\frac{a+b+3}{4},\frac{a+b+5}{4}; \\ \frac{3}{2},\frac{a+b-1}{4},\frac{a+b+1}{4}; \end{matrix}\ z^2\right]+$$

$$+\frac{(a+b+1)(a\,b\,z)\,\Gamma(\frac{a+b-3}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{a+1}{2})\,\Gamma(\frac{b+1}{2})}{}_3F_2\left[\begin{matrix} \frac{a+2}{2},\frac{b+2}{2},\frac{a+b+5}{4}; \\ \frac{3}{2},\frac{a+b+1}{4}; \end{matrix}\ z^2\right]+$$

$$+\frac{2\,(a+1)(b+1)z\,\Gamma(\frac{a+b-3}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{a}{2})\,\Gamma(\frac{b}{2})}{}_2F_1\left[\begin{matrix} \frac{a+3}{2},\frac{b+3}{2}; \\ \frac{3}{2}; \end{matrix}\ z^2\right], \qquad (10)$$

$$\left(\frac{a+b-3}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-\ ;\ |z|<1, \text{or } z\in\right.$$
$$\left.\{-1,1\}\ ,\ \Re(a+b)<-3, \text{or } z\in\{-i,i\}\ ,\ \Re(a+b)<-1\right).$$

Now put $z=0$ in equation (10), we have

$${}_2F_1\left[\begin{matrix} a,b; \\ \frac{a+b-3}{2}; \end{matrix}\ \frac{1}{2}\right]=\frac{\Gamma(\frac{a+b-3}{2})\Gamma(\frac{1}{2})}{2}\times$$

$$\times\left\{\frac{(a+b)^2+4(ab-a-b)+3}{2\Gamma(\frac{a+1}{2})\,\Gamma(\frac{b+1}{2})}+\frac{4(a+b-1)}{\Gamma(\frac{a}{2})\,\Gamma(\frac{b}{2})}\right\}, \qquad (11)$$

$$\left(\frac{a+b-3}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

4 Few Applications in Clausen's series ${}_3F_2(-1)$

Setting $z=i$ or $z=-i$ in two transformations (3), (8); under the common convergence condition ($\Re(a+b)<1$) in (3), (8); equating their right hand sides and separating real, imaginary parts, further making suitable adjustments in the parameters a and b, we get

$${}_3F_2\left[\begin{matrix} a,b,\frac{2a+2b+3}{4}; \\ \frac{1}{2},\frac{2a+2b-1}{4}; \end{matrix}\ -1\right]=\frac{(2b-1)}{(2a+2b-1)}\,{}_3F_2\left[\begin{matrix} a,b,\frac{2b+1}{2}; \\ \frac{1}{2},\frac{2b-1}{2}; \end{matrix}\ -1\right]+$$

$$+\frac{(2a)}{(2a+2b-1)}\,{}_2F_1\left[\begin{matrix} a+1,b; \\ \frac{1}{2}; \end{matrix}\ -1\right], \qquad (12)$$

$$\left(\Re(a+b) < \tfrac{1}{2}\ ;\ \tfrac{2a+2b-1}{4}, \tfrac{2b-1}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right),$$

and

$${}_3F_2\left[\begin{matrix} a,b,\frac{2a+2b+3}{4}; \\ \frac{3}{2},\frac{2a+2b-1}{4}; \end{matrix}\ -1\right] = \frac{(2b-1)}{(2a+2b-1)}\,{}_3F_2\left[\begin{matrix} a,b,\frac{2b+1}{2}; \\ \frac{3}{2},\frac{2b-1}{2}; \end{matrix}\ -1\right] +$$

$$+\frac{(2a)}{(2a+2b-1)}\,{}_2F_1\left[\begin{matrix} a+1,b; \\ \frac{3}{2}; \end{matrix}\ -1\right], \quad (13)$$

$$\left(\Re(a+b) < \tfrac{3}{2}\ ;\ \tfrac{2a+2b-1}{4}, \tfrac{2b-1}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

The unification of summations theorems (12) and (13) can be written as

$${}_3F_2\left[\begin{matrix} a,b,\frac{2a+2b+3}{4}; \\ d,\frac{2a+2b-1}{4}; \end{matrix}\ -1\right] = \frac{(2b-1)}{(2a+2b-1)}\,{}_3F_2\left[\begin{matrix} a,b,\frac{2b+1}{2}; \\ d,\frac{2b-1}{2}; \end{matrix}\ -1\right] +$$

$$+\frac{(2a)}{(2a+2b-1)}\,{}_2F_1\left[\begin{matrix} a+1,b; \\ d; \end{matrix}\ -1\right], \quad (14)$$

$$\left(\Re(a+b-d) < 0\ ;\ d, \tfrac{2a+2b-1}{4}, \tfrac{2b-1}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

Now using our reduction formula (4) in ${}_3F_2$ of right hand side of the equation (14), after simplifications, we get

$${}_3F_2\left[\begin{matrix} a,b,\frac{2a+2b+3}{4}; \\ d,\frac{2a+2b-1}{4}; \end{matrix}\ -1\right] = \frac{(2b-2a-1)}{(2a+2b-1)}\,{}_2F_1\left[\begin{matrix} a,b; \\ d; \end{matrix}\ -1\right] + \frac{4a}{(2a+2b-1)}\,{}_2F_1\left[\begin{matrix} a+1,b; \\ d; \end{matrix}\ -1\right], \quad (15)$$

$$\left(\Re(a+b-d) < 0\ ;\ d, \tfrac{2a+2b-1}{4} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

We have verified the reduction formula (15) by taking suitable values of a,b and d.

Now put $d = 1+a-b$ in the equation (15) and apply Kummer's first summation theorem [7, p.489, Entry 7.3.6.2] and associated theorem [7, p.489, Entry 7.3.6.1], recorded in Prudnikov *et al.*, after simplification, we get

$${}_3F_2\left[\begin{matrix} a,b,\frac{2a+2b+3}{4}; \\ 1+a-b,\frac{2a+2b-1}{4}; \end{matrix}\ -1\right] = \frac{\Gamma(1+a-b)\Gamma(\frac{1}{2})}{2^a(2a+2b-1)}\times$$

$$\times\left\{\frac{2b-1}{\Gamma(\frac{a+1}{2})\Gamma(\frac{a-2b+2}{2})} + \frac{2a}{\Gamma(\frac{a-2b+1}{2})\Gamma(\frac{a+2}{2})}\right\}, \quad (16)$$

$$\left(\Re(b) < \tfrac{1}{2}\ ;\ 1+a-b, (\tfrac{2a+2b-1}{4}) \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

When $d = 1+a-b-m$ in the equation (15) and use summation theorems of Choi, Rathi and Malani [3, pp.1523-24, Equation 2.2] and authors [8, p.14, Equation 3.1], after simplification, we get

$${}_3F_2\left[\begin{matrix} a,b,\frac{2a+2b+3}{4}; \\ 1+a-b-m,\frac{2a+2b-1}{4}; \end{matrix}\ -1\right] = \frac{\Gamma(1+a-b-m)}{2(2a+2b-1)\Gamma(a)}\times$$

$$\times\left((2b-2a-1)\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{\Gamma(\frac{r+a}{2})}{\Gamma(\frac{r+a+2-2b-2m}{2})}\right\}+\right.$$

$$\left.+4\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{\Gamma(\frac{a+r+1}{2})}{\Gamma(\frac{a+r+1-2b-2m}{2})} + \frac{\Gamma(\frac{a+r+2}{2})}{\Gamma(\frac{a+r+2-2b-2m}{2})}\right]\right\}\right), \quad (17)$$

$$\left(\Re(b) < \tfrac{-m+1}{2}\ ;\ a, 1+a-b-m, (\tfrac{2a+2b-1}{4}) \in \mathbb{C}\backslash\mathbb{Z}_0^-\ ;\ m \in \mathbb{N}_0\right).$$

If we put $d = 1+a-b+m$ in the equation (15) and apply summation theorems of Choi, Rathi and Malani [3, p.1524, Equation 2.3] and authors [8, p.14, Equation 3.2], after simplification, we get

$${}_3F_2\left[\begin{matrix} a,b,\frac{2a+2b+3}{4}; \\ 1+a-b+m,\frac{2a+2b-1}{4}; \end{matrix}\ -1\right] = \frac{\Gamma(1+a-b+m)}{2(2a+2b-1)\Gamma(a)}\times$$

$$\times\left(\frac{(2b-2a-1)}{(1-b)_m}\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(-1)^r\Gamma(\frac{r+a}{2})}{\Gamma(\frac{r+a+2-2b}{2})}\right\}+\right.$$

$$\left.+\frac{4}{(-b)_m}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{(-1)^r\Gamma(\frac{a+r+1}{2})}{\Gamma(\frac{a+r+1-2b}{2})} + \frac{(-1)^r\Gamma(\frac{a+r+2}{2})}{\Gamma(\frac{a+r+2-2b}{2})}\right]\right\}\right), \quad (18)$$

$$\left(\Re(b) < \tfrac{m+1}{2}\ ;\ -b, a, 1+a-b+m, (\tfrac{2a+2b-1}{4}) \in \mathbb{C}\backslash\mathbb{Z}_0^-\ ;\ m \in \mathbb{N}_0\right).$$

The summation theorems (14), (15), (16), (17) and (18) are believed to be new and are not recorded in the tables of Prudnikov *et al.* [7, pp.546–547] and Brychkov [2, pp.589–600] and other literature on hypergeometric functions.

We conclude our present investigation by observing that several other corollaries and consequences of hypergeometric transformation (5), can also be deduced in an analogous manner.

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M. I. Qureshi received the PhD degree in Mathematics at Aligarh Muslim University India. He is working at Jamia Millia Islamia. His research interests are in the areas of applied mathematics, special finctions and manuscript of Ramanujan. He has published more then hundred research articles in reputed international journals of mathematical and engineering sciences.

M. S. Baboo is Assistant Professor of Mathematical at Sharda University Greater Noida. His main research interests are of applied mathematics, special finctions and manuscript of Ramanujan.