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On $\varphi-$Convex Stochastic Processes and Integral Inequalities Related

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Abstract: In this paper the concept of $\phi-$convex stochastic process is introduced and certain algebraic properties are deduced. Also, some mean square integral inequalities of Hermite-Hadamard type are established. In addition, various mean square integral inequalities are investigated to find upper estimates using of the weighted arithmetic mean, the weighted power mean of order p and the logarithmic mean.

Keywords: $\phi-$convexity, Mean square integral inequalities, Stochastic processes.

1 Introduction

The study of stochastic processes began in the late 1930's and the introduction of stochastic convexity appeared in 1980 [1], where K. Nikodem presented this notion and a generalization of theorems was proved by B. Nagy [2] in the setting of a study of the Cauchy equation. For other results related to stochastic processes, see [3, 4] where further references are presented.

Similarly, the concept of convexity has had a great evolution because of its wide application in various fields of science, including those where the fractional and quantum calculus are applied [5–11]. In the last decades generalizations of the convexity, such as log-convexity, $s-$convexity in the first and second sense , Wright convexity, $E-$convexity, $m-$convexity, $\varphi-$convexity, $GA-$convexity, $(s,\varphi)-$convexity and others [12–23] have arisen.

Some authors have related these concepts with the stochastic processes. For example, A. Skrowonski explored J-convex stochastic processes and Wright convex stochastic processes [21, 24]. Other authors have obtained some further results in this area. For example, D. Kotrys investigated convex and strongly convex stochastic processes [25–27], E. Set E. et al. handled $s-$convex stochastic processes in the second sense [28], S. Maden et al. worked on $s-$convex stochastic processes in the first sense [18], N. Okur et al. investigated harmonically convex stochastic processes [29] and M. Tomar et al. worked on log-convex stochastic processes [30]. Furthermore, the works of Vivas-Cortez, Hernández Hernández and Gómez [31–35] addressed the $(m,h_1,h_2)-$convex stochastic processes in the setting of fractional calculus.

Following the path outlined by the aforementioned authors, the present paper aims to introduce the concept of $\varphi-$convex stochastic process, demonstrate some properties, and relate it to the inequalities of the Hermite Hadamard type and other inequalities associated with special means.

2 Preliminaries

The following notions correspond to mathematical fundaments on stochastic processes and the generalized convexity related to them. For elementary calculus associated with stochastic processes, we encourage the reader to review the following texts [36–38].

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Definition 1. *Let $(\Omega,\mathscr{A},P)$ be an arbitrary probability space. A function $X:\Omega\to\mathbb{R}$ is called a random variable if it is $\mathscr{A}$-measurable. Let $I\subset\mathbb{R}$ be time. A collection of random variable $X(t,w), t\in I$ with values in $\mathbb{R}$ is called a stochastic processes.*

1. *If $X(t,w)$ takes values in $S=\mathbb{R}^d$, then it is called vector-valued stochastic process.*
2. *If the time I is a discrete subset of $\mathbb{R}$, then $X(t,w)$ is called a discrete time stochastic process.*
3. *If the time I is an interval, $\mathbb{R}^+$ or $\mathbb{R}$, then it is called a stochastic process with continuous time.*

Definition 2. *Let (Ω,A,P) be a probability space and $I\subset\mathbb{R}$ be an interval. We say that the stochastic process $X:I\times\Omega\to\mathbb{R}$ is called*

1. *Continuous in probability in interval I if for all $t_0\in I$, it has*
$$P-\lim_{t\to t_0}X(t,\cdot)=X(t_0,\cdot),$$
where $P-\lim$ denotes the limit in probability;
2. *Mean-square continuous in the interval I if for all $t_0\in I$*
$$P-\lim_{t\to t_0}\mathbb{E}(X(t,\cdot)-X(t_0,\cdot))=0,$$
where $\mathbb{E}(X(t,\cdot))$ denotes the expectation value of the random variable $X(t,\cdot)$;
3. *Increasing (decreasing) if for all $u,v\in I$ such that $t<s$,*
$$X(u,\cdot)\leq X(v,\cdot),\quad (X(u,\cdot)\geq X(v,\cdot))$$
4. *Monotonic if it's increasing or decreasing;*
5. *Differentiable at a point $t\in I$ if there exists a random variable $X'(t,\cdot):\Omega\to\mathbb{R}$, such that*
$$X'(t,\cdot)=P-\lim_{t\to t_0}\frac{X(t,\cdot)-X(t_0,\cdot)}{t-t_0}.$$

We say that a stochastic process $X:I\times\Omega\to\mathbb{R}$ is continuous (differentiable) if it is continuous (differentiable) at every point of the interval I (See [21, 25, 36]).

Definition 3. *Let (Ω,A,P) be a probability space $T\subset\mathbb{R}$ be an interval with $E(X(t)^2)<\infty$ for all $t\in T$. Let $[a,b]\subset T, a=t_0<t_1<...<t_n=b$ be a partition of $[a,b]$ and $\theta_k\in[t_{k-1},t_k]$ for $k=1,2,...,n$. A random variable $Y:\Omega\to\mathbb{R}$ is called mean-square integral of the process $X(t,\cdot)$ on $[a,b]$ if the following identity holds:*

$$\lim_{n\to\infty}E[X(\theta_k)(t_k-t_{k-1})-Y]^2=0$$

Then we can write

$$\int_a^b X(t,\cdot)dt=Y(\cdot)(a.e.).$$

Also, mean square integral operator is increasing, i.e.,

$$\int_a^b X(t,\cdot)dt\leq\int_a^b Z(t,\cdot)dt(a.e.)$$

where $X(t,\cdot)\leq Z(t,\cdot)$ in $[a,b]$ ([24]).

In this paper, we will consider the stochastic processes that is with continuous time and mean-square continuous.

As mentioned in the introductory section, several notions of stochastic generalized convexity had been introduced. The following is a brief compilation of these concepts.

Definition 4. *([25])Let $(\Omega,\mathscr{A},P)$ be a probability space and $I\subset\mathbb{R}$ be an interval. It is said that a stochastic process $X:I\times\Omega\to\mathbb{R}$ is convex if for all $u,v\in I$ and $t\in[0,1]$ the following inequality holds almost everywhere*

$$X(tu+(1-t)v,\cdot)\leq tX(u,\cdot)+(1-t)X(v,\cdot).\qquad(1)$$

Now, we give the well-known Hermite-Hadamard integral inequality for convex stochastic processes (see [25]).

Theorem 1. *If $X:I\times\Omega\to\mathbb{R}$ is Jensen-convex and mean square continuous in the interval I, then for any $u,v\in I$, we have*

$$X\left(\frac{u+v}{2},\cdot\right)\leq\frac{1}{u-v}\int_u^v X(t,\cdot)dt\leq\frac{X(u,\cdot)+X(v,\cdot)}{2}$$

Definition 5. *Let $(\Omega,\mathscr{A},P)$ be a probability space, $I\subset\mathbb{R}$ be an interval and let $\varphi:R\times R\to R$ be a real valued function of two variables. It is said that a stochastic process $X:I\times\Omega\to\mathbb{R}$ is $\varphi-$convex if for all $u,v\in I$ and $t\in[0,1]$ the following inequality holds almost everywhere*

$$X(tu+(1-t)v,\cdot)\leq X(v,\cdot)+t\varphi(X(u,\cdot),X(v,\cdot)).$$

It must be noted that if the real valued function of two variables $\varphi:R\times R\to R$ is defined by $\varphi(x,y)=x-y$ then Definition 5 coincides with Definition 4.

*Example 1.*Let $X:I\times\Omega\to\mathbb{R}$ be a stochastic process defined by

$$X(t,\cdot)=A(\cdot)e^{kt},$$

then since the exponential function is convex, it implies that

$$\begin{aligned}X(ta+(1-t)b,\cdot)&=A(\cdot)e^{k(ta+(1-t)b)}\\&\leq A(\cdot)\left(te^{ka}+(1-t)e^{kb}\right)\\&=tX(a,\cdot)+(1-t)X(b,\cdot)\end{aligned}$$

showing that it is a convex stochastic process, and also

$$\begin{aligned}X(ta+(1-t)b,\cdot)&\leq X(b,\cdot)+t(X(a,\cdot)-X(b,\cdot))\\&=X(b,\cdot)+t\varphi(X(a,\cdot),X(b,\cdot))\end{aligned}$$

showing that X is a $\varphi-$convex stochastic process where $\varphi(x,y)=x-y$. In addition, if $\varphi_1(x,y)\geq\varphi(x,y)$, then X is φ_1-convex stochastic process.

Example 2. Let $X : \mathbb{R} \times \Omega \to \mathbb{R}_+$ defined by $X(t,\cdot) = f(t)A(\cdot)$ where

$$f(t) = \begin{cases} t & \text{if } 0 < t < 1 \\ 1 & \text{if } t > 1 \end{cases}$$

and A is a random variable. Let φ be defined by

$$\varphi(x,y) = \begin{cases} x+y, & \text{if } x \le y \\ 2(x+y) & \text{if } x > y \end{cases}.$$

Then, X is a $\varphi-$convex stochastic process, it is not convex.

Concerning the function φ, it is a necessary definition that will be useful for the development of this work.

Definition 6. *The function $\varphi : R \times R \to R$ is said to be*

(i) non negatively homogeneous if $k\varphi(x,y) = \varphi(kx,ky)$ for all $x,y \in R$ and $k \ge 0$
(ii) additive if $\varphi(x_1+x_2,y_1+y_2) = \varphi(x_1,y_1)+\varphi(x_2,y_2)$ for all $x_1,x_2,y_1,y_2 \in R$

3 Main Results

Proposition 1. *Let $\varphi : R \times R \to R$ be an additive and non negatively homogeneous real valued function of two variables, $X_1, X_2 : I \times \Omega \to R_+$ be $\varphi-$ convex stochastic processes and $k \ge 0$. Then, (X_1+X_2) and kX_1 are a $\varphi-$ convex stochastic processes.*

Proof. Let $X_1, X_2 : I \times \Omega \to R_+$ be $\varphi-$ convex stochastic processes, then

$$\begin{aligned}
&(X_1+X_2)(ta+(1-t)b,\cdot) \\
&= X_1(ta+(1-t)b,\cdot)+X_2(ta+(1-t)b,\cdot) \\
&\le X_1(b,\cdot)+t\varphi(X_1(a,\cdot),X_1(b,\cdot)) \\
&\qquad +X_2(b,\cdot)+t\varphi(X_2(a,\cdot),X_2(b,\cdot)) \\
&= (X_1(b,\cdot)+X_2(b,\cdot)) \\
&\qquad +t(\varphi(X_1(a,\cdot),X_1(b,\cdot))+\varphi(X_2(a,\cdot),X_2(b,\cdot))) \\
&= (X_1(b,\cdot)+X_2(b,\cdot)) \\
&\qquad +t(\varphi(X_1(a,\cdot)+X_2(a,\cdot),X_1(b,\cdot)+X_2(b,\cdot))) \\
&= (X_1+X_2)(b,\cdot)+t\varphi((X_1+X_2)(a,\cdot),(X_1+X_2)(b,\cdot)).
\end{aligned}$$

Now, if $k>0$ then

$$\begin{aligned}
kX_1&(ta+(1-t)b,\cdot) \\
&\le k(X_1(b,\cdot)+t\varphi(X_1(a,\cdot),X_1(b,\cdot))) \\
&= kX_1(b,\cdot)+kt\varphi(X_1(a,\cdot),X_1(b,\cdot)) \\
&= kX_1(b,\cdot)+t\varphi(kX_1(a,\cdot),kX_1(b,\cdot)).
\end{aligned}$$

The proof is complete.

Corollary 1. *Let $\varphi : R \times R \to R$ be an additive and non negatively homogeneous real valued function of two variables, $X_1, X_2, \ldots, X_n : I \times \Omega \to R_+$ be $\varphi-$ convex stochastic processes and $k_1, k_2, \ldots, k_n$ real non negative numbers. Then, $X(t,\cdot) = \sum_{i=1}^{n} k_i X_i(t,\cdot)$ is a $\varphi-$ convex stochastic process.*

Proposition 2. *Let $\varphi : R_+ \times R_+ \to R$ be a real valued function defined by $\varphi(x,y) = \alpha x + \beta y$, with $\alpha \in [0,\infty)$ and $\beta \in [-1,\infty)$. Let $X,Y : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be $\varphi-$ convex stochastic processes such that for each $t \in \mathbb{R}_+$, $\max\{X(t,\cdot),Y(t,\cdot)\}$ exists in $\mathbb{R}$. Then, the stochastic process $Z : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ defined by $Z(t,\cdot) = \max\{X(t,\cdot),Y(t,\cdot)\}$ is $\varphi-$ convex stochastic process.*

Proof. Let $X,Y : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be $\varphi-$ convex stochastic processes. Given $a,b \in R_+$ and $t \in [0,1]$ it implies that

$$X(ta+(1-t)b,\cdot) \le X(b,\cdot)+t\varphi(X(a,\cdot),X(b,\cdot))$$

and

$$Y(ta+(1-t)b,\cdot) \le Y(b,\cdot)+t\varphi(Y(a,\cdot),Y(b,\cdot)).$$

Then,

$$\begin{aligned}
&Z(ta+(1-t)b,\cdot) \\
&= \max\{X(ta+(1-t)b,\cdot),Y(ta+(1-t)b,\cdot)\} \\
&\le \max\{X(b,\cdot)+t\varphi(X(a,\cdot),X(b,\cdot)), \\
&\qquad\qquad Y(b,\cdot)+t\varphi(Y(a,\cdot),Y(b,\cdot))\} \\
&= \max\{X(b,\cdot)+t(\alpha X(a,\cdot)+\beta X(b,\cdot)), \\
&\qquad\qquad Y(b,\cdot)+t(\alpha Y(a,\cdot)+\beta Y(b,\cdot))\} \\
&= \max\{(1+t\beta)X(b,\cdot)+t\alpha X(a,\cdot), \\
&\qquad\qquad (1+t\beta)Y(b,\cdot)+t\alpha Y(a,\cdot)\} \\
&\le (1+t\beta)\max\{X(b,\cdot),Y(b,\cdot)\} \\
&\qquad\qquad +t\alpha\max\{X(a,\cdot),Y(a,\cdot)\} \\
&= (1+t\beta)Z(b,\cdot)+t(\alpha Z(a,\cdot)+\beta Z(b,\cdot)) \\
&= Z(b,\cdot)+t\varphi(Z(a,\cdot),Z(b,\cdot)).
\end{aligned}$$

The proof is complete.

Proposition 3. *Let $\varphi : R_+ \times R_+ \to R$ be a real valued function of two variables, continuous in each coordinate and let $X_n : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be $\varphi-$ convex stochastic process for $n \in N$. Let $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that $X(t,\cdot) = \lim_{n\to\infty} X_n(t,\cdot)$. Then, X is $\varphi-$ convex stochastic process.*

Proof. Let $X_n : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be $\varphi-$ convex stochastic process for $n \in N$. Then, we have

$$\begin{aligned}
&X(ta+(1-t)b,\cdot) \\
&= \lim_{n\to\infty} X_n(ta+(1-t)b,\cdot) \\
&\le \lim_{n\to\infty}(X_n(b,\cdot)+t\varphi(X_n(a,\cdot),X_n(b,\cdot))) \\
&= \lim_{n\to\infty} X_n(b,\cdot)+t\lim_{n\to\infty}\varphi(X_n(a,\cdot),X_n(b,\cdot)) \\
&= \lim_{n\to\infty} X_n(b,\cdot)+t\varphi\left(\lim_{n\to\infty} X_n(a,\cdot),\lim_{n\to\infty} X_n(b,\cdot)\right) \\
&= X(b,\cdot)+t\varphi(X(a,\cdot),X(b,\cdot))
\end{aligned}$$

The proof is complete.

3.1 Hermite – Hadamard type inequalities

The following two Theorems show a left side and right side inequality of Hermite – Hadamard type, respectively.

Theorem 2. *Let $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a $\varphi-$convex stochastic process . If $a,b \in \mathbb{R}_+$ with $a < b$ and X is mean square integrable, then the following inequality holds almost everywhere*

$$\begin{aligned} &X\left(\frac{a+b}{2},\cdot\right) - \frac{1}{2(b-a)}\int_a^b \varphi\left(X\left(u,\cdot\right),X\left(b+a-u,\cdot\right)\right)du \\ &\le \frac{1}{b-a}\int_a^b X(u,\cdot)du \qquad (2)\\ &\le \frac{X\left(a\,,\cdot\right)+X\left(b\,,\cdot\right)}{2} \\ &\quad + \frac{\left(\varphi\left(X\left(a\,,\cdot\right),X\left(b\,,\cdot\right)\right)+\varphi\left(X\left(b\,,\cdot\right),X\left(a\,,\cdot\right)\right)\right)}{2} \end{aligned}$$

Proof. Let $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a $\varphi-$convex stochastic process. From Definition 5 with $t = 1/2$, it follows that

$$\begin{aligned} &X\left(\frac{a+b}{2},\cdot\right) \\ &= X\left(\frac{ta+(1-t)b+(1-t)a+tb}{2},\cdot\right) \\ &\le X\left((1-t)a+tb,\cdot\right) \\ &\qquad + \frac{1}{2}\varphi\left(X\left(ta+(1-t)b,\cdot\right),X\left((1-t)a+tb,\cdot\right)\right) \end{aligned}$$

Integrating over $t \in [0,1]$ and making use of the change of variable $u = ta+(1-t)b$, the following is obtained

$$\begin{aligned} &X\left(\frac{a+b}{2},\cdot\right) \\ &\le \int_0^1 X\left((1-t)a+tb,\cdot\right)dt \\ &\qquad + \frac{1}{2}\int_0^1 \varphi\left(X\left(ta+(1-t)b,\cdot\right),X\left((1-t)a+tb,\cdot\right)\right)dt \\ &= \frac{1}{b-a}\int_a^b X(u,\cdot)du \\ &\qquad + \frac{1}{2(b-a)}\int_a^b \varphi\left(X\left(u,\cdot\right),X\left(b+a-u,\cdot\right)\right)du \end{aligned}$$

so,

$$\begin{aligned} &X\left(\frac{a+b}{2},\cdot\right) - \frac{1}{2(b-a)}\int_a^b \varphi\left(X\left(u,\cdot\right),X\left(b+a-u,\cdot\right)\right)du \\ &\le \frac{1}{b-a}\int_a^b X(u,\cdot)du, \end{aligned}$$

obtaining the left side of the inequality (2) in this way.

Also, it implies that

$$X\left(ta+(1-t)b,\cdot\right) \le X\left(b\,,\cdot\right) + t\varphi\left(X\left(a\,,\cdot\right),X\left(b\,,\cdot\right)\right)$$

and

$$X\left((1-t)a+tb,\cdot\right) \le X\left(a\,,\cdot\right) + t\varphi\left(X\left(b\,,\cdot\right),X\left(a\,,\cdot\right)\right).$$

Adding these inequalities, the following is obtained

$$\begin{aligned} &X\left(ta+(1-t)b,\cdot\right) + X\left((1-t)a+tb,\cdot\right) \\ &\quad \le \left(X\left(a\,,\cdot\right)+X\left(b\,,\cdot\right)\right) \\ &\qquad + t\left(\varphi\left(X\left(a\,,\cdot\right),X\left(b\,,\cdot\right)\right)+\varphi\left(X\left(b\,,\cdot\right),X\left(a\,,\cdot\right)\right)\right) \end{aligned}$$

Integrating over $t \in [0,1]$

$$\begin{aligned} &\frac{1}{b-a}\int_a^b X(u,\cdot)du \\ &\quad \le \frac{X\left(a\,,\cdot\right)+X\left(b\,,\cdot\right)}{2} \\ &\qquad + \frac{\left(\varphi\left(X\left(a\,,\cdot\right),X\left(b\,,\cdot\right)\right)+\varphi\left(X\left(b\,,\cdot\right),X\left(a\,,\cdot\right)\right)\right)}{2}. \end{aligned}$$

The proof is complete.

Remark. If $\varphi(x,y) = x-y$, the inequality for convex stochastic process is obtained

$$X\left(\frac{a+b}{2},\cdot\right) \le \frac{1}{b-a}\int_a^b X(u,\cdot)dt \le \frac{X\left(a\,,\cdot\right)+X\left(b\,,\cdot\right)}{2},$$

making coincidence with the result obtained by D. Kotrys in [25].

Theorem 3. *Let $X,Y : I \times \Omega \to \mathbb{R}$ be a $\varphi-$convex stochastic process . If $a,b \in I$ with $a < b$ and X,Y are mean square integrable, then the following inequality holds almost everywhere*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b X\left(u\,,\cdot\right)Y\left(u\,,\cdot\right)du \\ &\le X(b,\cdot)\left[Y(b,\cdot) + \frac{\varphi\left(Y(a,\cdot),Y(b,\cdot)\right)}{2}\right] \\ &+ \varphi\left(X(a,\cdot),X(b,\cdot)\right)\left[\frac{Y(b,\cdot)}{2} + \frac{\varphi\left(Y(a,\cdot),Y(b,\cdot)\right)}{3}\right]. \end{aligned}$$

*Proof.*Let $X,Y : \mathbb{I} \times \Omega \to \mathbb{R}$ be a $\varphi-$convex stochastic process, then for all $a,b \in I$ with $a < b$ and $t \in [0,1]$ it follows that

$$X\left(ta+(1-t)b\,,\cdot\right) \le X(b,\cdot) + t\varphi\left(X(a,\cdot),X(b,\cdot)\right) \qquad (3)$$

and

$$Y\left(ta+(1-t)b\,,\cdot\right) \le Y(b,\cdot) + t\varphi\left(Y(a,\cdot),Y(b,\cdot)\right). \qquad (4)$$

Multiplying (3) and (4), we have

$$\begin{aligned} &X\left(ta+(1-t)b\,,\cdot\right)Y\left(ta+(1-t)b\,,\cdot\right) \\ &\le \left(X(b,\cdot)+t\varphi\left(X(a,\cdot),X(b,\cdot)\right)\right)\times \\ &\qquad \left(Y(b,\cdot)+t\varphi\left(Y(a,\cdot),Y(b,\cdot)\right)\right) \\ &= X(b,\cdot)Y(b,\cdot) + X(b,\cdot)t\varphi\left(Y(a,\cdot),Y(b,\cdot)\right) \\ &\qquad + Y(b,\cdot)t\varphi\left(X(a,\cdot),X(b,\cdot)\right) \\ &\qquad\qquad + t^2\varphi\left(X(a,\cdot),X(b,\cdot)\right)\varphi\left(Y(a,\cdot),Y(b,\cdot)\right), \end{aligned}$$

Integrating over $t \in [0,1]$, the following is obtained

$$\begin{aligned}
&\int_0^1 X(ta+(1-t)b,\cdot)Y(ta+(1-t)b,\cdot)\,dt \\
&\le X(b,\cdot)Y(b,\cdot)+\frac{X(b,\cdot)\varphi(Y(a,\cdot),Y(b,\cdot))}{2} \\
&\quad+\frac{Y(b,\cdot)\varphi(X(a,\cdot),X(b,\cdot))}{2} \\
&\quad\quad+\frac{\varphi(X(a,\cdot),X(b,\cdot))\,\varphi(Y(a,\cdot),Y(b,\cdot))}{3} \\
&=X(b,\cdot)\left[Y(b,\cdot)+\frac{\varphi(Y(a,\cdot),Y(b,\cdot))}{2}\right] \\
&+\varphi(X(a,\cdot),X(b,\cdot))\left[\frac{Y(b,\cdot)}{2}+\frac{\varphi(Y(a,\cdot),Y(b,\cdot))}{3}\right].
\end{aligned}$$

With the change $u = ta+(1-t)b$, it follows that

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b X(u,\cdot)Y(u,\cdot)\,du \\
&\quad\le X(b,\cdot)\left[Y(b,\cdot)+\frac{\varphi(Y(a,\cdot),Y(b,\cdot))}{2}\right] \\
&+\varphi(X(a,\cdot),X(b,\cdot))\left[\frac{Y(b,\cdot)}{2}+\frac{\varphi(Y(a,\cdot),Y(b,\cdot))}{3}\right].
\end{aligned}$$

The proof is complete.

Corollary 2. *Let $X,Y : I\times\Omega \to \mathbb{R}$ be a convex stochastic process . If $a,b \in I$ with $a<b$ and X,Y are mean square integrable, then the following inequality holds almost everywhere*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b X(u,\cdot)Y(u,\cdot)\,du \\
&\quad\le \frac{X(a,\cdot)Y(a,\cdot)+X(b,\cdot)Y(b,\cdot)}{3} \\
&\quad\quad+\frac{X(a,\cdot)Y(b,\cdot)+X(b,\cdot)Y(a,\cdot)}{6}.
\end{aligned}$$

Proof. Letting $\varphi(x,y) = x-y$ in Theorem 3, it follows the desired result.
The proof is complete.

Theorem 4. *Let $X,Y : I\times\Omega \to \mathbb{R}$ be a $\varphi-$convex stochastic process . If $a,b \in I$ with $a<b$ and X,Y are mean square integrable, then the following inequality holds almost everywhere*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b X(u,\cdot)Y(a+b-u,\cdot)\,du \\
&\quad\le X(b,\cdot)\left[Y(a,\cdot)+\frac{\varphi(Y(b,\cdot),Y(a,\cdot))}{2}\right] \\
&\quad+\varphi(X(a,\cdot),X(b,\cdot))\left[\frac{Y(a,\cdot)}{2}+\frac{\varphi(Y(b,\cdot),Y(a,\cdot))}{3}\right].
\end{aligned}$$

Proof. Let $X,Y : \mathbb{I}\times\Omega \to \mathbb{R}$ be a $\varphi-$convex stochastic process, then for all $a,b \in I$ with $a<b$ and $t \in [0,1]$, it follows that

$$X(ta+(1-t)b,\cdot) \le X(b,\cdot)+t\varphi(X(a,\cdot),X(b,\cdot)) \quad (5)$$

and

$$Y(tb+(1-t)a,\cdot) \le Y(a,\cdot)+t\varphi(Y(b,\cdot),Y(a,\cdot)). \quad (6)$$

Multiplying (5) and (6), it follows that

$$\begin{aligned}
&X(ta+(1-t)b,\cdot)Y(tb+(1-t)a,\cdot) \\
&\le [X(b,\cdot)+t\varphi(X(a,\cdot),X(b,\cdot))]\times \\
&\qquad [Y(a,\cdot)+t\varphi(Y(b,\cdot),Y(a,\cdot))] \\
&= X(b,\cdot)Y(a,\cdot)+X(b,\cdot)t\varphi(Y(b,\cdot),Y(a,\cdot)) \\
&\quad +t\varphi(X(a,\cdot),X(b,\cdot))Y(a,\cdot) \\
&\qquad +t^2\varphi(X(a,\cdot),X(b,\cdot))\,\varphi(Y(b,\cdot),Y(a,\cdot))
\end{aligned}$$

Integrating over $t \in [0,1]$ and letting the change of varible $u = ta+(1-t)b$

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b X(u,\cdot)Y(a+b-u,\cdot)\,du \\
&\le X(b,\cdot)Y(a,\cdot)+\frac{X(b,\cdot)\varphi(Y(b,\cdot),Y(a,\cdot))}{2} \\
&\quad+\frac{\varphi(X(a,\cdot),X(b,\cdot))Y(a,\cdot)}{2} \\
&\qquad+\frac{\varphi(X(a,\cdot),X(b,\cdot))\,\varphi(Y(b,\cdot),Y(a,\cdot))}{3} \\
&=X(b,\cdot)\left[Y(a,\cdot)+\frac{\varphi(Y(b,\cdot),Y(a,\cdot))}{2}\right] \\
&\quad+\varphi(X(a,\cdot),X(b,\cdot))\left[\frac{Y(a,\cdot)}{2}+\frac{\varphi(Y(b,\cdot),Y(a,\cdot))}{3}\right]
\end{aligned}$$

The proof is complete.

Corollary 3. *Let $X,Y : I\times\Omega \to \mathbb{R}$ be a convex stochastic process . If $a,b \in I$ with $a<b$ and X,Y are mean square integrable, then the following inequality holds almost everywhere*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b X(u,\cdot)Y(a+b-u,\cdot)\,du \\
&\quad\le \frac{X(a,\cdot)Y(a,\cdot)+X(b,\cdot)Y(b,\cdot)}{6} \\
&\quad\quad+\frac{X(b,\cdot)Y(a,\cdot)+X(a,\cdot)Y(b,\cdot)}{3}.
\end{aligned}$$

Theorem 5. *Let $X,Y : I\times\Omega \to \mathbb{R}$ be a $\varphi-$convex stochastic process . If $a,b \in I$ with $a<b$ and X,Y are mean square integrable, then the following inequality holds almost everywhere*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b X(u,\cdot)Y(a+b-u)\,du \\
&\quad\le \frac{5}{8}\left((X(b,\cdot))^2+(Y(a,\cdot))^2\right) \\
&\quad\quad+\frac{14}{48}\left((\varphi(X(a,\cdot),X(b,\cdot)))^2+(\varphi(Y(b,\cdot),Y(a,\cdot)))^2\right).
\end{aligned}$$

Proof. Let $X,Y : I\times\Omega\to\mathbb{R}$ be a $\varphi-$convex stochastic processes. Using the change of variables $u = ta+(1-t)b$ and recalling that $(x-y)^2\geq 0$, we have

$$\frac{1}{b-a}\int_a^b X(u,\cdot)Y(a+b-u)\,du$$

$$=\int_0^1 X(ta+(1-t)b,\cdot)\,Y(tb+(1-t)a,\cdot)\,dt$$

$$\leq\frac{1}{2}\int_0^1 (X(ta+(1-t)b,\cdot))^2+(Y(tb+(1-t)a,\cdot))^2\,dt$$

$$\leq\frac{1}{2}\int_0^1 (X(b,\cdot)+t\varphi(X(a,\cdot),X(b,\cdot)))^2\,dt$$

$$+\frac{1}{2}\int_0^1 (Y(a,\cdot)+t\varphi(Y(b,\cdot),Y(a,\cdot)))^2\,dt$$

$$=\frac{1}{2}(X(b,\cdot))^2+\frac{1}{4}X(b,\cdot)\,\varphi(X(a,\cdot),X(b,\cdot))$$

$$+\frac{1}{6}(\varphi(X(a,\cdot),X(b,\cdot)))^2$$

$$+\frac{1}{2}(Y(a,\cdot))^2+\frac{1}{4}Y(a,\cdot)\,\varphi(Y(b,\cdot),Y(a,\cdot))$$

$$+\frac{1}{6}(\varphi(Y(b,\cdot),Y(a,\cdot)))^2$$

$$=\frac{(X(b,\cdot))^2+(Y(a,\cdot))^2}{2}$$

$$+\frac{(\varphi(X(a,\cdot),X(b,\cdot)))^2+(\varphi(Y(b,\cdot),Y(a,\cdot)))^2}{6}$$

$$+\frac{X(b,\cdot)\,\varphi(X(a,\cdot),X(b,\cdot))+Y(a,\cdot)\,\varphi(Y(b,\cdot),Y(a,\cdot))}{4}$$

$$\leq\frac{(X(b,\cdot))^2+(Y(a,\cdot))^2}{2}$$

$$+\frac{(\varphi(X(a,\cdot),X(b,\cdot)))^2+(\varphi(Y(b,\cdot),Y(a,\cdot)))^2}{6}$$

$$+\frac{(X(b,\cdot))^2+(\varphi(X(a,\cdot),X(b,\cdot)))^2}{8}$$

$$+\frac{(Y(a,\cdot))^2+(\varphi(Y(b,\cdot),Y(a,\cdot)))^2}{8}$$

$$=\frac{5}{8}\left((X(b,\cdot))^2+(Y(a,\cdot))^2\right)$$

$$+\frac{14}{48}\left((\varphi(X(a,\cdot),X(b,\cdot)))^2+(\varphi(Y(b,\cdot),Y(a,\cdot)))^2\right).$$

The proof is complete.

Corollary 4. *Let $X,Y : I\times\Omega\to\mathbb{R}$ be a convex stochastic process . If $a,b\in I$ with $a<b$ and X,Y are mean square integrable, then the following inequality holds almost everywhere*

$$\frac{1}{b-a}\int_a^b X(u,\cdot)Y(a+b-u)\,du$$

$$\leq\frac{7}{11}\left((X(b,\cdot))^2+(Y(a,\cdot))^2\right)$$

$$+\frac{7}{24}\left((X(a,\cdot))^2+(Y(b,\cdot))^2\right)$$

$$-\frac{7}{12}(X(a,\cdot)X(b,\cdot)+Y(b,\cdot)Y(a,\cdot)).$$

3.2 Some integral inequalities and some special means

Lemma 1. *Let $X : I\times\Omega\to\mathbb{R}$ be a differentiable stochastic process, where I is an interval include in $\mathbb{R}_+$, and $a,b\in I$ with $a<b$. If X' is mean square integrable, then*

$$bX(b,\cdot)-aX(a,\cdot)-\int_a^b X(u,\cdot)\,du$$

$$=\frac{b-a}{4}\left[\int_0^1((1+t)b+(1-t)a)X'\left(\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right),\cdot\right)dt\right.$$

$$\left.+\int_0^1((1-t)b+(1+t)a)X'\left(\left(\frac{1-t}{2}b+\frac{1+t}{2}a\right),\cdot\right)dt\right].$$

Proof. Integrating by parts the first integral inside the brackets, it follows that

$$I_1=\int_0^1((1+t)b+(1-t)a)X'\left(\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right),\cdot\right)dt$$

$$=\frac{2bX(b,\cdot)-(b+a)X\left(\frac{b+a}{2}\right)}{\frac{b-a}{2}}-\frac{2}{\frac{b-a}{2}}\int_{a+b/2}^b X(u,\cdot)\,du \quad (7)$$

similarly, the second integral is

$$I_2=\int_0^1((1-t)b+(1+t)a)X'\left(\left(\frac{1-t}{2}b+\frac{1+t}{2}a\right),\cdot\right)dt$$

$$=\frac{-2aX(a,\cdot)+(b+a)X\left(\frac{b+a}{2}\right)}{\frac{b-a}{2}}-\frac{2}{\frac{b-a}{2}}\int_a^{a+b/2} X(u,\cdot)\,du \quad (8)$$

Adding 7 and 8, it follows that

$$\frac{b-a}{4}(I_1+I_2)=bX(b,\cdot)-aX(a,\cdot)-\int_a^b X(u,\cdot)\,du.$$

The proof is complete.

Theorem 6. *Let $X : I\times\Omega\to\mathbb{R}$ be a differentiable stochastic process, where I is an interval include in $\mathbb{R}_+$, and $a,b\in I$ with $a<b$. If X' is mean square integrable*

and $|X'|^q$ is $\varphi-$convex, for $q>1$, then the following inequality holds almost everywhere

$$\left|bX(b,\cdot)-aX(a,\cdot)-\int_a^b X(u,\cdot)\,du\right|\frac{b-a}{4}\times$$
$$\left[\left(\frac{3b+a}{2}\right)^{1-1/q}\times\right.$$
$$\left(\frac{3b+a}{2}|X'(b,\cdot)|^q+\frac{2b+a}{6}\varphi\left(|X'(a,\cdot)|^q,|X'(b,\cdot)|^q\right)\right)^{1/q}$$
$$+\left(\frac{3a+b}{2}\right)^{1-1/q}\times$$
$$\left.\left(\frac{3a+b}{2}|X'(b,\cdot)|^q+\frac{6a+b}{3}\varphi\left(|X'(a,\cdot)|^q,|X'(b,\cdot)|^q\right)\right)^{1/q}\right].$$

Proof. Using Lemma 1 and the Hölder inequality, it follows that

$$\left|bX(b,\cdot)-aX(a,\cdot)-\int_a^b X(u,\cdot)\,du\right|\le\frac{b-a}{4}\times$$
$$\left[\int_0^1|((1+t)b+(1-t)a)|\left|X'\left(\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right),\cdot\right)\right|dt\right.$$
$$\left.+\int_0^1|((1-t)b+(1+t)a)|\left|X'\left(\left(\frac{1-t}{2}b+\frac{1+t}{2}a\right),\cdot\right)\right|dt\right]$$
$$\le\frac{b-a}{4}\times$$
$$\left[\left(\int_0^1(t(b-a)+a+b)\,dt\right)^{1-1/q}\times\right.$$
$$\left(\int_0^1(t(b-a)+b+a)\left|X'\left(\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right),\cdot\right)\right|^q dt\right)^{1/q}$$
$$+\left(\int_0^1(a+b-t(b-a))\,dt\right)^{1-1/q}\times$$
$$\left.\left(\int_0^1(a+b-t(b-a))\left|X'\left(\left(\frac{1-t}{2}b+\frac{1+t}{2}a\right),\cdot\right)\right|^q dt\right)^{1/q}\right].\tag{9}$$

For each integral in 9, we have

$$\int_0^1(t(b-a)+a+b)dt=\frac{3b+a}{2},\tag{10}$$

$$\int_0^1(a+b-t(b-a))dt=\frac{3a+b}{2},\tag{11}$$

so, using the $\varphi-$convexity of $|X'|^q$

$$\int_0^1(t(b-a)+b+a)\left|X'\left(\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right),\cdot\right)\right|^q dt$$
$$\le\int_0^1(t(b-a)+b+a)\,|X'(b,\cdot)|^q$$
$$+\left(\frac{1-t}{2}\right)\varphi\left(|X'(a,\cdot)|^q,|X'(b,\cdot)|^q\right)dt$$
$$\le|X'(b,\cdot)|^q\int_0^1(t(b-a)+b+a)\,dt$$
$$+\varphi\left(|X'(a,\cdot)|^q,|X'(b,\cdot)|^q\right)\int_0^1\left(\frac{1-t}{2}\right)(t(b-a)+b+a)\,dt$$
$$=\frac{3b+a}{2}|X'(b,\cdot)|^q+\frac{2b+a}{6}\varphi\left(|X'(a,\cdot)|^q,|X'(b,\cdot)|^q\right)\tag{12}$$

similarly

$$\int_0^1(a+b-t(b-a))\left|X'\left(\left(\frac{1-t}{2}b+\frac{1+t}{2}a\right),\cdot\right)\right|^q dt$$

$$\le\frac{3a+b}{2}|X'(b,\cdot)|^q+\frac{6a+b}{3}\varphi\left(|X'(a,\cdot)|^q,|X'(b,\cdot)|^q\right)\tag{13}$$

Replacing (10), (11), (12) and (13) in (9) the desired result is obtained.
The proof is complete.

Recall that the weighted arithmetic mean of two numbers a and b is defined by

$$A_{w_1,w_2}(a,b)=\frac{w_1a+w_2b}{w_1+w_2},$$

the weighted power mean of order $p,(p\neq 0)$, of two distinct numbers a and b by

$$M_{p;w_1,w_2}(a,b)=\left(\frac{w_1a^p+w_2b^p}{w_1+w_2}\right)^{1/p}$$

and the logarithmic mean for two positive numbers a and b is given for $a=b$ by $L_p(a,a)=a$ and for $a\neq b$ by

$$L_p(a,b)=\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}\quad\text{if }\ p\neq -1,0$$

Corollary 5. *Let $X:I\times\Omega\to\mathbb{R}$ be a differentiable stochastic process, where I is an interval include in $\mathbb{R}_+$, and $a,b\in I$ with $a<b$. If X' is mean square integrable and $|X'|^q$ is convex, for $q>1$, then the following inequality holds almost everywhere*

$$\left|bX(b,\cdot)-aX(a,\cdot)-\int_a^b X(u,\cdot)\,du\right|\le\frac{b-a}{2}\times$$
$$A_{3/2,1/2}(a,b)\left[M_{q;w_1,w_2}\left(|X'(a,\cdot)|,|X'(b,\cdot)|\right)\right.$$
$$\left.+M_{q;w_2,w_1}\left(|X'(a,\cdot)|,|X'(b,\cdot)|\right)\right],$$

where

$$w_1=\frac{2b+a}{6}\ \text{ and }\ w_2=\frac{7b+2a}{6}$$

Proof. Letting $\varphi(x,y)=x-y$ in the previous Theorem 6, the following is obtained

$$\left|bX(b,\cdot)-aX(a,\cdot)-\int_a^b X(u,\cdot)\,du\right|\le\frac{b-a}{4}\times$$
$$\left[\left(\frac{3b+a}{2}\right)^{1-1/q}\left(\frac{2b+a}{6}|X'(a,\cdot)|^q+\frac{7b+2a}{6}|X'(b,\cdot)|^q\right)^{1/q}\right.$$
$$\left.+\left(\frac{3a+b}{2}\right)^{1-1/q}\left(\frac{7a+2b}{6}|X'(a,\cdot)|^q+\frac{2a+b}{2}|X'(b,\cdot)|^q\right)^{1/q}\right]$$

and using the weighted arithmetic mean of a and b, and the weighted power mean of $|X'(a,\cdot)|$ and $|X'(b,\cdot)|$ with

$$w_1 = \frac{2b+a}{6} \text{ and } w_2 = \frac{7b+2a}{6}.$$

The proof is complete.

Theorem 7. *Let $X: I\times\Omega \to \mathbb{R}$ be a differentiable stochastic process, where I is an interval include in $\mathbb{R}_+$, and $a,b\in I$ with $a<b$. If X' is mean square integrable and $|X'|^q$ is $\varphi-$convex, for $q>1$ and $1/p+1/q=1$, then the following inequality holds almost everywhere*

$$\left|bX(b,\cdot)-aX(a,\cdot)-\int_a^b X(u,\cdot)\,du\right| \le \frac{(b-a)}{4}\times$$
$$\left[L_p(a+b,2b)\left(|X'(b,\cdot)|^q+\frac{1}{4}\varphi\left(|X'(a,\cdot)|^q,|X'(b,\cdot)|^q\right)\right)^{1/q}\right.$$
$$\left.+L_p(2a,a+b)\left(|X'(b,\cdot)|^q+\frac{3}{4}\varphi\left(|X'(a,\cdot)|^q,|X'(b,\cdot)|^q\right)\right)^{1/q}\right].$$

Proof. Using Lemma 1 and the Hölder inequality, it implies that

$$\left|bX(b,\cdot)-aX(a,\cdot)-\int_a^b X(u,\cdot)\,du\right| \le \frac{b-a}{4}\times$$
$$\left[\int_0^1 |((1+t)b+(1-t)a)|\left|X'\left(\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right),\cdot\right)\right|dt\right.$$
$$\left.+\int_0^1 |((1-t)b+(1+t)a)|\left|X'\left(\left(\frac{1-t}{2}b+\frac{1+t}{2}a\right),\cdot\right)\right|dt\right]$$
$$\le \frac{b-a}{4}\times$$
$$\left[\left(\int_0^1 ((1+t)b+(1-t)a)^{p)}\,dt\right)^{1/p}\times\right.$$
$$\left(\int_0^1 \left|X'\left(\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right),\cdot\right)\right|^q dt\right)^{1/q}$$
$$+\left(\int_0^1 ((1-t)b+(1+t)a)^{p)}\,dt\right)^{1/p}\times$$
$$\left.\left(\int_0^1 \left|X'\left(\left(\frac{1-t}{2}b+\frac{1+t}{2}a\right),\cdot\right)\right|^q dt\right)^{1/q}\right] \tag{14}$$

For each integrals in the inequality (14), we have

$$\int_0^1 ((1+t)b+(1-t)a)^p\,dt = \frac{(2b)^{p+1}}{(p+1)(b-a)}-\frac{(a+b)^{p+1}}{(p+1)(b-a)}, \tag{15}$$

$$\int_0^1 ((1-t)b+(1+t)a)^p\,dt = \frac{(a+b)^{p+1}}{(p+1)(b-a)}-\frac{(2a)^{p+1}}{(p+1)(b-a)}, \tag{16}$$

and using the $\varphi-$convexity of $|X'|^q$, the following is obtained

$$\int_0^1 \left|X'\left(\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right),\cdot\right)\right|^q dt \tag{17}$$
$$\le \int_0^1 |X'(b,\cdot)|^q+\frac{1-t}{2}\varphi\left(|X'(a,\cdot)|^q,|X'(b,\cdot)|^q\right)dt$$
$$= |X'(b,\cdot)|^q+\frac{1}{4}\varphi\left(|X'(a,\cdot)|^q,|X'(b,\cdot)|^q\right),$$

and similarly

$$\int_0^1 \left|X'\left(\left(\frac{1-t}{2}b+\frac{1+t}{2}a\right),\cdot\right)\right|^q dt$$
$$\le |X'(b,\cdot)|^q+\frac{3}{4}\varphi\left(|X'(a,\cdot)|^q,|X'(b,\cdot)|^q\right) \tag{18}$$

Replacing (15), (16), (17) and (18) in inequality (14) the desired result is obtained. The proof is complete.

Corollary 6. *Let $X: I\times\Omega \to \mathbb{R}$ be a differentiable stochastic process, where I is an interval include in $\mathbb{R}_+$, and $a,b\in I$ with $a<b$. If X' is mean square integrable and $|X'|^q$ is convex, for $q>1$ and $1/p+1/q=1$, then the following inequality holds almost everywhere*

$$\left|bX(b,\cdot)-aX(a,\cdot)-\int_a^b X(u,\cdot)\,du\right| \le \frac{(b-a)}{4}\times$$
$$\left[L_p(a+b,2b)M_{q;1/4,3/4}\left(|X'(a,\cdot)|,|X'(b,\cdot)|\right)\right.$$
$$\left.+L_p(2a,a+b)M_{q;3/4,1/4}\left(|X'(a,\cdot)|,|X'(b,\cdot)|\right)\right].$$

Proof. Letting $\varphi(x,y)=x-y$ in Theorem 7, we have

$$\left|bX(b,\cdot)-aX(a,\cdot)-\int_a^b X(u,\cdot)\,du\right| \le \frac{(b-a)}{4}\times$$
$$\left[L_p(a+b,2b)\left(\frac{1}{4}|X'(a,\cdot)|^q+\frac{3}{4}|X'(b,\cdot)|^q\right)^{1/q}\right.$$
$$\left.+L_p(2a,a+b)\left(\frac{3}{4}|X'(a,\cdot)|^q+\frac{1}{4}|X'(b,\cdot)|^q\right)^{1/q}\right].$$

and using the weighted power mean of $|X'(a,\cdot)|$ and $|X'(b,\cdot)|$ with

$$w_1=\frac{1}{4} \text{ and } w_2=\frac{3}{4}$$

the desired result is obtained.

4 Conclusion

In this paper, the concept of $\phi-$convex stochastic process was introduced and certain algebraic properties were deduced. Also, some mean square integral inequalities of Hermite – Hadamard type were established. In addition, various mean square integral inequalities were investigated.

This work is expected to serve as a motivation for further research in the area. A similar study using fractional integrals is significant.

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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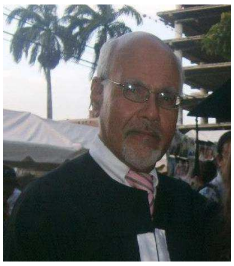

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