Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/120606

Ostrowski Type Inequalities for Functions Whose Second Derivatives are Convex Generalized.

Yenny Rangel-Oliveros and Miguel Vivas-Cortez[*]

Facultad de Ciencias Exactas y Naturales, Pontificia Universidad Católica del Ecuador (PUCE), Escuela de Ciencias Físicas y Matemática. Sede Quito, Ecuador.

Received: 2 Jun. 2018, Revised: 3 Sep. 2018, Accepted: 13 Sep. 2018
Published online: 1 Nov. 2018

Abstract: In this paper we introduce the notion of $s-\varphi-$convex functions as generalization of convex functions. Some basic results under various conditions for the function φ are investigated. Moreover, we establish Ostrowski type inequalities for twice differentiable mappings which are $s-\varphi-$convex.

Keywords: $\varphi-$convex function, $s-\varphi-$convex function, $s-$convex function, Ostrowski inequality.

1 Introduction

The Ostrowski's inequality was introduced by Alexander Ostrowski in [1], and with the passing of the years, generalizations on the same, involving derivatives of the function under study, have taken place.

Ostrowski's Inequality. Let $f: I \subset [0,+\infty) \to \mathbb{R}$ be a differentiable function on *int*(I), such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f'(x)| \leq M$ for all $x \in [a,b]$, then the inequality:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq M(b-a)\left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right], \tag{1}$$

holds for all $x \in [a,b]$. Recently, several generalizations of the Ostrowski integral inequality are considered by many authors; for instance covering the following concepts: functions of bounded variation, Lipschitzian, monotonic, absolutely continuous and n times differentiable mappings with error estimates with some special means together with some numerical quadrature rules. For recent results and generalizations concerning Ostrowski's inequality, we refer the reader to the recent papers [1,2,3, 26,27]. The convex functions play a significant role in many fields, for example in biological system, economy, optimization and so on [13,24]. And many important inequalities are established for these classes of functions. Also the evolution of the concept of convexity has had a great impact in the community of investigators. In recent years, for example, generalized concepts such as s-convexity (see[8]), h-convexity (see [25,28]), m-convexity (see [6,11]), MT- convexity (see[17]) and others, as well as combinations of these new concepts have been introduced.

The role of convex sets, convex functions and their generalizations are important in applied mathematics specially in nonlinear programming and optimization theory. For example in economics, convexity plays a fundamental role in equilibrium and duality theory. The convexity of sets and functions have been the object of many studies in recent years. But in many new problems encountered in applied mathematics, the notion of convexity is not enough to reach favorite results and hence it is necessary to extend the notion of convexity to the new generalized notions. Recently, several extensions have been considered for the classical convex functions such that some of these new concepts are based on extension of the domain of a convex function (a convex set) to a generalized form and some of them are new definitions that there is no generalization on the domain but on the form of the definition. Some new generalized concepts in this point of view are pseudo-convex functions [18], quasi-convex functions [4], invex functions [14], preinvex functions [20], B-vex functions [16], B-preinvex functions [7] and E-convex functions [29].

[*] Corresponding author e-mail: mjvivas@puce.edu.ec

2 Preliminaries

This section contains definitions and properties of generalized convexity. Recall that a real-valued function f defined in a real interval J is said to be convex if for all $x,y\in J$ and for any $t\in[0,1]$ the inequality

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y) \tag{2}$$

holds. If inequality (2) is strict when we say that f is strictly convex, and if inequality (2) is reversed the function f is said to be concave. In [15] Hudzik H. and Maligranda L. introduced the following generalized concept.

Definition 1. *Let $0<s\leq 1$. The function $f:I\subset\mathbb{R}\to\mathbb{R}$ is called a s-convex function in first sense if*

$$f(tx+(1-t)y)\leq t^{s}f(x)+(1-t^{s})f(y) \tag{3}$$

holds for all $x,y\in I$ and $t\in[0,1]$.

Besides in [12], Gordji M. E., Delavar M. R., De la Sen M. introduced the definition $\varphi-$ convex function, where $\varphi:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a bifunction except for special case.

Definition 2. *A function $f:I\subset\mathbb{R}\to\mathbb{R}$ is called convex with respect to φ (briefly $\varphi-$ convex), if*

$$f(tx+(1-t)y)\leq f(y)+t\varphi(f(x),f(y)), \tag{4}$$

for all $x,y\in I$ and $t\in[0,1]$.

In fact above definition geometrically says that if a function is $\varphi-$convex on I, then it's graph between any $x,y\in I$ is on or under the path starting from $(y,f(y))$ and ending at $(x,f(y)+\varphi(f(x),f(y)))$. If $f(x)$ should be the end point of the path for every $x,y\in I$, then we have $\varphi(x,y)=x-y$ and the function reduces to a convex one. Note that by taking $x=y$ in (4) we get $t\varphi(f(x),f(x))\geq 0$ for any $x\in I$ and $t\in[0,1]$ which implies that

$$\varphi(f(x),f(x))\geq 0$$

for any $x\in I$. Also if we take $t=1$ in (4) we get

$$f(x)-f(y)\leq\varphi(f(x),f(y))$$

for any $x,y\in I$. If $f:I\to\mathbb{R}$ is a convex function and $\varphi:I\times I\to\mathbb{R}$ is an arbitrary bifunction that satisfies

$$\varphi(x,y)\geq x-y$$

for any $x,y\in I$, then

$$\begin{aligned}f(tx+(1-t)y)&\leq f(y)+t[f(x)-f(y)]\\&\leq f(y)+t\varphi(f(x),f(y)),\end{aligned}$$

showing that f is $\varphi-$convex.

With this, we introduce the notion of $s-\varphi$-convex functions as a generalization of $s-$convex functions in first sense combined with the definition 2.

Definition 3. *Let $0<s\leq 1$. A function $f:I\subset\mathbb{R}\to\mathbb{R}$ is called $s-\varphi-$convex with respect to bifunction $\varphi:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ (briefly $\varphi-$convex), if*

$$f(tx+(1-t)y)\leq f(y)+t^{s}\varphi(f(x),f(y)) \tag{5}$$

for all $x,y\in I$ and $t\in[0,1]$.

Remark 1.
i) If we take $s=1$ in (5), then we have the definition of φ-convex function.
ii) If we take $\varphi(x,y)=x-y$ in (5), then the definition of $s-\varphi-$convex function is reduced to the definition of s-convex function on the first sense.

Example 1. Let $f(x)=x^2$, then f is convex and $\frac{1}{2}$-φ-convex with $\varphi(u,v)=2u+v$, indeed

$$\begin{aligned}f(tx+(1-t)y)&=(tx+(1-t)y)^2\\&=t^2x^2+2t(1-t)xy+(1-t)^2y^2\\&\leq y^2+tx^2+2txy\\&=y^2+t^{\frac{1}{2}}\left[t^{\frac{1}{2}}x^2+2t^{\frac{1}{2}}xy\right].\end{aligned}$$

On the other hand;

$$\begin{aligned}0<t<1&\Longrightarrow 0<t^{\frac{1}{2}}<1\\&\Longrightarrow t^{\frac{1}{2}}x^2+2t^{\frac{1}{2}}xy\leq x^2+2xy\leq x^2+x^2+y^2.\end{aligned}$$

Hence,

$$\begin{aligned}f(tx+(1-t)y)&\leq y^2+t^{\frac{1}{2}}\left[2x^2+y^2\right]\\&=f(y)+t^{\frac{1}{2}}\varphi(f(x),f(y)).\end{aligned}$$

Example 2. Let $f(x)=x^n$ and $0<s\leq 1$, then f is convex and s-φ-convex with

$$\varphi(u,v)=\sum_{k=1}^{n}\binom{n}{k}v^{1-\frac{k}{n}}\left(u^{\frac{1}{n}}-v^{\frac{1}{n}}\right)^{n},$$

indeed

$$\begin{aligned}f(tx+(1-t)y)&=f(y+t(x-y))=(y+t(x-y))^n\\&=y^n+\sum_{k=1}^{n}\binom{n}{k}y^{n-k}(t(x-y))^n\\&=y^n+t^s\left[\sum_{k=1}^{n}\binom{n}{k}t^{n-s}y^{n-k}(x-y)^n\right]\\&\leq y^n+t^s\left[\sum_{k=1}^{n}\binom{n}{k}(y^n)^{\frac{n-k}{n}}\left((x^n)^{\frac{1}{n}}-(y^n)^{\frac{1}{n}}\right)^{n}\right].\end{aligned}$$

The following results are $s-\varphi-$convex versions of some basic theorems and propositions related to $\varphi-$convex functions.

Definition 4. *The function φ is said to be*
(i) nonnegatively homogeneous if $\varphi(\gamma x,\gamma y)=\gamma\varphi(x,y)$ for all $x,y\in\mathbb{R}$ and all $\gamma\geq 0$.
(ii) additive if $\varphi(x_1,y_1)+\varphi(x_2,y_2)=\varphi(x_1+x_2,y_1+y_2)$

for all $x_1,x_2,y_1,y_2 \in \mathbb{R}$.
(iii) nonnegatively linear it satisfies conditions (i) and (ii).
(iv) nondecreasing in first variable if $x \le y$ *implies that* $\varphi(x,z) \le \varphi(y,z)$, *for all* $x,y,z \in \mathbb{R}$.
(v) nonnegatively sublinear in first variable if $\varphi(\gamma x + y,z) \le \gamma\varphi(x,z) + \varphi(y,z)$, *for all* $x,y,z \in \mathbb{R}$ *and* $\gamma \ge 0$.

The proof of Propositions 1, 2 and Theorem 1 is straightforward.

Proposition 1.*Let* $0 < s \le 1$. *Consider* $s-\varphi-$*convex function* $f : I \subset \mathbb{R} \to \mathbb{R}$ *such that* φ *is nonnegatively homogeneous. Then for any* $\gamma \ge 0$, *the function* $\gamma f : I \subset \mathbb{R} \to \mathbb{R}$ *is* $s-\varphi-$*convex.*

Proposition 2.*Let* $0 < s \le 1$. *Consider two* $s-\varphi-$*convex functions* $f,g : I \subset \mathbb{R} \to \mathbb{R}$ *such that* φ *is additive. Then* $f+g : I \subset \mathbb{R} \to \mathbb{R}$ *is* $s-\varphi-$*convex.*

Theorem 1.*Let* $0 < s \le 1$. *Consider* $s-\varphi-$*convex functions* $f_i : I \subset \mathbb{R} \to \mathbb{R}$ *for* $i = 1,\dots,n$ *such that* φ *is nonnegatively linear. Then for* $\gamma_i \ge 0$, $i = 1,\dots,n$, *the function* $f = \sum_{i=1}^n \gamma_i f_i : I \subset \mathbb{R} \to \mathbb{R}$ *is* $s-\varphi-$*convex.*

The class of $s-\varphi-$convex functions with special conditions are closed under Sup operation.

Theorem 2.*Let* $0 < s \le 1$. *Suppose that* $\{f_j : I \to \mathbb{R}, j \in J\}$ *is a nonempty collection of* $s-\varphi-$*convex functions such that*
(a) there exist $\alpha \in [0,\infty]$ *and* $\beta \in [-1,\infty]$ *such that* $\varphi(x,y) = \alpha x + \beta y$ *for all* $x,y \in \mathbb{R}$,
(b) For each $x \in I$, $sup_{j\in J} f_j(x)$ *exists in* $\mathbb{R}$.
Then the function $f : I \to \mathbb{R}$ *defined by* $f(x) = sup_{j\in J} f_j(x)$ *for each* $x \in I$, *is* $s-\varphi-$*convex.*

*Proof.*For any $x,y \in I$ and $\lambda \in [0,1]$, we can drive following relations

$$\begin{aligned} f(tx+(1-t)y) &= sup_{j\in J} f_j(tx+(1-t)y) \\ &\le sup_{j\in J}\{f_j(y)+t^s\varphi(f_j(x),f_j(y))\} \\ &= sup_{j\in J}\{f_j(y)+t^s(\alpha f_j(x)+\beta f_j(y))\} \\ &\le (1+\beta t^s)sup_{j\in J}\{f_j(y)\}+\alpha t^s sup_{j\in J}\{f_j(x)\} \\ &= (1+\beta t^s)f(y)+\alpha t^s f(x) \\ &= f(y)+t^s(\beta f(y)+\alpha f(x)) \\ &= f(y)+t^s\varphi(f(x),f(y)). \end{aligned}$$

The proof is complete.

Proposition 3.*Let* $0 < s \le 1$. *If* $f : I \subset \mathbb{R} \to \mathbb{R}$ *is* $s-\varphi-$*convex and attains a local minimun at* $x_0 \in I$, *then* $\varphi(f(x), f(x_0)) \ge 0$, *for any* $x \in I$.

*Proof.*Suppose that f has a local minimun at $x_0 \in I$. For any $x \in I$ we can find $t > 0$ sufficiently small such that $tx+(1-t)x_0 \in N_r(x_0)$. Therefore we get

$$f(x_0) \le f(tx+(1-t)x_0) \le f(x_0)+t^s\varphi(f(x),f(x_0))$$

This implies that

$$t^s\varphi(f(x),f(x_0)) \ge 0,$$

the proof is complete.

Proposition 4.*Let* $0 < s \le 1$. *Any* $s-\varphi-$*convex function* $f : [a,b] \to \mathbb{R}$ *with respect to a bifunction* φ *bounded from above on* $f([a,b]) \times f([a,b])$, *has lower and upper bounds. That is, there are* $m > 0$ *and* $M > 0$ *such that* $m \le f(x) \le M$, *for all* $x \in [a,b]$.

*Proof.*Suppose that M_φ is upper bound of φ on $f([a,b]) \times f([a,b])$. Consider any $x = ta+(1-t)b \in [a,b]$ with $t \in [0,1]$. In fact, we have

$$\begin{aligned} f(x) = f(ta+(1-t)b) &\le f(b)+t^s\varphi(f(a),f(b)) \\ &\le max\{f(b), f(b)+\varphi(f(a),f(b))\} \\ &\le max\{f(b), f(b)+M_\varphi\}. \end{aligned}$$

Now set $M = max\{f(b), f(b)+M_\varphi\}$. For lower bound of f consider an arbitrary point in the form $\frac{a+b}{2}-t$ in $[a,b]$, then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{a+b}{4}+\frac{t}{2}+\frac{a+b}{4}-\frac{t}{2}\right) \\ &= f\left(\frac{1}{2}\left(\frac{a+b}{2}+t\right)+\frac{1}{2}\left(\frac{a+b}{2}-t\right)\right) \\ &\le f\left(\frac{a+b}{2}-t\right)+\frac{1}{2^s}\varphi(f\left(\frac{a+b}{2}+t\right), f\left(\frac{a+b}{2}-t\right)) \\ &\le f\left(\frac{a+b}{2}-t\right)+KM_\varphi, \end{aligned}$$

where K is an upper bound of $\frac{1}{2^s}$ on $[0,1]$. Now consider $m = f\left(\frac{a+b}{2}\right)-KM_\varphi$, and the statement is proved.

Definition 5.*A function* $f : [a,b] \to \mathbb{R}$ *is said to satisfy a Hölder condition, or is* α*-Hölder continuous on* $[a,b]$, *when there are nonnegative real constants* K,α *such that*

$$|f(x)-f(y)| \le K|x-y|^\alpha, \tag{6}$$

for all $x,y \in [a,b]$.

Lemma 1.*Suppose that* $f : I \to \mathbb{R}$ *is an* $s-\varphi-$*convex function and* φ *is bounded from above on* $f(I) \times f(I)$. *Then* f *satisfies the Hölder condition on any closed interval* $[a,b]$ *contained in* I°, *the interior of* I. *Hence,* f *is uniformly continuous on* $[a,b]$ *and continuous on* I°.

*Proof.*Let M_φ be the upper bound of φ on $f(I) \times f(I)$. Consider closed interval $[a,b]$ in I° and choose $\varepsilon > 0$ such that $[a-\varepsilon, b+\varepsilon]$ belongs to I. Suppose that x,y are distinct points of $[a,b]$. Set $z = y+\frac{\varepsilon}{|y-x|}(y-x)$ and $t = \frac{|y-x|}{\varepsilon+|y-x|}$. So it is not hard to see that $z \in [a-\varepsilon, b+\varepsilon]$ and $y = tz+(1-t)x$. Then

$$\begin{aligned} f(y) &= f(tz+(1-t)x) \\ &\le f(x)+t^s\varphi(f(z),f(x)) \\ &\le f(x)+t^sM_\varphi. \end{aligned}$$

This implies that

$$\begin{aligned} f(y)-f(x) \le t^sM_\varphi &= \frac{|y-x|^s}{(\varepsilon+|y-x|)^s}M_\varphi \\ &\le \frac{|y-x|^s}{\varepsilon^s}M_\varphi, \end{aligned}$$

if $K = \frac{M_\varphi}{\varepsilon^s}$ then

$$(f(y) - f(x)) \le K|y-x|^s.$$

Also if we change the place of x, y in above argument we have $f(x) - f(y) \le K|y-x|^s$. Therefore $|f(x) - f(y)| \le K|y-x|^s$. Hence f is s-Hölder continuous, and as consequence f is uniformly continuous on $[a,b] \subset I^\circ$. Finally since $[a,b]$ is arbitrary on I°, then f is continuous on I°.

Corollary 1. *Suppose that $f : I \to \mathbb{R}$ is an s-φ-convex function and φ is bounded from above on $f(I) \times f(I)$, then f is integrable on $f(I) \times f(I)$.*

Corollary 2. *Any s-φ-convex function f, with bifunction φ bounded on a subset $I \subset \mathbb{R}$ admits a uniformly continuous extention to $\mathbb{R}$, which is Hölder continuous with constant C and exponent α. The largest such extension is:*

$$f^*(x) = \inf_{y\in\mathbb{R}} \{f(y) + C|x-y|^\alpha\}.$$

Proof. It's inmediate since all $s-\varphi-$convex function is a Hölder contiuous function.

The next result establishes an Ostrowski type inequality for $s-\varphi-$convex functions.

Corollary 3. *Suppose that $f : [a,b] \to \mathbb{R}$ is an $s-\varphi-$convex function. Then for all $x \in [a,b]$ we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \frac{K}{s+1}\left(\left(\frac{b-x}{b-a}\right)^{s+1} + \left(\frac{x-a}{b-a}\right)^{s+1} \right)(b-a)^s.$$

Proof. It's inmediate since all $s-\varphi-$convex function is a Hölder contiuous function and by theorem 3 in [10].

3 Main Results

In this section, we give some integral approximation of $f \in C^2([a,b])$ such that $f'' \in L([a,b])$ using the following lemma as the main tool (see [9]).

Lemma 2. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping such that f' is absolutely continuous on $[a,b]$. Then for all $x \in [a,b]$ we have the identity*

$$\int_a^b f(t)dt = \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1} + (-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x) + \int_a^b K_2(x,t) f''(t)dt,$$

where the kernel $K_2 : [a,b]^2 \to \mathbb{R}$ is given by

$$K_2(x,t) = \begin{cases} \frac{(t-a)^2}{2!} & \text{if } t \in [a,x] \\ \frac{(t-b)^2}{2!} & \text{if } t \in (x,b] \end{cases}$$

with $x \in [a,b]$.

Theorem 3. *Let $f : I \to \mathbb{R}$ be twice differentiable mapping on $[a,b]$ such that $f'' \in L([a,b])$ and $0 < s \le 1$. If $|f''|$ is s-φ-convex, then the following inequality*

$$\begin{aligned}&\left|\int_a^b f(t)dt - \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1} + (-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right| \\ &\le \frac{(x-a)^3}{2}\left(\frac{1}{3}\left|f''(a)\right| + \frac{A_\varphi(x)}{s+3}\right) \\ &+ \frac{(b-x)^3}{2}\left[\frac{1}{3}|f''(x)| + \sum_{k=0}^{2}\binom{2}{k}(-1)^k\frac{B_\varphi(x)}{k+s+1}\right],\end{aligned}$$

holds for all $x \in [a,b]$, where $A_\varphi(x) = \varphi\left(\left|f''(a)\right|, \left|f''(x)\right|\right)$ and $B_\varphi(x) = \varphi\left(\left|f''(x)\right|, \left|f''(b)\right|\right)$.

Proof. From Lemma 2, properties of modulus, making the changes of variables $u = (1-t)a + tx$ in the first integral and $u = (1-t)x + tb$ in the second integral we have that,

$$\begin{aligned}&\left|\int_a^b f(t)dt - \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1} + (-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right| \\ &\le \int_a^x \frac{(u-a)^2}{2}\left|f''(u)\right| du + \int_x^b \frac{(b-u)^2}{2}\left|f''(u)\right| du \\ &= \frac{(x-a)^3}{2}\int_0^1 t^2 \left|f''((1-t)a+tx)\right| dt \\ &+ \frac{(b-x)^3}{2}\int_0^1 (1-t)^2 \left|f''((1-t)x+tb)\right| dt.\end{aligned}$$

Since $|f''|$ is s-φ-convex, (5) gives

$$\begin{aligned}&\left|\int_a^b f(t)dt - \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1} + (-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right| \\ &\le \frac{(x-a)^3}{2}\int_0^1 t^2\left(\left|f''(a)\right| + t^s\varphi\left(|f''(a)|, |f''(x)|\right)\right) dt \\ &+ \frac{(b-x)^3}{2}\int_0^1 (1-t)^2\left(\left|f''(x)\right| + t^s\varphi\left(|f''(x)|, |f''(b)|\right)\right) dt \\ &= \frac{(x-a)^3}{2}\left(\frac{1}{3}\left|f''(a)\right| + \frac{A_\varphi(x)}{s+3}\right) \\ &+ \frac{(b-x)^3}{2}\left[\frac{1}{3}|f''(x)| + \sum_{k=0}^{2}\binom{2}{k}(-1)^k\frac{B_\varphi(x)}{s+k+1}\right],\end{aligned}$$

which is the desired result. The proof is completed.

Remark 2. If in Theorem 3 we choose $x = \frac{a+b}{2}$, then we have

$$\begin{aligned}&\left|\int_a^b f(t)dt - (b-a)f\left(\frac{a+b}{2}\right) - \frac{(b-a)^3}{4!}f''\left(\frac{a+b}{2}\right)\right| \\ &\le \frac{(b-a)^3}{16}\left[\frac{1}{3}\left|f''(a)\right| + \frac{A_\varphi(\frac{a+b}{2})}{s+3} + \frac{1}{3}\left|f''\left(\frac{a+b}{2}\right)\right| + \sum_{k=0}^{2}\binom{2}{k}(-1)^k\frac{B_\varphi(\frac{a+b}{2})}{k+s+1}\right],\end{aligned}$$

Corollary 4. *Let $f: I \to \mathbb{R}$ be twice differentiable mapping on $[a,b]$ such that $f'' \in L([a,b])$ and $0 < s \le 1$. If $|f''|$ is s-convex in the first sense, we have the following estimate*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\le \frac{s}{6(s+3)}|f''(a)| + \frac{(b-x)^3}{2}|f''(b)|\sum_{k=0}^{2}\binom{2}{k}\frac{(-1)^k}{k+s+1} +$$
$$\left[\frac{(x-a)^3}{2(s+3)} + \frac{(b-x)^3}{2}\left(\frac{1}{3} - \sum_{k=0}^{2}\binom{2}{k}\frac{(-1)^k}{k+s+1}\right)\right]|f''(x)|.$$

Proof. Taking $\varphi(u,v) = u - v$ in Theorem 3.

Corollary 5. *Let $f: I \to \mathbb{R}$ be twice differentiable mapping on $[a,b]$ such that $f'' \in L([a,b])$. If $|f''|$ is φ-convex, then the following inequality*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\le \frac{(x-a)^3}{2}\left(\frac{1}{3}\left|f''(a)\right| + \frac{1}{4}\varphi\left(|f''(a)|,|f''(x)|\right)\right)$$
$$+ \frac{(b-x)^3}{2}\left[\frac{1}{3}|f''(x)| + \frac{1}{12}\varphi\left(|f''(x)|,|f''(b)|\right)\right]$$

holds for all $x \in [a,b]$.

Proof. Taking $s = 1$ in Theorem 3.

Corollary 6. *Let $f: I \to \mathbb{R}$ be twice differentiable mapping on $[a,b]$ such that $f'' \in L([a,b])$. If $|f''|$ is convex, we have the following estimate*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\le \frac{1}{24}\left((x-a)^3|f''(a)| + (b-x)^3|f''(b)|\right)$$
$$+ \frac{1}{8}\left((x-a)^3 + (b-x)^3\right)|f''(x)|.$$

Proof. Taking $s = 1$ in Corollary 4.

Theorem 4. *Let $f: I \to \mathbb{R}$ be twice differentiable mapping on $[a,b]$ such that $f'' \in L([a,b])$, $0 < s \le 1$ and let $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $|f''|^q$ is s-φ-convex, then the following inequality*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\le \frac{(x-a)^3}{2(s+1)^{\frac{1}{q}}(2p+1)^{\frac{1}{p}}}\left((s+1)|f''(a)|^q + A_\varphi(x)\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^3}{2(s+1)^{\frac{1}{q}}(2p+1)^{\frac{1}{p}}}\left((s+1)|f''(x)|^q + B_\varphi(x)\right)^{\frac{1}{q}},$$

holds for all $x \in [a,b]$, where $A_\varphi(x) = \varphi\left(\left|f''(a)\right|^q, \left|f''(x)\right|^q\right)$ and $B_\varphi(x) = \varphi\left(\left|f''(x)\right|^q, \left|f''(b)\right|^q\right)$.

Proof. From Lemma 2, properties of modulus, and Hölder's inequality, we have

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\le \int_a^x \frac{(u-a)^2}{2}\left|f''(u)\right|du + \int_x^b \frac{(b-u)^2}{2}\left|f''(u)\right|du$$
$$= \frac{(x-a)^3}{2}\int_0^1 t^2\left|f''((1-t)a+tx)\right|dt$$
$$+ \frac{(b-x)^3}{2}\int_0^1 (1-t)^2\left|f''((1-t)x+tb)\right|dt$$
$$\le \frac{(x-a)^3}{2}\left(\int_0^1 t^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''((1-t)a+tx)\right|^q dt\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^3}{2}\left(\int_0^1 (1-t)^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''((1-t)x+tb)\right|^q dt\right)^{\frac{1}{q}}$$
$$= \frac{(x-a)^3}{2(2p+1)^{\frac{1}{p}}}\left(\int_0^1\left|f''((1-t)a+tx)\right|^q dt\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^3}{2(2p+1)^{\frac{1}{p}}}\left(\int_0^1\left|f''((1-t)x+tb)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f''|^q$ is s-φ-convex, we deduce

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\le \frac{(x-a)^3}{2(2p+1)^{\frac{1}{p}}}\left(\int_0^1\left(|f''(a)|^q + t^s\varphi\left(|f''(a)|^q,|f''(x)|^q\right)\right)dt\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^3}{2(2p+1)^{\frac{1}{p}}}\left(\int_0^1\left(|f''(x)|^q + t^s\varphi\left(|f''(x)|^q,|f''(b)|^q\right)\right)dt\right)^{\frac{1}{q}}$$
$$= \frac{(x-a)^3}{2(s+1)^{\frac{1}{q}}(2p+1)^{\frac{1}{p}}}\left((s+1)|f''(a)|^q + \varphi\left(|f''(a)|^q,|f''(x)|^q\right)\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^3}{2(s+1)^{\frac{1}{q}}(2p+1)^{\frac{1}{p}}}\left((s+1)|f''(x)|^q + \varphi\left(|f''(x)|^q,|f''(b)|^q\right)\right)^{\frac{1}{q}}.$$

Remark 3. If in Theorem 4 we choose $x = \frac{a+b}{2}$, then we have

$$\left|\int_a^b f(t)dt - (b-a)f\left(\frac{a+b}{2}\right) - \frac{(b-a)^3}{4!}f''\left(\frac{a+b}{2}\right)\right|$$
$$\le \frac{(b-a)^3}{32(s+1)^{\frac{1}{q}}(2p+1)^{\frac{1}{p}}}$$
$$\left[\left((s+1)|f''(a)|^q + A_\varphi(\frac{a+b}{2})\right)^{\frac{1}{q}} + \left((s+1)|f''(\frac{a+b}{2})|^q + B_\varphi(\frac{a+b}{2})\right)^{\frac{1}{q}}\right],$$

Corollary 7. *Let $f: I \to \mathbb{R}$ be twice differentiable mapping on $[a,b]$ such that $f'' \in L([a,b])$, $0 < s \le 1$ and let $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $|f''|^q$ is s-convex in the first sense, then* *the following inequality holds*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\leq \frac{(x-a)^3}{2(s+1)^{\frac{1}{q}}(2p+1)^{\frac{1}{p}}}\left(s|f''(a)|^q+|f''(x)|^q\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^3}{2(s+1)^{\frac{1}{q}}(2p+1)^{\frac{1}{p}}}\left(s|f''(x)|^q+|f''(b)|^q\right)^{\frac{1}{q}}. \qquad (7)$$

*Proof.*Taking $\varphi(u,v)=u-v$ in Theorem 4.

Corollary 8.*Let $f:I\to\mathbb{R}$ be twice differentiable mapping on $[a,b]$ such that $f''\in L([a,b])$, $0<s\leq 1$ and let $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $|f''|^q$ is s-convex in the first sense, then the following inequality holds.*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\leq \frac{(x-a)^3}{2(s+1)^{\frac{1}{q}}(2p+1)^{\frac{1}{p}}}\left(s|f''(a)|+|f''(x)|\right)$$
$$+ \frac{(b-x)^3}{2(s+1)^{\frac{1}{q}}(2p+1)^{\frac{1}{p}}}\left(s|f''(x)|+|f''(b)|\right).$$

*Proof.*Taking $\varphi(u,v)=u-v$ in Theorem 4, we obtain (7). Then using the following algebraic inequality for all $a,b\geq 0$, and $0\leq\alpha\leq 1$ we have $(a+b)^\alpha\leq a^\alpha+b^\alpha$, we get the desired result.

Theorem 5.*Let $f:I\to\mathbb{R}$ be twice differentiable mapping on $[a,b]$ such that $f''\in L([a,b])$, $0<s\leq 1$ and let $q>1$. If $|f''|^q$ is s-φ-convex, then the following inequality*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\leq \frac{3^{\frac{1}{q}}(x-a)^3}{6}\left(\frac{1}{3}\left|f''(a)\right|^q+\frac{A_\varphi(x)}{s+3}\right)^{\frac{1}{q}}$$
$$+ \frac{3^{\frac{1}{q}}(b-x)^3}{6}\left(\frac{1}{3}\left|f''(x)\right|^q+B_\varphi(x)\sum_{k=0}^{2}\binom{2}{k}\frac{(-1)^k}{s+k+1}\right)^{\frac{1}{q}},$$

holds for all $x\in[a,b]$, where $A_\varphi(x)=\varphi\left(\left|f''(a)\right|^q,\left|f''(x)\right|^q\right)$ and $B_\varphi(x)=\varphi\left(\left|f''(x)\right|^q,\left|f''(b)\right|^q\right)$.

*Proof.*From Lemma 2, properties of modulus, and power mean inequality, we have

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\leq \int_a^x\frac{(u-a)^2}{2}\left|f''(u)\right|du+\int_x^b\frac{(b-u)^2}{2}\left|f''(u)\right|du$$
$$= \frac{(x-a)^3}{2}\int_0^1 t^2\left|f''((1-t)a+tx)\right|dt$$
$$+ \frac{(b-x)^3}{2}\int_0^1(1-t)^2\left|f''((1-t)x+tb)\right|dt$$
$$\leq \frac{(x-a)^3}{2}\left(\int_0^1 t^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left|f''((1-t)a+tx)\right|^q dt\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^3}{2}\left(\int_0^1(1-t)^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^2\left|f''((1-t)x+tb)\right|^q dt\right)^{\frac{1}{q}}$$
$$= \frac{3^{\frac{1}{q}}(x-a)^3}{6}\left(\int_0^1 t^2\left|f''((1-t)a+tx)\right|^q dt\right)^{\frac{1}{q}}$$
$$+ \frac{3^{\frac{1}{q}}(b-x)^3}{6}\left(\int_0^1(1-t)^2\left|f''((1-t)x+tb)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f''|^q$ is s-φ-convex, we deduce

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\leq \frac{3^{\frac{1}{q}}(x-a)^3}{6}\left(\left|f''(a)\right|^q\int_0^1 t^2dt+A_\varphi(x)\int_0^1 t^{2+s}dt\right)^{\frac{1}{q}}$$
$$+ \frac{3^{\frac{1}{q}}(b-x)^3}{6}\left(\left|f''(x)\right|^q\int_0^1(1-t)^2dt+B_\varphi(x)\int_0^1 t^s(1-t)^2dt\right)^{\frac{1}{q}}$$
$$= \frac{3^{\frac{1}{q}}(x-a)^3}{6}\left(\frac{1}{3}\left|f''(a)\right|^q+\frac{A_\varphi(x)}{s+3}\right)^{\frac{1}{q}}$$
$$+ \frac{3^{\frac{1}{q}}(b-x)^3}{6}\left(\frac{1}{3}\left|f''(x)\right|^q+B_\varphi(x)\sum_{k=0}^{2}\binom{2}{k}\frac{(-1)^k}{k+s+1}\right)^{\frac{1}{q}}.$$

The proof is completed.

Remark 4. If in Theorem 5 we choose $x=\frac{a+b}{2}$, then we have

$$\left|\int_a^b f(t)dt-(b-a)f\left(\frac{a+b}{2}\right)-\frac{(b-a)^3}{4!}f''\left(\frac{a+b}{2}\right)\right|\leq\frac{3^{\frac{1}{q}}(b-a)^3}{48}$$
$$\left[\left(\frac{1}{3}\left|f''(a)\right|^q+\frac{A_\varphi(\frac{a+b}{2})}{s+3}\right)^{\frac{1}{q}}+\left(\frac{1}{3}\left|f''\left(\frac{a+b}{2}\right)\right|^q+B_\varphi\left(\frac{a+b}{2}\right)\sum_{k=0}^{2}\binom{2}{k}\frac{(-1)^k}{s+k+1}\right)^{\frac{1}{q}}\right],$$

Corollary 9.*Let $f:I\to\mathbb{R}$ be twice differentiable mapping on $[a,b]$ such that $f''\in L([a,b])$,and let $q>1$. If $|f''|^q$ is φ-convex, then the following inequality*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\leq \frac{3^{\frac{1}{q}}(x-a)^3}{6}\left(\frac{1}{3}\left|f''(a)\right|^q+\frac{1}{4}\varphi\left(\left|f''(a)\right|^q,\left|f''(x)\right|^q\right)\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^3}{6}\left(\left|f''(x)\right|^q+\frac{1}{4}\varphi\left(\left|f''(x)\right|^q,\left|f''(b)\right|^q\right)\right)^{\frac{1}{q}},$$

holds for all $x\in[a,b]$.

Corollary 10. *Let $f : I \to \mathbb{R}$ be twice differentiable mapping on $[a,b]$ such that $f'' \in L([a,b])$, $0 < s \le 1$ and let $q > 1$. If $|f''|^q$ is s-convex in the first sense, then the following inequality*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\le \frac{3^{\frac{1}{q}}(x-a)^3}{6(s+3)^{\frac{1}{q}}}\left(\frac{s}{3}\left|f''(a)\right|^q+\left|f''(x)\right|^q\right)^{\frac{1}{q}} + \frac{3^{\frac{1}{q}}(b-x)^3}{6}$$
$$\left(\frac{1}{3}\left|f''(x)\right|^q + \left[\left|f''(b)\right|^q - \left|f''(x)\right|^q\right]\sum_{k=0}^{2}\binom{2}{k}\frac{(-1)^k}{k+s+1}\right)^{\frac{1}{q}},$$

holds for all $x \in [a,b]$.

Proof. Taking $\varphi(u,v) = u - v$ in Theorem 5.

Theorem 6. *Let $f : I \to \mathbb{R}$ be twice differentiable mapping on $[a,b]$ such that $f'' \in L([a,b])$, $0 < s \le 1$ and let $q > 1$. If $|f''|^q$ is s-φ-convex, then the following inequality*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\le \frac{(x-a)^3}{2}\left(\frac{1}{2q+1}\left|f''(a)\right|^q + \frac{A_\varphi(x)}{2q+s+1}\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^3}{2}\left(\frac{1}{2q+1}\left|f''(x)\right|^q + \sum_{k=0}^{2q}\binom{2q}{k}\frac{(-1)^k}{k+s+1}B_\varphi(x)\right)^{\frac{1}{q}},$$

holds for all $x \in [a,b]$, where $A_\varphi(x) = \varphi\left(\left|f''(a)\right|^q, \left|f''(x)\right|^q\right)$ and $B_\varphi(x) = \varphi\left(\left|f''(x)\right|^q, \left|f''(b)\right|^q\right)$.

Proof. From Lemma 2, properties of modulus, and power mean inequality, we have

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\le \int_a^x \frac{(u-a)^2}{2}\left|f''(u)\right|du + \int_x^b \frac{(b-u)^2}{2}\left|f''(u)\right|du$$
$$= \frac{(x-a)^3}{2}\int_0^1 t^2\left|f''((1-t)a+tx)\right|dt$$
$$+ \frac{(b-x)^3}{2}\int_0^1 (1-t)^2\left|f''((1-t)x+tb)\right|dt$$
$$\le \frac{(x-a)^3}{2}\left(\int_0^1 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{2q}\left|f''((1-t)a+tx)\right|^q dt\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^3}{2}\left(\int_0^1 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)^{2q}\left|f''((1-t)x+tb)\right|^q dt\right)^{\frac{1}{q}}$$
$$= \frac{(x-a)^3}{2}\left(\int_0^1 t^{2q}\left|f''((1-t)a+tx)\right|^q dt\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^3}{2}\left(\int_0^1 (1-t)^{2q}\left|f''((1-t)x+tb)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f''|^q$ is s-φ-convex, we deduce

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\le \frac{(x-a)^3}{2}\left(\left|f''(a)\right|^q\int_0^1 t^{2q}dt + A_\varphi(x)\int_0^1 t^{2q+s}dt\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^3}{2}\left(\left|f''(x)\right|^q\int_0^1 (1-t)^{2q}dt + B_\varphi(x)\int_0^1 t^s(1-t)^{2q}dt\right)^{\frac{1}{q}}$$
$$= \frac{(x-a)^3}{2}\left(\frac{1}{2q+1}\left|f''(a)\right|^q + \frac{A_\varphi(x)}{2q+s+1}\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^3}{2}\left(\frac{1}{2q+1}\left|f''(x)\right|^q + \sum_{k=0}^{2q}\binom{2q}{k}\frac{(-1)^k}{k+s+1}B_\varphi(x)\right)^{\frac{1}{q}}.$$

Which in the desired result.

Remark 5. If in Theorem 6 we choose $x = \frac{a+b}{2}$, then we have

$$\left|\int_a^b f(t)dt - (b-a)f\left(\frac{a+b}{2}\right) - \frac{(b-a)^3}{4!}f''\left(\frac{a+b}{2}\right)\right|$$
$$\le \frac{(b-a)^3}{16}\left(\frac{1}{2q+1}\left|f''(a)\right|^q + \frac{A_\varphi\left(\frac{a+b}{2}\right)}{2q+s+1}\right)^{\frac{1}{q}}$$
$$+ \frac{(b-a)^3}{16}\left(\frac{1}{2q+1}\left|f''\left(\frac{a+b}{2}\right)\right|^q + \sum_{k=0}^{2q}\binom{2q}{k}\frac{(-1)^k}{k+s+1}B_\varphi\left(\frac{a+b}{2}\right)\right)^{\frac{1}{q}}.$$

Corollary 11. *Let $f : I \to \mathbb{R}$ be twice differentiable mapping on $[a,b]$ such that $f'' \in L([a,b])$, and let $q > 1$. If $|f''|^q$ is φ-convex, then the following inequality*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\le \frac{(x-a)^3}{2}\left(\frac{1}{2q+1}\left|f''(a)\right|^q + \frac{A_\varphi(x)}{2(q+1)}\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^3}{2}\left(\frac{1}{2q+1}\left|f''(x)\right|^q + \frac{B_\varphi(x)}{2(2q+1)(q+1)}\right)^{\frac{1}{q}},$$

holds for all $x \in [a,b]$, where $A_\varphi(x) = \varphi\left(\left|f''(a)\right|^q, \left|f''(x)\right|^q\right)$ and $B_\varphi(x) = \varphi\left(\left|f''(x)\right|^q, \left|f''(b)\right|^q\right)$.

Corollary 12. *Let $f : I \to \mathbb{R}$ be twice differentiable mapping on $[a,b]$ such that $f'' \in L([a,b])$, $0 < s \le 1$ and let $q > 1$. If $|f''|^q$ is s-convex in the first sense, then the following inequality*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{2}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\le \frac{(x-a)^3}{2}\left(\frac{1}{2q+1}\left|f''(a)\right|^q + \frac{\left|f''(x)\right|^q - \left|f''(a)\right|^q}{2q+s+1}\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^3}{2}\left(\frac{1}{2q+1}\left|f''(x)\right|^q + C_{f,s,q}\right)^{\frac{1}{q}},$$

holds for all $x \in [a,b]$, *where* $C_{f,s,q} = \sum_{k=0}^{2q} \binom{2q}{k} \frac{(-1)^k}{k+s+1} \left(\left| f''(b) \right|^q - \left| f''(x) \right|^q \right)$.

Proof. Taking $\varphi(u,v) = u - v$ in Theorem 6.

4 Applications for some particular mappings

In this section we give some applications for the special case where the function $\varphi(f(x), f(y)) = f(x) - f(y)$, in this case we have that f is s-convex in the first sense.

Example 3. Let $s \in (0,1)$ and $p,q,r \in \mathbb{R}$, we define the function $f : [0,+\infty) \to \mathbb{R}$ as

$$f(t) = \begin{cases} p & \text{if } t = 0 \\ qt^s + r & \text{if } t > 0 \end{cases},$$

we have that if $q \geq 0$ and $r \leq p$, then f is s-convex in the first sense (see [15]). If we do $\varphi(f(x), f(y)) = f(x) - f(y)$, then f is s-φ-convex, but is not φ-convex because f is not convex.

Example 4. In the previous example if $s = \frac{1}{2}$, $p = 1$, $q = 2$ and $r = 1$ we have that $f : [0,+\infty) \to \mathbb{R}$, $f(t) = 2t^{\frac{1}{2}} + 1$ is $\frac{1}{2}$-φ-convex. Then if we define $g : [0,+\infty) \to \mathbb{R}$, $g(t) = \frac{8}{15} t^{\frac{5}{2}} + \frac{t^2}{2}$, we have to $g''(t) = 2t^{\frac{1}{2}} + 1$ is $\frac{1}{2}$-φ-convex in $[0,+\infty)$ with $\varphi(f(x), f(y)) = f(x) - f(y)$. Using Theorem 3, for $a,b \in [0,+\infty)$ with $a < b$ and $x \in [a,b]$, we get

$$\left| 16\left(b^{\frac{7}{2}} - a^{\frac{7}{2}}\right) + 35(a^3 - b^3)\sqrt{x} + 35(b^2 - a^2)x^{\frac{3}{2}} + 21(a-b)x^{\frac{5}{2}} \right|$$
$$\leq \frac{35}{2}\left(1 + 2\sqrt{b}\right)(b-x)^3 - \frac{10}{3}\left(\frac{7}{2} + \sqrt{a} + 6\sqrt{b}\right)(a-x)^3.$$

Remark 6. In particular if we choose $a = 0$ and $b = 1$, we have for $x \in [0,1]$, we get a graphic representation of the Example 4

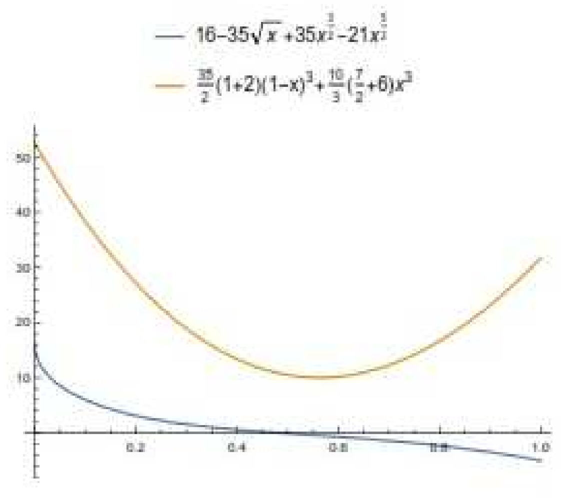

Fig. 1: Graphical representation of the inequality of example 4.

Example 5. If we define $g(t) = \frac{t^4}{12}$ we have that $g''(t)$ is $\frac{1}{2}$-φ- convex with $\varphi(u,v) = 2u + v$ (see example 1) and by Theorem 3, for $a,b \in \mathbb{R}$ with $a < b$ and $x \in [a,b]$, we have that

$$\left| \frac{b^5 - a^5}{60} - \frac{(b-a)}{12}x^4 - \left[\frac{b^2 - 2x(b-a) - a^2}{3} \right] x^3 - \left[\frac{(b-x)^3 + (x-a)^3}{6} \right] x^2 \right|$$
$$\leq (x-a)^3 \left[x^2 - \frac{5}{6}a^2 \right] + (b-x)^3 \left[\frac{19}{210}x^2 + \frac{8}{105}b^2 \right].$$

Moreover, if choose $x = \frac{a+b}{2}$, we obtain by **Remark 2** that

$$\left| \frac{b^5 - a^5}{60} - \frac{(b-a)(a+b)^4}{192} - \frac{(b-a)^3(a+b)^2}{96} \right|$$
$$\leq \frac{(b-a)^3}{16} \left[\frac{a^2}{3} + \frac{9a^2 + 2ab + b^2}{14} + \frac{(a+b)^2}{12} + \frac{8a^2 + 16ab + 24b^2}{105} \right].$$

Then

$$\left| (a-b)^5 \right| \leq \frac{(b-a)^3}{7}(477a^2 + 194ab + 161b^2).$$

Therefore

$$(a-b)^2 \leq \frac{477a^2 + 194ab + 161b^2}{7}.$$

Example 6. Consider $g : \mathbb{R} \to \mathbb{R}$, $g(t) = e^t$. Then we choose $s = 1$ and $\varphi(u,v) = u - v$ in Corollary 4, we have

$$\left| e^b - e^a - \sum_{k=0}^{2} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] e^x \right|$$
$$\leq \frac{e^a}{24} + \frac{(b-x)^3}{24}e^b + \left[(x-a)^3 + (b-x)^3 \right] \frac{e^x}{8}.$$

Choosing $a = 0$ and $b = 1$, we have for all $x \in [0,1]$

$$\left| e - 1 - \sum_{k=0}^{2} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] e^x \right|$$
$$\leq \frac{1}{24} + \frac{(1-x)^3}{24}e + \left[x^3 + (1-x)^3 \right] \frac{e^x}{8}.$$

Moreover, if we choose $x = \frac{1}{2}$, we get

$$\left| e - 1 - \sum_{k=0}^{2} \left[\frac{1+(-1)^k}{2^{k+1}(k+1)!} \right] \sqrt{e} \right| \leq \frac{192 + e + 6\sqrt{e}}{4608}.$$

*Example 7.*If we define $g(t) = \frac{36}{91}\sqrt[3]{2}\ t^{\frac{13}{6}}$ we have that $|g''(t)|^3$ is $\frac{1}{2}$-φ-convex with $\varphi(u,v) = u - v$ (see example 1) and by Theorem 4, for $a,b \in \mathbb{R}$ with $a < b$ and $x \in [a,b]$, we have

$$\left| \frac{216}{1729}[b^{\frac{19}{6}} - a^{\frac{19}{6}}] - \frac{36x^{\frac{13}{6}}}{91}(b-a) + \frac{6x^{\frac{7}{6}}}{14}[(b-x)^2 - (x-a)^2] + \frac{x^{\frac{1}{6}}}{6}[(b-x)^3 + (x-a)^3] \right|$$
$$\leq \frac{(x-a)^3}{2\sqrt[3]{48}} \left(a^{\frac{1}{2}} + 2x^{\frac{1}{2}} \right)^{\frac{1}{3}} + \frac{(b-x)^3}{2\sqrt[3]{48}} \left(x^{\frac{1}{2}} + 2b^{\frac{1}{2}} \right)^{\frac{1}{3}}.$$

5 Conclusions

In this paper we have introduced the concept of $s-\varphi-$convex functions as a generalization of convex functions, we study some properties of these functions and we have established new Ostrowski's inequality given by Badreddine Meftah in [19] for $s-\varphi-$convex functions. We expect that the ideas and techniques used in this paper may inspire interested readers to explore some new applications of these newly-introduced functions in various fields of pure and applied sciences.

Acknowledgements.

We want to give thanks to the Dirección de investigación from Pontificia Universidad Católica del Ecuador for technical support to our research project entitled: “Algunas desigualdades de funciones convexas generalizadas”.

References

[1] R. Agarwal, M. Luo and R. Raina, On Ostrowski Type inequalities, Fasciculli Mathematici, No. 56, pp. 5-27 (2016).
[2] M. Alomari and M. Darus, Ostrowski type inequalities for quasi-convex functions with applications to special means, RGMIA Res. Rep. Coll, Vol. 2, No. 3 (2010).
[3] M. Alomari, M. Darus, S. Dragomir and P. Cerone, Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense, Appl. Math. Lett, No. 23, pp. 1071-1076 (2010).
[4] K. Arrow and A. Enthoven, Quasiconcave Programming, Econometrica, No. 29, pp. 779-800 (1961).
[5] M. Badreddine, New Ostrowski's inequalties, Revista Colombiana de Matemáticas, No. 51, pp. 57-69 (2017).
[6] R. Bai, F. Qi and B. Xi, Hermite-Hadamard type inequalities for the m- and (α,m)-logarithmically convex functions, Filomat, Vol. 27, No. 1, pp. 1-7.
[7] C. Bector and C. Singh, B-vex functions, J. Optimiz. Theory. App., No. 71, pp. 237-254 (1991).
[8] W. Breckner, Stetigkeitsaussagen fur eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen R aumen, Pub. Inst. Math., No. 23, pp. 13-20 (1978).
[9] P. Cerone, S. Dragomir, and J. Roumeliotis, Some Ostrowski type inequalities for n-time differentiable mappings and applications, Demonstratio Math, Vol. 32, No. 4, pp. 697-712 (1999).
[10] S. Dragomir, An Ostrowski Type inequality for convex functions, Univ. Beograd. Publ. Elektrotehn, No. 16, pp. 12-25 (2005).
[11] S. Dragomir, On some new inequalities of Hermite-Hadamard type for m-convex functions, Tamkang Journal of Mathematics, Vol. 33, No. 1, pp. 55-65 (2002).
[12] M. Gordji, M. Delavar, and M. De La Sen, On φ-convex functions, J. Math. Inequal., Vol. 10, No. 1, pp. 173-183 (2016).
[13] M. Grinalatt, J. Linnainmaa, Jensen's Inequality, parameter uncertainty and multiperiod investment, Review of Asset Pricing Studies, No. 1, pp. 1-34 (2011).
[14] M. Hanson, On sufficiency of the Kuhn-Tucker conditions, J. Optimiz. Theory. App., Vol. 80, pp. 545-550 (1981).
[15] H. Hudzik, and L. Maligranda, Some remarks on $s-$convex functions, Aequationes Mathematicae No. 48, pp. 100-111 (1994).
[16] X. Li, J. Dong and Q. Liu, Lipschitz B-vex functions and nonsmooth programming, J. Optimiz. Theory. App., No. 3, pp. 557-573 (1997).
[17] W. Liu, W. Wen and J. Park, Hermite-Hadamard type Inequalities for MT-convex functions via classical integrals and fractional integrals, J. Nonlinear Sci.Appl., No. 9, pp. 766-777 (2016).
[18] O. Mangasarian, Pseudo-Convex Functions, SIAM. J. Control, No. 3, pp. 281-290 (1965).
[19] B. Meftah, New Ostrowski's inequalties, Revista Colombiana de Matemáticas, No. 51, pp. 57-69 (2017).
[20] S. Mohan and S. Neogy, On invex sets and preinvex functions, J. Math. Anal. Appl., No. 189, pp. 901-908 (1995).
[21] A. Ostrowski, Uber die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, Comment. Math. Hel., No. 10, pp. 226-227 (1938).
[22] M. Özdemir, C. Yildiz, A. Akdemir and E. Set, On some inequalities for s-convex functions and applications, J. Inequal and Appl., No. 48, pp. 1-11 (2013).
[23] J. Pecaric, F. Proschan, and Y. Tong, Convex functions, partial orderings, and statistical applications, Mathematics in Science and Engineering, Academic Press, Inc., Boston, No. 187 (1992).
[24] J. Ruel, M. Ayres, Jensen's inequality predicts effects of environmental variations, Trends in Ecology and Evolution, Vol. 14, No. 9, pp. 361-366 (1999).
[25] M. Sarikaya, H. Filiz, and M. Kiris On some generalized integral inequalities for Riemann Lioville Fractional Integral, Filomat, pp. 1307-1314 (2015).
[26] E. Set, New inequalities of Ostrowski type for mapping whose derivatives are s-convex in the second sense via fractional integrals, Comput. Math. Appl., No. 63, pp. 1147-1154 (2012).

[27] M. Tunç, Ostrowski type inequalities via h-convex functions with applications to special means, Journal of Inequalities and Applications, No. 326 (2013).
[28] S. Varoŝanec, On h-convexity, J. Math. Anal. Appl., Vol. 326, No. 1 (2007).
[29] E. Youness, E-convex sets, E-convex functions and E-convex programming, J. Optimiz. Theory. App., No. 102, pp. 439-450 (1999).

Yenny C. Rangel O. earned his Ph.D. degree from Universidad Complutense de Madrid, España (2009) in the field Pure Mathematics (Mathematical Analysis), She has vast experience of teaching and research at university levels. It covers many areas of Mathematical such as Inequalities, Functional Analysis, Real Analysis and Geometry. Se has written and published several research articles in reputed international journals of mathematical. She was Professor in Decanato de Ciencias y Tecnología of Universidad Centroccidental Lisandro Alvarado (UCLA), Barquisimeto, Lara state, Venezuela, and Ocasional Professor in Universidad Yachay Tech , Ibarra, Ecuador. Actually is Professor and Researcher in Pontificia Universidad Católica del Ecuador. Sede Quito, Ecuador.

Miguel J. Vivas C. earned his Ph.D. degree from Universidad Central de Venezuela, Caracas, Distrito Capital (2014) in the field Pure Mathematics (Nonlinear Analysis), and earned his Master Degree in Pure Mathematics in the area of Differential Equations (Ecological Models). He has vast experience of teaching and research at university levels. It covers many areas of Mathematical such as Inequalities, Bounded Variation Functions and Ordinary Differential Equations. He has written and published several research articles in reputed international journals of mathematical and textbooks. He was Titular Professor in Decanato de Ciencias y Tecnología of Universidad Centroccidental Lisandro Alvarado (UCLA), Barquisimeto, Lara state, Venezuela, and invited Professor in Facultad de Ciencias Naturales y Matemáticas from Escuela Superior Politécnica del Litoral (ESPOL), Guayaquil, Ecuador, actually is Principal Professor and Researcher in Pontificia Universidad Católica del Ecuador. Sede Quito, Ecuador.