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Atomic Solution for Fractional Non-linear Burgers Equation of Conformable Type

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Abstract: In this paper we present a new method for solving the fractional Burgers equation

$$u_t^{\alpha} - uu_x^{\alpha} = u_x^{\alpha\alpha}, \tag{1}$$

where u is a continuously differentiable function on $[0,\infty)$. In this method an exact solution is obtained to the above equation using tensor product techniques. Exact solutions of nonlinear partial differential equations similar to this one are hard to find.

Keywords: Conformable fractional derivative, Burgers equation, tensor product, atomic solution.

1 Introduction

In 1915 an English mathematician named Harry Bateman [1], while studying the motion of viscous fluid, introduced a very important nonlinear partial differential equation.

$$\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x} = v\frac{\partial^2 u}{\partial x^2}. \tag{2}$$

After that in 1948, a Dutch physicist Johannes Martinus, Burgers [2], studied this equation extensively, later named after him as Burgers equation.

Burgers equation forms a standard test problem for PDE solvers. In addition, it is used in boundary Layer calculation of a viscous fluid flow and in other various fields such as gas dynamics, aspect turbulence, nonlinear wave propagation, traffic flow shock wave theory, cosmology, and molecular interfaces. Burgers equation has been solved numerically by different methods see [3,4,5,6,7,8]. Fractional Burgers equation has also attracted many Mathematicians who gave numerical solution by different techniques see [9,10,11,12].

In this paper we tackle the fractional Burger's equation by using the tensor product techniques to get an exact solution.

2 Fractional derivative

In recent years, many differential equations have been generalized to a random (non-integer) order as fractional differential equations. Due to their ability to model complex phenomena, they attracted much attention and were used widely in engineering, science, and other fields.

Fractional derivatives got different definitions as Riemann-Liouville, Caputo, Caputo-Frabrizio, Hadamard, and others [13]. In this paper, we focus on the conformable derivative which was originally defined by Khalil and others, [14,15] using the limit approach. Here, we briefly recall some definitions and properties of such fractional derivative.

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Definition 1. *[15] Let h be a function defined from $[0,\infty)$ into $\mathbb{R}$. The α^{th} order fractional derivative of h is defined by*

$$D^{\alpha}(h)(t)=\lim_{\varepsilon\to 0}\frac{h(t+\varepsilon t^{1-\alpha})-h(t)}{\varepsilon}, \tag{3}$$

for all $t>0$, $\alpha\in(0,1]$. If h is $\alpha-$differentiable in some $(0,b)$, $b>0$ and $\lim_{t\to 0^+}D^{\alpha}(h)(t)$ exists, then $D^{\alpha}(h)(0)=\lim_{t\to 0^+}D^{\alpha}(h)(t)$.

Let $h^{(\alpha)}(t)$ stands for $D^{\alpha}(h)(t)$. If $\alpha=1$, Definition 1, turns to be the definition of usual first order derivative,

$$h'(t)=\lim_{\varepsilon\to 0}\frac{h(t+\varepsilon)-h(t)}{\varepsilon}. \tag{4}$$

In 2017, S.H. Al-Sharif and A. Malkawi gave a generalization of the previous definition as follows.

Definition 2. *[14] Let h be a function defined from $[0,\infty)$ into $\mathbb{R}$, the α^{th} order fractional derivative of h is defined by*

$$h^{(\alpha)}(t)=\lim_{\varepsilon\to 0}\frac{h(tq(\varepsilon t^{-\alpha}))-h(t)}{\varepsilon}, \tag{5}$$

where q is any positive continuously differentiable function such that $q(0)=q'(0)=1$, for all $t>0$, $\alpha\in(0,1]$.
If h is $\alpha-$differentiable in $(0,b), b>0$ and $\lim_{t\to 0^+}h^{(\alpha)}(t)$ exists, then $h^{(\alpha)}(0)=\lim_{t\to 0^+}h^{(\alpha)}(t)$.

Theorem 1. *[14] Let $\alpha\in(0,1]$, h and w be two $\alpha-$differentiable functions at a point $t>0$. Then*

1. $(sh+zw)^{(\alpha)}(t)=sh^{(\alpha)}(t)+zw^{(\alpha)}(t)$, *for all* $s,\ z\in\mathbb{R}$.
2. $K^{(\alpha)}=0$, *for all constant functions* $h(t)=K$.
3. $(t^n)^{(\alpha)}=nt^{n-\alpha}$.
4. $(hw)^{(\alpha)}(t)=w(t)h^{(\alpha)}(t)+h(t)(w)^{(\alpha)}(t)$.
5. $D^{\alpha}(\frac{h}{w})(t)=\dfrac{w(t)h^{(\alpha)}(t)-h(t)w^{(\alpha)}(t)}{w^2(t)}$.
6. *If h is differentiable, then* $h^{(\alpha)}(t)=t^{1-\alpha}\frac{d}{dt}(h(t))$.

As a consequence of Theorem 1, we have the following theorem.

Theorem 2. *[14] Let $b\in\mathbb{R}$ and $\alpha\in(0,1]$. Then*

1. $(e^{bt})^{(\alpha)}=bt^{1-\alpha}e^{bt}$.
2. $(\sin(bt))^{(\alpha)}=bt^{1-\alpha}\cos(bt)$.
3. $(\cos(bt))^{(\alpha)}=-bt^{1-\alpha}\sin(bt)$.
4. $(\frac{1}{\alpha}t^{\alpha})^{(\alpha)}=1$.

Let P^*, Q^* be the duals of the two Banach spaces P and Q respectively. For $(p,q)\in P\times Q$, define the linear operator $p\otimes q$ by

$$p\otimes q: P^*\longrightarrow Q,$$

as

$$(p\otimes q)(p^*)=\langle p,p^*\rangle q, \tag{6}$$

where $\langle p,p^*\rangle$ denotes $p^*(p)$. The expression $p\otimes q$ is called an atom. It can be shown easily that $p\otimes q$ is bounded and has a norm $||p\otimes q||=||p||\,||q||$.

The following Lemma provides some properties of these atoms.

Lemma 1. *[16] For any $p,\ t\in P$ and $q,\ z\in Q$, then we have,*

1. $\beta(p\otimes q)=\beta p\otimes q=p\otimes\beta q$, *where β any scalar.*
2. $(p+t)\otimes q=p\otimes q+t\otimes q$.
3. $p\otimes(q+z)=p\otimes q+p\otimes z$.

4. $p \otimes 0 = 0 \otimes q = 0 \otimes 0$.

Definition 3. *[16] The tensor product $P \otimes Q := span\{p \otimes q : p \in P,\ q \in Q\}$, is a subspace of $\mathscr{L}(P^*, Q)$; where $\mathscr{L}(P^*, Q)$ denotes the space of all continuous linear operators from P^* into Q. $P \otimes Q$ represents the subspace of finite rank operators in $\mathscr{L}(P^*, Q)$.*

Notice that, for any element $T \in P \otimes Q$, then $T = \sum_{j=1}^{m} p_j \otimes q_j;\ p_j \in P,\ q_j \in Q,\ 1 \le j \le m$, for $m \in \mathbb{N}$. Moreover, T can be also represented as,

$$T = \sum_{j=1}^{n} \lambda_j (p_j \otimes q_j), ||p_j|| = ||q_j|| = 1, \text{ for some } n \in \mathbb{N} \text{ and } \lambda_j' \text{ s are scalars.}$$

Two more properties of such operators are the followings,

$$\beta \sum_{j=1}^{m} p_j \otimes q_j = \sum_{j=1}^{m} \beta p_j \otimes q_j = \sum_{j=1}^{m} p_j \otimes \beta q_j, \text{ for any scalar } \beta, \tag{7}$$

and

$$\sum_{j=1}^{k} p_j \otimes q_j + \sum_{j=k+1}^{m} p_j \otimes q_j = \sum_{j=1}^{m} p_j \otimes q_j, \text{ for } k < m. \tag{8}$$

In spite of the different norms that can be imposed on $P \otimes Q$, the projective and the injective norms remain the most popular ones.

Definition 4. *[16] Let $T = \sum_{j=1}^{m} p_j \otimes q_j \in P \otimes Q$, define the **projective norm** on $P \otimes Q$ as*

$$||T||_{\wedge} = \inf \left\{ \sum_{j=1}^{m} ||p_j||\ ||q_j|| \ :\ p_j \in P,\ q_j \in Q \right\}. \tag{9}$$

Definition 5. *[16] Let $S = \sum_{j=1}^{m} p_j \otimes q_j \in P \otimes Q$, define the **injective norm** on $P \otimes Q$ as*

$$||S||_{\vee} = \sup \left\{ \sum_{j=1}^{m} |\langle p_j, p^* \rangle \langle q_j, q^* \rangle|,\ \forall\ p^* \otimes q^* \in P^* \otimes Q^* :\ ||p^*|| = ||q^*|| = 1 \right\}. \tag{10}$$

Note that, the spaces $(P \otimes Q, ||.||_{\vee})$ and $(P \otimes Q, ||.||_{\wedge})$ are not complete. The completion of $P \otimes Q$ with respect to the injective (respectively, projective) norm, is denoted by $P \overset{\vee}{\otimes} Q$ (respectively, $P \overset{\wedge}{\otimes} Q$). For more on the theory of tensor products we refer to [16].

It is known [16], that if S is a compact Hausdorff space and Q is a Banach space, then $C(S, Q)$ is isometrically isomorphic to $C(S) \otimes Q$. If $Q = C(T)$ then $C(S \times T)$ is isometrically isomorphic to $C(S) \overset{\vee}{\otimes} C(T)$. Hence, for any $u(s,t) \in C(S \times T)$, $u(s,t)$ can be seen as an element of $C(S) \overset{\vee}{\otimes} C(T)$ in the form $\sum_{j=1}^{\infty} R_j \otimes W_j;\ R_j \in C(S)$ and $W_j \in C(T)$. If the sum is finite, $j = 1, 2, 3, \ldots, m$, then the solution $u(s,t) = \sum_{j=1}^{m} R_j \otimes W_j$ is represented by a finite rank operator. If $m = 1$, then the solution (if exists) to the differential equation, which is represented by one atom $u(s,t) = R \otimes W$, is called an **atomic solution**.

It had been proved that atomic solutions for certain ordinary and fractional differential equations exist such as Gardner equation, first and second order abstract Cauchy problems. For that see [17, 18, 19]. In this paper, (sections 3 and 4), we prove that fractional Burgers equation has an atomic solution.

3 Linear form

In this section an atomic solution of a linear form of fractional partial differential equations is presented.

The following Lemma plays a great role in the success of the atomic solution method to prove the existence of analytic solutions to some partial differential equations.

Lemma 2. *[19] Let $p_1 \otimes q_1$ and $p_2 \otimes q_2$ be two non zero atoms in $P \otimes Q$. Then the following are equivalent:*

1. *$p_1 \otimes q_1 + p_2 \otimes q_2 = p_3 \otimes q_3$ is a non zero atom.*
2. *p_1, p_2 or q_1, q_2 are linearly dependent.*

Theorem 3. *If $u \in C(S \times T)$, where $S,\ T \in \{[0,1],\ [0,\infty)\}$ and the second fractional partial derivatives of u exist for some α in $(0,1)$, then the differential equation*

$$u_t^{(\alpha)} + u_x^{(\alpha)} = u_x^{(\alpha\alpha)}, \tag{11}$$

has an atomic solution.

Proof. Let $u(x,t) = R \otimes W$, $R = R(x)$, and $W = W(t)$. Then $u_t^{(\alpha)} = R \otimes W^{(\alpha)}$, $u_x^{(\alpha)} = R^{(\alpha)} \otimes W$ and $u_x^{(\alpha\alpha)} = R^{(\alpha\alpha)} \otimes W$.

So, the equation (11) becomes

$$\begin{aligned} R \otimes W^{(\alpha)} + R^{(\alpha)} \otimes W &= R^{(\alpha\alpha)} \otimes W \\ R^{(\alpha\alpha)} \otimes W - R^{(\alpha)} \otimes W &= R \otimes W^{(\alpha)} \\ [R^{(\alpha\alpha)} - R^{(\alpha)}] \otimes W &= R \otimes W^{(\alpha)}. \end{aligned}$$

Now, since two atoms are equal then $R^{(\alpha\alpha)} - R^{(\alpha)} = R$ and $W = W^{(\alpha)}$.

In the first step, we solve $W = W^{(\alpha)}$, as follows.

$$\begin{aligned} W^{-1}W^{(\alpha)} &= 1 \\ \int W^{-1} d_\alpha W &= \int d_\alpha t \\ ln(W) &= (\frac{1}{\alpha})t^\alpha \\ W(t) &= e^{\frac{t^\alpha}{\alpha}} \end{aligned}$$

As a second step, we solve $R^{(\alpha\alpha)} - R^{(\alpha)} = R$, equivalently $R^{(\alpha\alpha)} - R^{(\alpha)} - R = 0$. Let us assume that $R = e^{\eta \frac{x^\alpha}{\alpha}}$, this implies $R^{(\alpha)} = \eta e^{\eta \frac{x^\alpha}{\alpha}}$ and $R^{(\alpha\alpha)} = \eta^2 e^{\eta \frac{x^\alpha}{\alpha}}$, then

$$\begin{aligned} R^{(\alpha\alpha)} - R^{(\alpha)} - R &= 0, \\ \eta^2 e^{\eta \frac{x^\alpha}{\alpha}} - \eta e^{\eta \frac{x^\alpha}{\alpha}} - e^{\eta \frac{x^\alpha}{\alpha}} &= 0, \\ e^{\eta \frac{x^\alpha}{\alpha}}[\eta^2 - \eta - 1] &= 0. \end{aligned}$$

Since $e^{\eta \frac{x^\alpha}{\alpha}} \neq 0$, then $\eta^2 - \eta - 1 = 0$, and we get the two roots,

$$\eta_1 = \frac{1+\sqrt{5}}{2}, \text{ and } \eta_2 = \frac{1-\sqrt{5}}{2}.$$

So $R(x) = c_1 e^{\eta_1 \frac{x^\alpha}{\alpha}} + c_2 e^{\eta_2 \frac{x^\alpha}{\alpha}} = c_1 e^{(\frac{1+\sqrt{5}}{2})\frac{x^\alpha}{\alpha}} + c_2 e^{(\frac{1-\sqrt{5}}{2})\frac{x^\alpha}{\alpha}}$.

Then $u(x,t) = R \otimes W$, where $W(t) = e^{(\frac{1}{\alpha})t^\alpha}$ and $R(x) = c_1 e^{(\frac{1+\sqrt{5}}{2})\frac{x^\alpha}{\alpha}} + c_2 e^{(\frac{1-\sqrt{5}}{2})\frac{x^\alpha}{\alpha}}$.

Theorem 4. *Let $u \in C(S \times T)$, where $S,T \in \{[0,1],[0,\infty)\}$. If u is twice $\alpha-$ differentiable for some $\alpha \in (0,1)$ and f is a continuous function of the variable t, then the fractional partial differential equation*

$$u_t^{(\alpha)} + f u_x^{(\alpha)} = u_x^{(\alpha\alpha)}, \tag{12}$$

has an atomic solution.

Proof. Let $u(x,t) = R \otimes W$, where R and W are functions of x and t respectively, then $u_t^{(\alpha)} = R \otimes W^{(\alpha)}$, $u_x^{(\alpha)} = R^{(\alpha)} \otimes W$ and $u_x^{(\alpha\alpha)} = R^{(\alpha\alpha)} \otimes W$

So equation (12) becomes

$$\begin{aligned} R \otimes W^{(\alpha)} + f[R^{(\alpha)} \otimes W] &= R^{(\alpha\alpha)} \otimes W, \\ R \otimes W^{(\alpha)} + R^{(\alpha)} \otimes fW &= R^{(\alpha\alpha)} \otimes W, \\ R^{(\alpha\alpha)} \otimes W - R^{(\alpha)} \otimes fW &= R \otimes W^{(\alpha)}. \end{aligned}$$

Now, by using Lemma 2 we have either $R^{(\alpha\alpha)} = R^{(\alpha)}$ or $W = -fW$.

Case (1). If $W = -fW$, then $f = -1$, implies $[R^{(\alpha\alpha)} + R^{(\alpha)}] \otimes W = R \otimes W^{(\alpha)}$, then $R^{(\alpha\alpha)} + R^{(\alpha)} = R$ and $W = W^{(\alpha)}$.

For $R^{(\alpha\alpha)} + R^{(\alpha)} = R$, we may assume $R = e^{\eta \frac{x^\alpha}{\alpha}}$. This implies $R^{(\alpha)} = \eta e^{\eta \frac{x^\alpha}{\alpha}}$ and $R^{(\alpha\alpha)} = \eta^2 e^{\eta \frac{x^\alpha}{\alpha}}$, hence

$$\begin{aligned} R^{(\alpha\alpha)} + R^{(\alpha)} - R &= 0, \\ \eta^2 e^{\eta \frac{x^\alpha}{\alpha}} + \eta e^{\eta \frac{x^\alpha}{\alpha}} - e^{\eta \frac{x^\alpha}{\alpha}} &= 0, \\ e^{\eta \frac{x^\alpha}{\alpha}} [\eta^2 + \eta - 1] &= 0. \end{aligned}$$

But $e^{\eta \frac{x^\alpha}{\alpha}} \neq 0$, then $\eta^2 + \eta - 1 = 0$ which implies $\eta_1 = \frac{-1-\sqrt{5}}{2}$ and $\eta_2 = \frac{-1+\sqrt{5}}{2}$. So $R(x) = c_1 e^{\eta_1 \frac{x^\alpha}{\alpha}} + c_2 e^{\eta_2 \frac{x^\alpha}{\alpha}}$ where $c_1, c_2 \in \mathbb{R}$, any two real numbers.

Also, $W = W^{(\alpha)}$, then

$$\begin{aligned} W^{(\alpha)} &= W \\ W^{-1} W^{(\alpha)} &= 1 \\ \int W^{-1} d_\alpha W &= \int d_\alpha t \\ ln(W) &= \frac{t^\alpha}{\alpha} \\ W(t) &= e^{\frac{t^\alpha}{\alpha}}. \end{aligned}$$

Therefore, $u(x,t) = R \otimes W$, where $W(t) = e^{\frac{t^\alpha}{\alpha}}$ and $R(x) = c_1 e^{\eta_1 \frac{x^\alpha}{\alpha}} + c_2 e^{\eta_2 \frac{x^\alpha}{\alpha}}$. But this solution is a special case when $f = -1$. We aim to obtain the general form of a solution that is dependent on $f(t)$, hence we have to go through the other case.

Case (2). If $R^{(\alpha\alpha)} = R^{(\alpha)}$, then assume $R = e^{\eta \frac{x^\alpha}{\alpha}}$, this implies $R^{(\alpha)} = \eta e^{\eta \frac{x^\alpha}{\alpha}}$ and $R^{(\alpha\alpha)} = \eta^2 e^{\eta \frac{x^\alpha}{\alpha}}$, hence

$$\begin{aligned} R^{(\alpha\alpha)} - R^{(\alpha)} &= 0, \\ \eta^2 e^{\eta \frac{x^\alpha}{\alpha}} - \eta e^{\eta \frac{x^\alpha}{\alpha}} &= 0, \\ e^{\eta \frac{x^\alpha}{\alpha}} [\eta^2 - \eta] &= 0. \end{aligned}$$

Since $e^{\eta \frac{x^\alpha}{\alpha}} \neq 0$, then $\eta^2 - \eta = 0$, which implies $\eta_1 = 0$ and $\eta_2 = 1$. So $R(x) = c_1 e^{\eta_1 \frac{x^\alpha}{\alpha}} + c_2 e^{\eta_2 \frac{x^\alpha}{\alpha}} = c_1 + c_2 e^{\frac{x^\alpha}{\alpha}}$, where c_1, c_2 are real numbers.

Therefore

$$\begin{aligned} R^{(\alpha\alpha)} \otimes W - R^{(\alpha)} \otimes fW &= R \otimes W^{(\alpha)}, \\ R^{(\alpha)} \otimes W - R^{(\alpha)} \otimes fW &= R \otimes W^{(\alpha)}, \\ R^{(\alpha)} \otimes [W - fW] &= R \otimes W^{(\alpha)}. \end{aligned}$$

Now, since we have two atoms are equal then $R^{(\alpha)} = R$ and $[W - fW] = W^{(\alpha)}$.

First step, when $R^{(\alpha)} = R$ then we must take $c_1 = 0$, and hence we have $R^{(\alpha\alpha)} = R^{(\alpha)} = R = c_2 e^{\frac{x^\alpha}{\alpha}}$.

Second step, for $W^{(\alpha)} = [W - fW]$, we get

$$\begin{aligned} W^{(\alpha)} &= [1-f]W \\ W^{-1}W^{(\alpha)} &= [1-f] \\ \int W^{-1} d_\alpha W &= \int [1-f]\, d_\alpha t \\ ln(W) &= \int [1-f]\, d_\alpha t \\ W(t) &= e^{\int [1-f] d_\alpha t}. \end{aligned}$$

Now to verify that $R \otimes W$ is a solution of (12). Set $R^{(\alpha\alpha)} = R^{(\alpha)} = R$, then

$$\begin{aligned} R \otimes W^{(\alpha)} + f[R^{(\alpha)} \otimes W] &= R^{(\alpha\alpha)} \otimes W^{(\alpha)} + R^{(\alpha\alpha)} \otimes fW \\ &= R^{(\alpha\alpha)} \otimes [W^{(\alpha)} + fW] \end{aligned}$$

and since $W^{(\alpha)} = [W - fW]$ implies $W = \left[W^{(\alpha)} + fW\right]$, then

$$R \otimes W^{(\alpha)} + f[R^{(\alpha)} \otimes W] = R^{(\alpha\alpha)} \otimes W.$$

Therefore, the atomic solution of (12) is $u(x,t) = R \otimes W$, where $W(t) = e^{\int [1-f] d_\alpha t}$ and $R(x) = c_2 e^{\frac{x^\alpha}{\alpha}}$.

4 Fractional non-linear Burgers equation

Theorem 5. *Let $u \in C(S \times T)$, where $S, T \in \{[0,1], [0,\infty)\}$. If the second fractional partial derivatives of u exist for some $\alpha \in (0,1)$, then the differential equation*

$$u_t^{(\alpha)} + u u_x^{(\alpha)} = u_x^{(\alpha\alpha)}, \tag{13}$$

has an atomic solution.

Proof. Let $u(x,t) = R \otimes W$, where R and W are functions of x and t respectively, then $u_t^{(\alpha)} = R \otimes W^{(\alpha)}$, $u_x^{(\alpha)} = R^{(\alpha)} \otimes W$ and $u_x^{(\alpha\alpha)} = R^{(\alpha\alpha)} \otimes W$. So, equation (13) becomes

$$\begin{aligned} R \otimes W^{(\alpha)} + [R \otimes W][R^{(\alpha)} \otimes W] &= R^{(\alpha\alpha)} \otimes W, \\ R \otimes W^{(\alpha)} + RR^{(\alpha)} \otimes W^2 &= R^{(\alpha\alpha)} \otimes W, \\ R^{(\alpha\alpha)} \otimes W - RR^{(\alpha)} \otimes W^2 &= R \otimes W^{(\alpha)}. \end{aligned} \tag{14}$$

Now, by using Lemma 2, we have either $R^{(\alpha\alpha)} = RR^{(\alpha)}$ or $W^2 = W$.

Case (1). If $R^{(\alpha\alpha)} = RR^{(\alpha)}$, then equation (14) becomes

$$\begin{aligned} RR^{(\alpha)} \otimes W - RR^{(\alpha)} \otimes W^2 &= R \otimes W^{(\alpha)}, \\ RR^{(\alpha)} \otimes (W - W^2) &= R \otimes W^{(\alpha)}. \end{aligned}$$

Hence, $RR^{(\alpha)} = R$ and $W - W^2 = W^{(\alpha)}$, since $R^{(\alpha\alpha)} = RR^{(\alpha)}$. Now, let $R^{(\alpha)} = y$ then $R^{(\alpha\alpha)} = y^{(\alpha)}$ and by chain rule we have $y^{(\alpha)} = y'(R)R^{(\alpha)} = y'y$. So

$$\begin{aligned} R^{(\alpha\alpha)} &= RR^{(\alpha)}, \\ y'y &= Ry, \\ \int dy &= \int RdR, \\ y &= \frac{1}{2}R^2, \\ R^{(\alpha)} &= \frac{1}{2}R^2, \\ \int R^{-2} d_\alpha R &= \frac{1}{2}\int d_\alpha x, \\ -R^{-1} &= (\frac{1}{2\alpha})x^\alpha, \\ R &= -2\alpha x^{-\alpha}. \end{aligned}$$

Since $R = -2\alpha x^{-\alpha}$ then $R^{(\alpha)} = 2\alpha^2 x^{-2\alpha}$ and $R^{(\alpha\alpha)} = -4\alpha^3 x^{-3\alpha}$.
This implies

$$RR^{(\alpha)} = (-2\alpha x^{-\alpha})(2\alpha^2 x^{-2\alpha}) = -4\alpha^3 x^{-3\alpha} = R^{(\alpha\alpha)} \neq R.$$

That gives a contradiction.

Case (2). If $W^2 = W$ then we have either $W = 1$ or $W = 0$. If $W = 0$ then we get a zero solution. On the other hand, if $W = 1$ then we have $W^{(\alpha)} = 0$ and so we get,

$$\begin{aligned} R^{(\alpha\alpha)} \otimes W - RR^{(\alpha)} \otimes W^2 &= R \otimes W^{(\alpha)}, \\ R^{(\alpha\alpha)} \otimes 1 - RR^{(\alpha)} \otimes 1 &= R \otimes 0, \\ (R^{(\alpha\alpha)} - RR^{(\alpha)}) \otimes 1 &= 0. \end{aligned}$$

This gives that $1 = 0$ or $R^{(\alpha\alpha)} - RR^{(\alpha)} = 0$, but $1 \neq 0$ then $R^{(\alpha\alpha)} - RR^{(\alpha)} = 0$, which implies that $R^{(\alpha\alpha)} = RR^{(\alpha)}$. Similarly as above we get $R = -2\alpha x^{-\alpha}$.

Now, to verify that $R \otimes W$ is a solution of (13), set $W = 1$ and $R = -2\alpha x^{-\alpha}$ then

$$\begin{aligned} R^{(\alpha\alpha)} \otimes W - RR^{(\alpha)} \otimes W^2 &= R^{(\alpha\alpha)} \otimes W - R^{(\alpha\alpha)} \otimes W^2 \\ &= R^{(\alpha\alpha)} \otimes (W - W^2). \end{aligned}$$

Since $W = W^2$ then we have,

$$\begin{aligned} R^{(\alpha\alpha)} \otimes W - RR^{(\alpha)} \otimes W^2 &= R^{(\alpha\alpha)} \otimes 0 \\ &= 0 \\ &= R \otimes 0 \\ &= R \otimes W^{(\alpha)}. \end{aligned}$$

Hence, $u(x,t) = R \otimes W$, where $W(t) = 1$ and $R(x) = -2\alpha x^{-\alpha}$, is the required atomic solution of (13).

5 Conclusion

In this paper, we studied the existence of an analytic solution to the fractional Burgers equation

$$u_t^{(\alpha)} + uu_x^{(\alpha)} = u_x^{(\alpha\alpha)}.$$

We proved that an atomic solution exists for this equation.

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