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On Distributions of Order Statistics for Nonidentically Distributed Variables

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Abstract: In this study, distribution and probability density functions of any *d* order statistics of *innid* continuous random variables are expressed. Then, some results connecting distributions of order statistics of *innid* random variables to that of order statistics of *iid* random variables are given.

Keywords: Order statistics, distribution function, probability density function, continuous random variable.

1 Introduction

Several identities and recurrence relations for probability density function (*pdf*) and distribution function (*df*) of order statistics of independent and identically-distributed (*iid*) random variables were established by numerous authors including Arnold et al.[1], Balasubramanian and Beg[2], David[3], and Reiss[4]. Furthermore, Arnold et al.[1], David[3], Gan and Bain[5], and Khatri[6] obtained the probability function (*pf*) and *df* of order statistics of *iid* random variables from a discrete parent. Corley[7] defined a multivariate generalization of classical order statistics for random samples from a continuous multivariate distribution. Goldie and Maller[8] derived expressions for generalized joint densities of order statistics of *iid* random variables in terms of Radon-Nikodym derivatives with respect to product measures based on *df*. Guilbaud[9] expressed the probability of the functions of independent but not necessarily identically distributed(*innid*) random vectors as a linear combination of probabilities of the functions of *iid* random vectors and thus also for order statistics of random variables.

Cao and West[10] obtained recurrence relationships among the distribution functions of order statistics arising from *innid* random variables. Vaughan and Venables[11] derived the joint *pdf* and marginal *pdf* of order statistics of *innid* random variables by means of permanents. Balakrishnan[12], and Bapat and Beg[13] obtained the joint *pdf* and *df* of order statistics of *innid* random variables by means of permanents. Childs and Balakrishnan[14] obtained, using multinomial arguments, the *pdf* of $X_{r:n+1}$ $(1 \leq r \leq n+1)$ by adding another independent random variable to the original *n* variables $X_1, X_2, \ldots, X_n$. Balasubramanian et al.[15] established the identities satisfied by distributions of order statistics from non-independent non-identical variables through operator methods based on the difference and differential operators. Also, marginal and joint distributions of order statistics from innid / iid continuous and discrete random variables / vectors are obtained in different ways by Güngör and Turan[16,17], Güngör[18,19,20], Güngör et al.[21], Yüzbaşı and Güngör[22], Bulut et al.[23], Yüzbaşı et al.[24] and Güngör and Bulut[25].

In general, distribution theory for order statistics is complicated when random variables are *innid*.

In this study, distributions of order statistics of *innid* continuous random variables are obtained.

From now on, subscripts and superscripts are defined in first place in which they are used and these definitions are valid unless they are redefined.

Let $X_1, X_2, \ldots, X_n$ be *innid* continuous random variables and $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be order statistics obtained by arranging the n X_i's in increasing order of magnitude.

Let F_i and f_i be *df* and *pdf* of X_i (i=1, 2,..., n), respectively. Moreover, $X^s_{1:n}$, $X^s_{2:n}$, ... ,$X^s_{n:n}$ are order

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statistics of *iid* random variables with *df* F^s and *pdf* f^s, respectively, defined by

$$F^s = \frac{1}{n_s}\sum_{i\in s} F_i \tag{1}$$

and

$$f^s = \frac{1}{n_s}\sum_{i\in s} f_i, \tag{2}$$

Here, s is a subset of the integers $\{1, 2,\ldots, n\}$ with $n_s \geq 1$ elements.

In follows, *df* and *pdf* of $X_{r_1:n}, X_{r_2:n}, \ldots, X_{r_d:n}$ $(1\leq r_1 < r_2 < \ldots < r_d \leq n,\ d=1, 2,\ldots, n)$ are given. For notational convenience we write $\sum\sum$ and $\sum_{m_d,\ldots,m_2,m_1}^{n,\ldots,m_3,m_2}$ instead of $\sum_{\kappa=1}^{n}(-1)^{n-\kappa}\frac{\kappa^n}{n!}\sum_{n_s=\kappa}$ and $\sum_{m_d=r_d}^{n}\cdots\sum_{m_2=r_2}^{m_3}\sum_{m_1=r_1}^{m_2}$ in the expressions below, respectively.

This paper is organized as follows. In section 2, we give theorems concerning *df* and *pdf* of order statistics of *innid* continuous random variables. In section 3, some results related to *df* and *pdf* are given.

2 Theorems for distribution and probability density functions

In this section, theorems related to *df* and *pdf* of $X_{r_1:n}, X_{r_2:n}, \ldots, X_{r_d:n}$ are given. Theorems connect *df* and *pdf* of order statistics of *innid* random variables to that of order statistics of *iid* random variables using Eq. (1) and Eq. (2).

The following theorem can be expressed for joint *df* of order statistics of *innid* continuous random variables.

Theorem 2.1.

$$\begin{aligned}F_{r_1,r_2,\ldots,r_d:n}(x_1,x_2,\ldots,x_d) = &\sum_{m_d,\ldots,m_2,m_1}^{n,\ldots,m_3,m_2} C\sum_P\left[\prod_{l=1}^{m_1}F_{j_l}(x_1)\right]\\ &\cdot\prod_{w=2}^{d+1}\sum_{t=m_{w-1}}^{m_w}(-1)^{m_w-t}\\ &\cdot\sum_{n_\tau=t-m_{w-1}}\left[\prod_{l=1}^{t-m_{w-1}}F_{\tau_l}(x_w)\right]\\ &\cdot\prod_{l=1}^{m_w-t}F_{\tau_l'}(x_{w-1}),\end{aligned} \tag{3}$$

where $x_1 < x_2 < \ldots < x_d$, $C = \left[\prod_{w=1}^{d+1}(m_w - m_{w-1})!\right]^{-1}$, $m_0 = 0$, $m_{d+1} = n$, $F_{j_l}(x_{d+1}) = 1$, $x_w \in R$, $\sum_P$ denotes sum over all $n!$ permutations $(j_1, j_2, \ldots, j_n)$ of $(1, 2, \ldots, n)$, and $\sum_{n_\tau=t-m_{w-1}}$ denotes sum over all $\binom{m_w-m_{w-1}}{t-m_{w-1}}$ subsets $\tau = \{\tau_1, \tau_2, \ldots, \tau_{t-m_{w-1}}\}$, $\tau' = \{\tau_1', \tau_2', \ldots, \tau_{m_w-t}'\}$ of $\tau\bigcup\tau' = \{j_{m_{w-1}+1}, j_{m_{w-1}+2}, \ldots, j_{m_w}\}$, $\tau\bigcap\tau' = \emptyset$.

Proof. It can be written

$$\begin{aligned}&F_{r_1,r_2,\ldots,r_d:n}(x_1,x_2,\ldots,x_d) =\\ &P\{X_{r_1:n}\leq x_1, X_{r_2:n}\leq x_2, \ldots, X_{r_d:n}\leq x_d\}.\end{aligned} \tag{4}$$

Eq. (4) can be expressed as

$$\begin{aligned}F_{r_1,r_2,\ldots,r_d:n}(x_1,x_2,\ldots,x_d) = &\sum_{m_d,\ldots,m_2,m_1}^{n,\ldots,m_3,m_2} C\sum_P\left[\prod_{l=1}^{m_1}F_{j_l}(x_1)\right]\\ &\cdot\left[\prod_{l=m_1+1}^{m_2}[F_{j_l}(x_2)-F_{j_l}(x_1)]\right]\ldots\\ &\cdot\prod_{l=m_d+1}^{n}[1-F_{j_l}(x_d)]\\ =&\sum_{m_d,\ldots,m_2,m_1}^{n,\ldots,m_3,m_2} C\sum_P\left[\prod_{l=1}^{m_1}F_{j_l}(x_1)\right]\\ &\cdot\prod_{w=2}^{d+1}\prod_{l=m_{w-1}+1}^{m_w}[F_{j_l}(x_w)-F_{j_l}(x_{w-1})].\end{aligned} \tag{5}$$

By considering expression of $F_{r_1,r_2,\ldots,r_d:n}(x_1,x_2,\ldots,x_d)$ in Eq. (5), writing

$$\begin{aligned}&\prod_{l=m_{w-1}+1}^{m_w}[F_{j_l}(x_w)-F_{j_l}(x_{w-1})] =\\ &\sum_{t=m_{w-1}}^{m_w}(-1)^{m_w-t}\sum_{n_\tau=t-m_{w-1}}\left[\prod_{l=1}^{t-m_{w-1}}F_{\tau_l}(x_w)\right]\prod_{l=1}^{m_w-t}F_{\tau_l'}(x_{w-1}),\end{aligned} \tag{6}$$

and using Eq. (6) in Eq. (5), Eq. (3) is obtained.

It can be written $C^{-1}\sum_{CP_{m_d,\ldots,m_2,m_1}}$ or $(n-m_d)!\sum_{P_{m_d}}$ instead of $\sum_P$ in the above theorem.

Here, $\sum_{CP_{m_d,\ldots,m_2,m_1}}$ denotes sum over all $n!$ permutations $(j_1, j_2, \ldots, j_n)$ of $(1, 2, \ldots, n)$ for which $j_1 < j_2 < \ldots < j_{m_1}$, $j_{m_1+1} < j_{m_1+2} < \ldots < j_{m_2}$, $\ldots$ and $j_{m_d+1} < j_{m_d+2} < \ldots < j_n$. Moreover, $\sum_{P_{m_d}}$ denotes sum over all permutations $(j_1, j_2, \ldots, j_{m_d})$ of $(1, 2, \ldots, n)$.

In theory of order statistics, it is usually assumed that $X_1, X_2, \ldots, X_n$ are identically distributed. However, in many practical situations, it is necessary to allow for nonidentically $F_1, F_2, \ldots, F_n$.

The following theorem is based on Theorem 2.1 in terms of *df* of order statistics of *iid* continuous random variables.

Theorem 2.2.

$$F_{r_1,r_2,\ldots,r_d:n}(x_1,x_2,\ldots,x_d)=\sum\sum\sum_{m_d,\ldots,m_2,m_1}^{n,\ldots,m_3,m_2} n!C[F^s(x_1)]^{m_1} \cdot\prod_{w=2}^{d+1}\sum_{t=m_{w-1}}^{m_w}(-1)^{m_w-t}\binom{m_w-m_{w-1}}{t-m_{w-1}} \cdot[F^s(x_w)]^{t-m_{w-1}}[F^s(x_{w-1})]^{m_w-t}. \tag{7}$$

Proof. Eq. (4) can be expressed as

$$F_{r_1,r_2,\ldots,r_d:n}(x_1,x_2,\ldots,x_d)= \sum\sum P\{X^s_{r_1:n}\leq x_1,\ X^s_{r_2:n}\leq x_2,\ldots,X^s_{r_d:n}\leq x_d\}. \tag{8}$$

Eq. (7) is obtained from Eq. (3) and Eq. (8). Thus, Eq. (7) is obtained.

We now express the following theorem for joint *pdf* of order statistics of *innid* continuous random variables.

Theorem 2.3.

$$f_{r_1,r_2,\ldots,r_d:n}(x_1,x_2,\ldots,x_d)=D\sum_P\left[\prod_{l=1}^{r_1-1}F_{j_l}(x_1)\right] \cdot\left[\prod_{w=2}^{d+1}\sum_{t=r_{w-1}}^{r_w-1}(-1)^{r_w-1-t}\sum_{n_\tau=t-r_{w-1}}\left(\prod_{l=1}^{t-r_{w-1}}F_{\tau_l}(x_w)\right) \prod_{l=1}^{r_w-1-t}F_{\tau'_l}(x_{w-1})\right]\prod_{w=1}^{d}f_{j_{r_w}}(x_w), \tag{9}$$

where $x_1<x_2<\ldots<x_d$, $D=\prod_{w=1}^{d+1}[(r_w-r_{w-1}-1)!]^{-1}$, $r_0=0$, $r_{d+1}=n+1$, and $\sum_{n_\tau=t-r_{w-1}}$ denotes sum over all $\binom{r_w-r_{w-1}}{t-r_{w-1}}$ subsets $\tau=\{\tau_1,\tau_2,\ldots,\tau_{t-r_{w-1}}\}$, $\tau'=\{\tau'_1,\tau'_2,\ldots,\tau'_{r_w-t}\}$ of $\tau\bigcup\tau'=\{j_{r_{w-1}+1},j_{r_{w-1}+2},\ldots,j_{r_w}\}$, $\tau\bigcap\tau'=\emptyset$.

Proof. Consider

$$P\{x_1<X_{r_1:n}\leq x_1+\delta x_1,x_2<X_{r_2:n}\leq x_2+\delta x_2, \ldots,x_d<X_{r_d:n}\leq x_d+\delta x_d\}. \tag{10}$$

Dividing Eq. (10) by $\prod_{w=1}^{d}\delta x_w$ and then letting $\delta x_1,\delta x_2,\ldots,\delta x_d$ tend to zero, we obtain

$$f_{r_1,r_2,\ldots,r_d:n}(x_1,x_2,\ldots,x_d)=D\sum_p\left[\prod_{l=1}^{r_1-1}[F_{j_l}(x_1)]\right]f_{j_{r_1}}(x_1) \cdot\left[\prod_{l=r_1+1}^{r_2-1}[F_{j_l}(x_2)-F_{j_l}(x_1)]\right]f_{j_{r_2}}(x_2) \cdot\ldots f_{j_{r_d}}(x_d)\prod_{l=r_d+1}^{n}[1-F_{j_l}(x_d)]. \tag{11}$$

Eq. (11) reduces to

$$f_{r_1,r_2,\ldots,r_d:n}(x_1,x_2,\ldots,x_d)=D\sum_P\left[\prod_{l=1}^{r_1-1}F_{j_l}(x_1)\right] \cdot\left[\prod_{w=2}^{d+1}\prod_{l=r_{w-1}+1}^{r_w-1}[F_{j_l}(x_w)-F_{j_l}(x_{w-1})]\right]\prod_{w=1}^{d}f_{j_{r_w}}(x_w). \tag{12}$$

By considering the expression of $f_{r_1,r_2,\ldots,r_d:n}(x_1,x_2,\ldots,x_d)$ in Eq. (12), writing

$$\prod_{l=r_{w-1}+1}^{r_w-1}[F_{j_l}(x_w)-F_{j_l}(x_{w-1})]= \sum_{t=r_{w-1}}^{r_w-1}(-1)^{r_w-1-t}\sum_{n_\tau=t-r_{w-1}}\left(\prod_{l=1}^{t-r_{w-1}}F_{\tau_l}(x_w)\right)\prod_{l=1}^{r_w-1-t}F_{\tau'_l}(x_{w-1}), \tag{13}$$

and using Eq. (13) in Eq. (12), Eq. (9) is obtained. We can write $D^{-1}\sum_{DP_{r_d,\ldots,r_2,r_1}}$ or $(n-r_d)!\sum_{P_{r_d}}$ instead of $\sum_P$ in the above theorem. Here, $\sum_{DP_{r_d,\ldots,r_2,r_1}}$ denotes sum over all $n!$ permutations $(j_1,j_2,\ldots,j_n)$ of $(1,2,\ldots,n)$ for which $j_1<j_2<\ldots<j_{r_1-1}$, $j_{r_1+1}<j_{r_1+2}<\ldots<j_{r_2-1}$, $\ldots$ $,j_{r_d+1}<j_{r_d+2}<\ldots<j_n$. Moreover, $\sum_{P_{r_d}}$ denotes sum over all permutations $(j_1,j_2,\ldots,j_{r_d})$ of $(1,2,\ldots,n)$.

The following theorem can be obtained from Eq. (10) in terms of *df* and *pdf* of *iid* continuous random variables.

Theorem 2.4.

$$f_{r_1,r_2,\ldots,r_d:n}(x_1,x_2,\ldots,x_d)=\sum\sum n!D[F^s(x_1)]^{r_1-1} \cdot\left[\prod_{w=2}^{d+1}\sum_{t=r_{w-1}}^{r_w-1}(-1)^{r_w-1-t}\binom{r_w-r_{w-1}-1}{t-r_{w-1}} \cdot[F^s(x_w)]^{t-r_{w-1}}[F^s(x_{w-1})]^{r_w-1-t}\right]\prod_{w=1}^{d}f^s(x_w) \tag{14}$$

Proof. Eq. (10) can be expressed as

$$\sum\sum P\{x_1<X^s_{r_1:n}\leq x_1+\delta x_1,x_2<X^s_{r_2:n}\leq x_2+\delta x_2, \ldots,x_d<X^s_{r_d:n}\leq x_d+\delta x_d\}. \tag{15}$$

Dividing Eq. (15) by $\prod_{w=1}^{d}\delta x_w$ and then letting $\delta x_1,\delta x_2,\ldots,\delta x_d$ tend to zero, Eq. (14) is obtained.

3 Results for distribution and probability density functions

In this section, some results related to *df* and *pdf* of $X_{r_1:n},X_{r_2:n},\ldots,X_{r_d:n}$ are given. Also, these results connect *df* and *pdf* of order statistics of *innid* random variables to that of order statistics of *iid* random variables.

We now obtain three results for *df* of order statistics of *innid* continuous random variables from Theorem 2.1 and Theorem 2.2.

Result 3.1.

$$F_{r_1:n}(x_1)=\sum_{m_1=r_1}^{n}\frac{1}{m_1!(n-m_1)!}\sum_P\left[\prod_{l=1}^{m_1}F_{j_l}(x_1)\right]\cdot\sum_{t=m_1}^{n}(-1)^{n-t}\sum_{n_{\tau'}=n-t}\prod_{l=1}^{n-t}F_{\tau'_l}(x_1)=\sum\sum\sum_{m_1=r_1}^{n}\binom{n}{m_1}[F^s(x_1)]^{m_1}\cdot\sum_{t=m_1}^{n}(-1)^{n-t}\binom{n-m_1}{t-m_1}[F^s(x_1)]^{n-t}. \tag{16}$$

Proof. In Eq. (3) and Eq. (7), if $d=1$, Eq. (16) is obtained.

Result 3.2.

$$F_{1:n}(x_1)=1-\frac{1}{n!}\sum_P\sum_{t=0}^{n}(-1)^{n-t}\sum_{n_{\tau'}=n-t}\prod_{l=1}^{n-t}F_{\tau'_l}(x_1)=\sum\sum\left[1-\sum_{t=0}^{n}(-1)^{n-t}\binom{n}{t}[F^s(x_1)]^{n-t}\right]. \tag{17}$$

Proof. In Eq. (16), if $r_1=1$, Eq. (17) is obtained.

Result 3.3.

$$F_{n:n}(x_1)=\frac{1}{n!}\sum_P\prod_{l=1}^{n}F_{j_l}(x_1)=\sum\sum[F^s(x_1)]^n. \tag{18}$$

Proof. In Eq. (16), if $r_1=n$, Eq. (18) is obtained.

Next results for *pdf* of order statistics of *innid* continuous random variables can be obtained from Theorem 2.3 and Theorem 2.4.

The following three results are given for *pdf* of single order statistic.

Result 3.4.

$$f_{r_1:n}(x_1)=\frac{1}{(r_1-1)!(n-r_1)!}\sum_P\left[\prod_{l=1}^{r_1-1}F_{j_l}(x_1)\right]\cdot\sum_{t=r_1}^{n}(-1)^{n-t}\sum_{n_{\tau'}=n-t}\left[\prod_{l=1}^{n-t}F_{\tau'_l}(x_1)\right]f_{j_{r_1}}(x_1)=\sum\sum r_1\binom{n}{r_1}[F^s(x_1)]^{r_1-1}\cdot\sum_{t=r_1}^{n}(-1)^{n-t}\binom{n-r_1}{t-r_1}[F^s(x_1)]^{n-t}f^s(x_1). \tag{19}$$

Proof. In Eq. (9) and Eq. (14), if $d=1$, Eq. (19) is obtained.

Result 3.5.

$$f_{1:n}(x_1)=\frac{1}{(n-1)!}\sum_P\sum_{t=1}^{n}(-1)^{n-t}\sum_{n_{\tau'}=n-t}\left[\prod_{l=1}^{n-t}F_{\tau'_l}(x_1)\right]f_{j_1}(x_1)=\sum\sum n\sum_{t=1}^{n}(-1)^{n-t}\binom{n-1}{t-1}[F^s(x_1)]^{n-t}f^s(x_1). \tag{20}$$

Proof. In Eq. (19), if $r_1=1$, Eq. (20) is obtained.

Result 3.6.

$$f_{n:n}(x_1)=\frac{1}{(n-1)!}\sum_P\left[\prod_{l=1}^{n-1}F_{j_l}(x_1)\right]f_{j_n}(x_1)=\sum\sum n[F^s(x_1)]^{n-1}f^s(x_1). \tag{21}$$

Proof. In Eq. (19), if $r_1=n$, Eq. (21) is obtained.

The following two results are given for joint *pdf* of two and more order statistics.

Result 3.7.

$$f_{1,n:n}(x_1,x_2)=\frac{1}{(n-2)!}\sum_P\sum_{t=1}^{n-1}(-1)^{n-1-t}\cdot\sum_{n_\tau=t-1}\left[\prod_{l=1}^{t-1}F_{\tau_l}(x_2)\right]\left[\prod_{l=1}^{n-1-t}F_{\tau'_l}(x_1)\right]\cdot f_{j_1}(x_1)f_{j_n}(x_2)=\sum\sum n(n-1)\sum_{t=1}^{n-1}(-1)^{n-1-t}\binom{n-2}{t-1}\cdot[F^s(x_2)]^{t-1}[F^s(x_1)]^{n-t-1}f^s(x_1)f^s(x_2). \tag{22}$$

Proof. In Eq. (9) and Eq. (14), if $d=2$, $r_1=1$ and $r_2=n$, Eq.(22) is obtained.

Result 3.8.

$$f_{1,2,\ldots,k:n}(x_1,x_2,\ldots,x_k)=\frac{1}{(n-k)!}\sum_P\sum_{t=k}^{n}(-1)^{n-t}\sum_{n_{\tau'}=n-t}\left[\prod_{l=1}^{n-t}F_{\tau'_l}(x_k)\right]f_{j_1}(x_1)f_{j_2}(x_2)\ldots f_{j_k}(x_k)=\sum\sum\frac{n!}{(n-k)!}\sum_{t=k}^{n}(-1)^{n-t}\binom{n-k}{t-k}\cdot[F^s(x_k)]^{n-t}f^s(x_1)f^s(x_2)\ldots f^s(x_k). \tag{23}$$

Proof. In Eq. (9) and Eq. (14), if $d=k$ and $r_1=1$, $r_2=2,\ldots$, $r_k=k$, Eq. (23) is obtained.

4 Conclusion

Some results connecting distributions of order statistics of *innid* random variables to that of order statistics of *iid* random variables are given.

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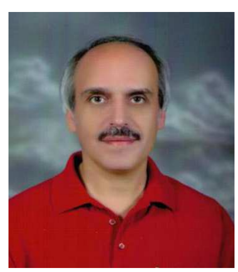

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