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Some generalized statistical convergent sequence spaces of fuzzy numbers via ideals

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Abstract: In this article we introduce the sequence spaces ${}_Fc^{I(S)}$ and ${}_Fc_0^{I(S)}$ of fuzzy numbers defined by I-statistical convergence using a difference operator. We study some basic topological and algebraic properties of these spaces. We also investigate the relations related to this spaces and some of their properties viz. solidity, symmetry, convergence free etc. and prove some inclusion results.

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1 Introduction

The idea of convergence of a real sequence had been extended to statistical convergence by Fast[4]. The notion of ideal convergence was introduced first by Kostyrko et. al. [10], as a generalization of statistical convergence. Later on it was further investigated from the sequence space point of view and linked to Summability theory by Łalt et. al.[11], Tripathy [2], Savas[3]. Recently E. Savas and P. Das [3] introduced the notion of I-statistical convergence of sequences of real numbers.

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh[6] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets. Matloka [7] introduced bounded and convergent sequences of fuzzy numbers, studied some of their properties and showed that every convergent sequence of fuzzy numbers is bounded. There are many applications of the sequences and difference sequences of numbers (real, complex and fuzzy numbers). For example sequences of numbers have unexpected and practical uses in many areas of science and engineering, including acoustics. They find application in measuring concert hall acoustics, radar echoes from planets, the travel times of deep-ocean sound waves for monitoring ocean temperature, and improving synthetic speech and the sounds associated with computer music. Furthermore, it is shown by Kawamura et. al.[5] that the earthquake ground motions have very simple conditioned fuzzy set rules with non-fuzzy parameters of the first and second order differences $\triangle X_i$ and $\triangle^2 X_i$ defined by membership functions. Therefore the difference sequences of fuzzy numbers are used, for example in the prediction of earthquake waves.

Let X is a non-empty set. Then a family of sets $I \subseteq 2^X$ is said to be an ideal if I is additive, i.e. $A,B \in I \Rightarrow A \cup B \in I$ and hereditary i.e.$A \in I, B \subset A \Rightarrow B \in I$.An ideal $I \subseteq 2^X$ is called non-trivial if $I \neq 2^X$.A non-trivial ideal I is called admissible iff $I \supset x : x \in X$.A non-trivial ideal I is maximal if there does not exist any non-trivial ideal $J \neq I$, containing I as a subset.For each ideal I there is a filter $F(I)$ corresponding to I i.e. $F(I) = K \subset N : K^c \in I$, where $K^c = N - K$. Throughout w denotes the class of all sequences. Now we shall give a brief introduction about the sequences of fuzzy real numbers.

Let D be the set of all closed and bounded intervals $X = [x_1, x_2]$ on the real line R. For $X,Y \in D$ we define $X \leq Y$ iff $x_1 \leq y_1$ and $x_2 \leq y_2$ and $d(X,Y)=\max\{|x_1 - y_1|, |x_2 - y_2|\}$ where $X = [x_1, x_2]$ and $Y = [y_1, y_2]$. Then it can be shown that (D,d) is a complete metric space.Also the relation $' <'$is a partial

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order relation on D. A fuzzy number X is a fuzzy subset of the real line R, i.e, a mapping $X : R \to I = [0,1]$ associating each real number t with its grade of membership $X(t)$. A fuzzy number X is normal if there exists $t_0 \in R$ such that $X(t_0) = 1$. A fuzzy number X is upper semi continuous if for each $\varepsilon > 0$, $X^{-1}([0,a+\varepsilon))$ is open in the usual topology for all $a \in [0,1)$.

Let $R(I)$ denote the set of all fuzzy numbers which are upper semi continuous and have a compact support ,i.e, if $X \in R(I)$then for any $\alpha \in [0,1]$, $[X]^\alpha$ is compact where $[X]^\alpha = \{t \in R : X(t) \geq \alpha\}$

The set R of all real numbers can be embedded into $R(I)$ if we define $\overline{r}(t)=\begin{cases} 1 & \text{for r}= t \\ 0 & \text{for } r \neq t \end{cases}$

The additive identity and multiplicative identity of $R(I)$ are $\underline{0}$ and $\underline{1}$ respectively. The arithmetic operators on $R(I)$ are defined as follows:

Let $X,Y \in R(I)$ and the α-level set $[X]^\alpha = [x_1^\alpha, x_2^\alpha]$ and $[Y]^\alpha = [y_1^\alpha, y_2^\alpha]$ and $\alpha \in [0,1]$. Then we define

$[X \oplus Y]^\alpha$=$[x_1^\alpha + y_1^\alpha, x_2^\alpha + y_2^\alpha]$

$[X \ominus Y]^\alpha$=$[x_1^\alpha - y_1^\alpha, x_2^\alpha - y_2^\alpha]$

$[X \otimes Y]^\alpha$=$(min\{x_i^\alpha y_j^\alpha\}, max\{x_i^\alpha y_j^\alpha\}), i,j = 1,2$

$[X^{-1}]^\alpha$=$[(x_2^\alpha)^{-1}, (x_1^\alpha)^{-1}]$, $x_i^\alpha > 0$ for all $\alpha \in [0,1]$

For $r \in R$ and $X \in R(I)$, the product rX is defined as

$rX(t)=\begin{cases} X(r^{-1}t) & \text{for r} \neq 0 \\ 0 & \text{for r=0} \end{cases}$

The absolute value$|X|(t)$ is defined by

$|X|(t)=\begin{cases} max\{X(t), X(-t)\} & \text{for t} > 0 \\ 0 & \text{for t} \leq 0 \end{cases}$

Let us define a mapping $\overline{d} : R(I) \times R(I) \longrightarrow R^+ \bigcup \{0\}$ by $\overline{d}(X,Y) = Sup_\alpha d([X]^\alpha, [Y]^\alpha)$ where $\alpha \in [0,1]$.

It can be shown that $(R(I), \overline{d})$is a complete metric space.

Let $w(F)$ denotes the set of all sequences of fuzzy numbers. The operator $\Delta : w(F) \to w(F)$ is defined by $\Delta^0(X_k) = X_k$, $(\Delta X_k) = \Delta X_k = X_k - X_{k+1}$ for all $k \in N$.

2 Definitions and preliminaries

Definition 2.1. [8] A sequence of fuzzy numbers $X = (X_k)$ is said to be statistical convergent to a fuzzy number X_0 if for each $\varepsilon > 0, \delta(A(\varepsilon)) = \delta(\{k \in N : \overline{d}(X_k, X_0) \geq \varepsilon\}) = 0$.

Definition 2.2. A sequence of fuzzy numbers $X = (X_k)$ is said to be I-convergent to a fuzzy number X_0 if for each $\varepsilon > 0$, $\{k \in N : \overline{d}(X_k, X_0) > \varepsilon\} \in I$. The fuzzy number X_0 is called the I-limit of the sequence (X_k) of fuzzy number and we write $I - limX_k = X_0$.

Definition 2.3. A sequence of fuzzy numbers $X = (X_k)$ is said to be I- bounded if there exists $M > 0$, such that $\{k \in N : \overline{d}(X_k, \overline{0}) > M\} \in I$.

Remark 2.4. Let $I = I_f = \{A \subset N : A \text{ is finite}\}$. Then I_f is non trivial admissible ideal of N and the corresponding convergence coincides with ordinary convergence. If $I = I_\delta = \{A \subset N : \delta(A) = 0\}$, where $\delta(A)$ denotes the asymptotic density of A. Then I_δ is a non-trivial admissible ideal of N and the corresponding convergence coincide with statistical convergence.

Lemma[1] A sequence space is normal implies that it is monotone.

Lemma[2] [10] If $I \subset 2^N$ is a maximal ideal then for each $A \subset N$, we have either $A \in I$ or $N - A \in I$.

Definition 2.5. [3] A sequence of fuzzy numbers $X = (X_k)$ is said to be I-statistically convergent to a fuzzy number X_0 if for each $\varepsilon > 0$, $\{n \in N : frac1n|\{k \leq N : \overline{d}(X_k, X_0) \geq \varepsilon\}| \geq \delta\} \in I$. The fuzzy number X_0 is called I-statistical limit of the sequence (X_k) of fuzzy number and we write $I - st.limX_k = X_0$.

Definition 2.6. A sequence space $E(F)$ is said to be solid (or normal) if $(Y_k) \in E(F)$ whenever $(X_k) \in E(F)$ and $\overline{d}(Y_k, \overline{0}) \leq \overline{d}(X_k, \overline{0})$ for all $k \in N$.

Definition 2.7. A sequence space $E(F)$ is said to be symmetric if $(X_k) \in E(F) \Rightarrow (X_{\pi(k)}) \in E(F)$ where $\pi(k)$ is a permutation of N.

Definition 2.8. A sequence space $E(F)$ is said to be monotone if $E(F)$ contains the canonical pre image of all its step spaces.

Definition 2.9. A sequence space $E(F)$ is said to be sequence algebra if $(X_k \bigotimes Y_k) \in E(F)$ whenever $(X_k), (Y_k) \in E(F)$.

Definition 2.10. A sequence space $E(F)$ is said to be convergence free if $(Y_k) \in \underline{E}(F)$ whenever $(X_k) \in E(F)$ and $(X_k) = \overline{0}$ implies $(Y_k) = \overline{0}$.

3 Main result

In this section we shall introduce the following new sequence spaces of fuzzy numbers and examine some properties of the resulting sequence spaces.

Let I be an admissible ideal of N and $X = (X_k)$ be a sequence of fuzzy numbers. We define the following sequence spaces of fuzzy number.

${}_Fc^{I(S)}(\Delta)$=$\{X = (X_k) : I\text{-st lim } \Delta X_k = X_0\}$.

$_Fc_0^{I(S)}(\Delta)$={$X=(X_k):I$-st lim $\Delta X_k=\overline{0}$}.

$_Fl_\infty(\Delta)$={$X=(X_k)$:$sup_k\ \overline{d}(\Delta X_k,\overline{0})<\infty$}.

$_Fm^{I(S)}(\Delta)$=$_Fc^{I(S)}(\Delta)\bigcap_F l_\infty(\Delta)$.

$_Fm_0^{I(S)}(\Delta)$=$_Fc_0^{I(S)}(\Delta)\bigcap_F l_\infty(\Delta)$.

From definition it is obvious that $_Fc_0^{I(S)}(\Delta)\subset_F c^{I(S)}(\Delta)\subset_F l_\infty(\Delta)$.

Example 3.1. Let $X=(X_k)$ be a sequence of fuzzy numbers defined by

$$X_k(t)=\left\{\begin{array}{l}(t-k)\quad \text{for } t\in[k,k+1]\\ (-t+k+2)\quad \text{for } t\in[k+1,k+2]\\ 0\ \text{, otherwise}\end{array}\right\},\quad \text{if}$$

$k=3^n(n=0,1,2,...)$.

$$X_k(t)=\left\{\begin{array}{l}(t+3)\quad \text{for } t\in[-3,-2]\\ (-t-1)\quad \text{for } t\in[-2,-1]\\ 0\ \text{, otherwise}\end{array}\right\}, \text{ if } k\neq 3^n \text{ and}$$

k is odd.

$$X_k(t)=\left\{\begin{array}{l}(t-6)\quad \text{for } t\in[6,7]\\ (-t+8)\quad \text{for } t\in[7,8]\\ 0\ \text{, otherwise}\end{array}\right\}, \text{ if } k\neq 3^n \text{ and k is}$$

even.

Then for $\alpha\in[0,1]$, α-level set of X_k and ΔX_k are respectively

$$[X_k]^\alpha=\begin{cases}k+\alpha,k+2-\alpha\ \text{, for } k=3^n\\ -3+\alpha,-1-\alpha\ \text{, for } k\neq 3^n \text{ and k is odd}\\ 6+\alpha,8-\alpha\ \text{, for } k\neq 3^n \text{ and k is even}\end{cases}$$

$$[\Delta X_k]^\alpha=\begin{cases}k-8+2\alpha,k-4-2\alpha\ \text{, for } k=3^n\\ -k+3+2\alpha,-k+7-2\alpha\ \text{, for } k+1=3^n\\ -11+2\alpha,-7-2\alpha\ \text{, for } k\neq 3^n,k+1\neq 3^n \text{ and k is odd}\\ 7+2\alpha,11-2\alpha\ \text{, for } k\neq 3^n,k+1\neq 3^n \text{ and k is even}\end{cases}$$

Now if we take $I=I_\delta$ (the ideal of density zero sets of N), then the sequence (X_k) is an example which is Δ-statistically bounded but not Δ-statistically convergent[8] and hence not I_Δ-statistically convergent.

Example 3.2. Let $I=I_f$ (the ideal of all finite subsets of N). Let us define a sequence of fuzzy numbers

Let $X_k(t)=\overline{1}$ for $k=2^n$,n=1,2,...

$$\text{otherwise, } X_k(t)=\begin{cases}\frac{k}{3}(t-2)+1\quad \text{for } t\in[\frac{2k-3}{k},2]\\ \frac{-k}{3}(t-2)+1\quad \text{for } t\in[2,\frac{2k+3}{2}]\\ 0\ \text{, otherwise}\end{cases}$$

Then

$$[X_k]^\alpha=\begin{cases}[1,1]\quad ,k=2^n,n=1,2,...\\ 2-\frac{3}{k}(1-\alpha),2+\frac{3}{k}(1-\alpha)\quad ,otherwise\end{cases}$$

$$[\Delta X_k]^\alpha=\begin{cases}-1-\frac{3}{k}(1-\alpha),-1+\frac{3}{k}(1-\alpha)\quad ,k=2^n\\ 1-\frac{3}{k}(1-\alpha),1+\frac{3}{k}(1-\alpha)\quad ,otherwise\end{cases}$$

$\overline{d}(\Delta X_k,\overline{0})=Sup_\alpha d([\Delta X_k]^\alpha,[0]^\alpha)=5$

Thus $(X_k)\in_F l^\infty(\Delta)$. But$(X_k)$ is not I_Δ- statistically convergent.

Remark 3.3. If $I=I_f$ then the sequence spaces $_Fc_0^{I(S)}(\Delta)$, $_Fc^{I(S)}(\Delta)$ coincide with the sequence spaces $_Fc_0(\Delta)$, $_Fc(\Delta)$, which were studied by Tripathy and Das[1] and many others.

Theorem 3.4. The spaces $_Fc^{I(S)}(\Delta)$, $_Fc_0^{I(S)}(\Delta)$ are linear space.

Proof. we will prove the result for $_Fc_0^{I(S)}(\Delta)$.

Let $X=(X_k)$, $Y=(Y_k)$ be any two element of $_Fc_0^{I(S)}(\Delta)$ and α,β be any scalar. Then

$$A(\varepsilon)=\{n\in N:\tfrac{1}{n}|\{k\leq n:\overline{d}(\Delta X_k,\overline{0})\geq\tfrac{\varepsilon}{2}\}|\geq\delta\}\in I.$$

$$B(\varepsilon)=\{n\in N:\tfrac{1}{n}|\{k\leq n:\overline{d}(\Delta Y_k,\overline{0})\geq\tfrac{\varepsilon}{2}\}|\geq\delta\}\in I.$$

Now,$C(\varepsilon)=\{n\in N:\frac{1}{n}|\{k\leq n:\overline{d}(\Delta(\alpha X_k\bigoplus\beta Y_k),\overline{0})\geq\frac{\varepsilon}{2}\}|\geq\delta\}\in I$
$\subseteq\{n\in N:\frac{1}{n}|\{k\leq n:|\alpha|\overline{d}(\Delta X_k,\overline{0})\geq\frac{\varepsilon}{2}\}|\geq\delta\}\cup\{n\in N:\frac{1}{n}|\{k\leq n:|\beta|\overline{d}(\Delta Y_k,\overline{0})\geq\frac{\varepsilon}{2}\}|\geq\delta\}$
i.e., $C(\varepsilon)\subseteq A(\frac{\varepsilon}{2|\alpha|})\cup B(\frac{\varepsilon}{2|\beta|})$
i.e, $C(\varepsilon)\in I$.
Hence $_Fc_0^{I(S)}(\Delta)$ is a linear space. Similarly we can prove that $_Fc^{I(S)}(\Delta)$ is linear space.

Theorem 3.5. The spaces $_Fc_0^{I(S)}(\Delta)$ and $_Fm_0^{I(S)}(\Delta)$ are normal and monotone.

Proof. Let $X=(X_k)$ be any element of $_Fc_0^{I(S)}(\Delta)$ and $Y=(Y_k)$ be any sequence such that $\overline{d}(\Delta X_k,\overline{0})\geq\overline{d}(\Delta Y_k,\overline{0})$, for all $k\in N$.Then for all $\varepsilon>0$,

$\{n\in N:\frac{1}{n}|\{k\leq n:\overline{d}(\Delta X_k,\overline{0})\geq\varepsilon\}|\geq\delta\}\supseteq\{n\in N:\frac{1}{n}|\{k\leq n:\overline{d}(\Delta Y_k,\overline{0})\geq\varepsilon\}|\geq\delta\}\in I$
Hence $Y=(Y_k)\in_F c_0^{I(S)}(\Delta)$.

Thus the spaces $_Fc_0^{I(S)}(\Delta)$ is normal and hence monotone. Similarly $_Fm_0^{I(S)}(\Delta)$ is normal and monotone.

Proposition 3.6. If I is not maximal ideal then the space $_Fc^{I(S)}(\Delta)$ is neither normal nor monotone.

Example 3.7. Let us consider a sequence of fuzzy number

$$X_k(t) = \begin{cases} \frac{1+t}{2} & \text{for } t \in [-1,1] \\ \frac{3-t}{2} & \text{for } t \in [1,3] \\ 0 & \text{, otherwise} \end{cases}$$

Then $(X_k) \in_F c^{I(S)}(\Delta)$.Since I is not maximal, by *lemma*2, there exists a subset K of N such that $K \notin I$ and $N-K \notin I$. Let us define a sequence $Y = (Y_k)$ by

$$Y_k = \begin{cases} X_k & \text{, k} \in K \\ 0 & \text{, otherwise} \end{cases}$$

Then (Y_k) belongs to the canonical pre image of the k-step spaces of $_F c^{I(S)}(\Delta)$, but $(Y_k) \notin_F c^{I(S)}(\Delta)$. Hence $_F c^{I(S)}(\Delta)$ is not monotone. Therefore by lemma1, $_F c^{I(S)}(\Delta)$ is not normal.

Proposition 3.8. If I is neither maximal nor $I = I_f$ then the spaces $_F c^{I(S)}(\Delta)$ and $_F c_0^{I(S)}(\Delta)$ are not symmetric.

Example 3.9. Let us consider a sequence of fuzzy number defined by $X = (X_k)$ where

$$X_k(t) = \begin{cases} 1+t & \text{, -1} \le t \le 0 \\ 1-t & \text{, } 0 \le t \le 1 \end{cases}$$

Then for $k \in A \in I$(an infinite set), $(X_k) \in_F c^{I(S)}(\Delta)$. Let $K \subset N$ be such that $K \notin I$ and $N-K \notin I$. Let us consider a sequence space $Y = (Y_k)$, a rearrangement of the sequence (X_k) defined by

$$Y_k = \begin{cases} X_k & \text{, k} \in K \\ 0 & \text{, otherwise} \end{cases}$$

Then $(Y_k) \notin_F c^{I(S)}(\Delta)$. Hence $_F c^{I(S)}(\Delta)$ is not symmetric. Similarly $_F c_0^{I(S)}(\Delta)$ is not symmetric.

Theorem 3.10. The spaces $_F c^{I(S)}(\Delta)$, $_F c_0^{I(S)}(\Delta)$, $_F m^{I(S)}(\Delta)$ and $_F m_0^{I(S)}(\Delta)$ are sequence algebra.

Proof. Let $(X_k),(Y_k) \in {_F c_0^{I(S)}}(\Delta)$ and $0 < \varepsilon < 1$.Then the result follows from the following inclusion relation: $\{n \in N : \frac{1}{n}|\{k \le n : \bar{d}(\Delta X_k \otimes \Delta Y_k, \bar{0}) \ge \varepsilon\}| \ge \delta\} \supset \{n \in N : \frac{1}{n}|\{k \le n : \bar{d}(\Delta X_k, \bar{0}) \ge \varepsilon\}| \ge \delta\} \cap \{n \in N : \frac{1}{n}|\{k \le n : \bar{d}(\Delta Y_k, \bar{0}) \ge \varepsilon\}| \ge \delta\}$.

Theorem 3.11. The spaces $_F c^{I(S)}(\Delta)$, $_F c_0^{I(S)}(\Delta)$ are not convergence free.

Proof. Let us consider a sequence of fuzzy number defined by

$$X_k(t) = \begin{cases} \frac{2+t}{4} & ,-2 \le t \le 2 \\ \frac{6-t}{4} & ,2 \le t \le 6 \\ 0 & \text{, otherwise} \end{cases}$$

Then $X_k(t) \in {_F c^{I(S)}}(\Delta)$. Let $Y_k(t) = \frac{1}{k}$ for all $k \in N$.Then $Y_k(t) \in {_F c^{I(S)}}(\Delta)$.But $X_k = \bar{0}$ does not implies $Y_k = \bar{0}$.Hence $_F c^{I(S)}(\Delta)$ is not convergence free. Similarly $_F c_0^{I(S)}(\Delta)$ is not convergence free.

Theorem 3.12. The sequence spaces $_F m^{I(S)}(\Delta)$ and $_F m_0^{I(S)}(\Delta)$ are complete metric spaces with respect to the metric defined by $\bar{d}(X,Y) = \bar{d}(X_1,Y_1)$+Sup$_k \bar{d}(\Delta X_k, \Delta Y_k)$,where $(X_k),(Y_k) \in {_F m^{I(S)}}(\Delta)$.

Proof. This can be proved easily by taking a Cauchy sequence and so omitted.

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