Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/010406

Controllability for a Class of Nonlocal Impulsive Neutral Fractional Functional Differential Equations

GangaRam Gautam * *and Jaydev Dabas*

Dept. Appl. Sci. Engin., IIT Roorkee, Saharanpur Campus-247001, India.

Received: 4 May 2015, Revised: 8 Jun. 2015, Accepted: 10 Jun. 2015
Published online: 1 Oct. 2015

Abstract: In this article, we have dealt with the controllability for impulsive (Imp.) neutral fractional functional integro-differential equations with state dependent delay (S-D Delay) subject to non-local conditions. We have obtained the appropriate conditions for Controllability result by using the classical fixed point technique and analytic operator theory under the more general conditions. At last, an example is presented to demonstrate the application of the obtained result.

Keywords: Fractional order differential equation, non-local initial conditions, contractions, impulsive conditions.

1 Introduction

In this article, we deliberate a class of neutral fractional (Frac.) functional (Func.) integro-differential equations (Diff. Eqns.) subject to impulsive (Imp.) and non-local conditions on complex Banach space $(X,\|\cdot\|_X)$

$$^{C}D_t^{\alpha}N[t,y] = AN[t,y] + Bu(t) + f(t,y_{\rho(t,y_t)}) + (q*g)(t), t\in J, t\neq t_k, \tag{1}$$
$$y(t) + h(y_{\rho_1},\ldots,y_{\rho_n})(t) = \phi(t),\ t\in(-\infty,0], \tag{2}$$
$$\Delta y(t_k) = I_k(y(t_k^-)), \quad k=1,2,\ldots,m, \tag{3}$$

where $^{C}D_t^{\alpha}$ denote the Caputo's Frac. derivative of order $\alpha\in(0,1); A: D(A)\subset X\to X$ is the closed linear operator sectorial type defined on $X; N[t,y] = y(t)+p(t,y_{\rho(t,y_t)})$ and $(q*g)(t) = \int_0^t q(t-s)g(s,y_{\rho(s,y_s)})ds$. The functions $f;g;p: J\times\mathfrak{B}_h\to X,\ q: J\to X,\ \rho: J\times\mathfrak{B}_h\to(-\infty,T]$ and $h:\mathfrak{B}_h{}^n\to X$ are given and satisfies some assumptions. The history function $y_t:(-\infty,0]\to X$ is demarcated by $y_t(\theta) = y(t+\theta),\ \theta\in(-\infty,0]$ fits in the abstract phase space $\mathfrak{B}_h$ and $J = (0,T],\ 0<T<\infty$, is an operational interval such that $0\le t_0<t_1<\cdots<t_m<t_{m+1}\le T$, are impulse points. $B: U\to X$ is a linear bounded operator, and the control map $u(\cdot)$ belong in Banach space $L^2(J,U)$ of admissible control maps with U as a given Banach space. The map $\phi(t)\in\mathfrak{B}_h; \Delta y(t_k) = y(t_k^+)-y(t_k^-),\ y(t_k^+)$ and $y(t_k^-)$, represents the right hand and left hand limits of function $y(t)$ at $t=t_k$ and $y(t_k^-) = y(t_k)$ and $I_k: X\to X,\ k=1,2,\ldots,m$, are continuous and bounded maps.

Frac. Diff. Eqns. originate in several fields as engineering, physics, biology, signal and image processing etc. so these equations become more naturalistic and practical than integer equations models. For more details descriptions one can see [3,25] and references therein. Imp. effects have a realistic role in the evolutionary processes owing to wide applications in science especially for population description, biological and social macro-systems. We mention the reader to see the papers [1,6,10,11,12,13,14,29] for more details and concept of Imp. effects.

The non-local conditions give improved results when compared to the normal local condition, for instance, to define the diffusion phenomenon of a slight quantity of gas in a apparent tube. For more details of these topics one can refer to [5,10,12,24,29]. For several decades Frac. Func. Diff. Equs. with S-D Delay are frequently applied in many fields, such as modeling of equations, panorama of natural phenomena and porous media [1,2,4,6,7,8,9,15,18,19,20,21,23].

Nowadays, controllability is one of the important ideas in mathematical control theory and has a chief role in many areas of science and technology. In Controllability systems control maps, which steers the solution of the problem from its primary state to last state, where the primary and last states may diverge over the whole space, deals existence results.

* Corresponding author e-mail: gangaiitr11@gmail.com

Many authors studied Controllability systems and established several results. In [26] H. Qin et al. studied the existence of PC-mild solutions for Imp. Frac. semi-linear integro-Diff. Eqns.

$$D_t^{\alpha}x(t) = Ax(t) + Bu(t) + f(t,x,Hx(t)), t \in J = [0,b], t \neq t_k,$$
$$\Delta x(t_k) = I_k(x(t_k^-)),\ k = 1,2,\dots,m,\ x(0) = x_0 \in X,$$

and then presented controllability results using fixed point theorem, C_0-semigroup theory, and the generalized Bellman inequality. H. Zhang et al. [30] studied the following problem

$$D_t^{\alpha}x(t) = Ax(t) + Bx(t-\tau) + Cu(t),\ t \in [0,T] \setminus \{t_1,t_2,\dots,t_i\},$$
$$\Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i(x(t_i^-)),\ i = 1,2,\dots,k,$$
$$x(t) = \phi(t),\ t \in [-\tau,0],$$

and established sufficient conditions of controllability standards. Z. Tai et al. [28] obtained the appropriate conditions for the controllability with Frac. calculus, C_0-semigroup theory and Krasnoselskii's theorem of the following problem

$$\frac{d^q}{dt^q}[x(t) - g(t,x_t)] = (Ax)(t) + (Bu)(t) + f(t,x_t,\int_0^t h(t,s,x_s)ds), t \in [0,T], t \neq t_k,$$
$$\Delta x(t_k) = I_k(x(t_k^-)),\ k = 1,2,\dots,m,\ x_0 = \phi \in \mathfrak{B}_v.$$

Controllability of Frac. Imp. neutral evolution integro-Diff. Eqns. in a Banach space has been mentioned in the paper [27] for the following system

$$\frac{d^q}{dt^q}[x(t) - g(t,x_t)] = A(t)x(t)(t) + f(t,x_t,\int_0^t h(t,s,x_s)ds) + (Gu)(t), t \in [0,T],$$
$$\Delta x(t_k) = I_k(x(t_k^-)),\ k = 1,2,\dots,m,\ x_0 = \phi \in \mathfrak{B}_v,\ t \neq t_k.$$

In paper [29] sufficient condition for the controllability is established by means of solution operator of the following problem

$$\frac{d^{\alpha}}{dt^{\alpha}}x(t) = Ax(t) + Bu(t) + f(t,x(t),x(a_1(t),),\dots,x(a_m(t))),\ t \in [0,T], t \neq t_i,$$
$$x(0) + g(x) = x_0,\ \Delta y(t_i) = I_i(y(t_i^-)),\ i = 1,2\dots,k.$$

Recently, sufficient conditions are derived by R. Ganesh et al. [17] for the exact controllability of nonlinear neutral Imp. Frac. Func. equation with infinite delay

$$D_t^{\alpha}[x(t) + g(t,x_t)] = A[x(t) + g(t,x_t)] + J_t^{1-\alpha}[Bu(t) + f(t,x_t,Hx(t))], t \in [0,T],$$
$$\Delta x(t_k) = I_k(x(t_k^-)),\ k = 1,2,\dots,m,\ x_0 = \phi \in \mathfrak{B}_h.$$

Very recently, author of the paper [22] remarks on some current results on exact controllability of abstract differential control systems with a linear part prevailed by a sectorial operator. Actually, author shows that the abstract control problems [17,26,28] are not exactly controllable because A is consider as unbounded operator, therefore the generated α-resolvent family is unbounded and due to this fact, results are absurd.

Our work is motivated by the mention work [17,26,27,28,29,30]. We followed the idea mentioned in [22] and applying it on (1)-(3) and obtained the sufficient conditions for non-local neutral Imp. Frac. Func. Diff. Eqns. with S-D Delay regarding infinite delay. In author knowledge this topics is unread yet. In this work, we established a general background to find the mild solutions for such Imp. Frac. integro-Diff. Eqns. and demarcated the mild solutions of the equations (1)-(3) by means of the idea presented in [14], in which the mild solutions are related with Mittag–Leffler map, resolvent operator and solution operator.

This work is divided in four sections. The second section offers some definitions and basic preliminaries to be used in proving our result. In the third section, we obtain the controllability results for the problem. The fourth section is concerned with an example.

2 Preliminaries and Definitions

Let $(X,\|\cdot\|_X)$ be a complex Banach space taking the norm

$$\|y\|_X = \sup\{|y(t)| : y \in X, t \in J,\}$$

and $L(X)$ denotes the Banach space of all bounded linear operators $K : X \to X$ taking the norm

$$\|K\|_{L(X)} = \sup\{\|Ky\|_X : \|y\|_X \leq 1, y \in X\}.$$

We did our computations in an abstract phase space $\mathfrak{B}_h$ due to infinite delay and $\mathfrak{B}'_h$ due to impulse effect which are same as described in [13].

Lemma 1. *From the paper [6] "If $y : (-\infty, T] \to X$ be a map s.t. $y_0 = \phi, y \in \mathfrak{B}'_h$, then for all $t \in J$, the following conditions holds:*

(C_1) $y_t \in \mathfrak{B}_h$.
(C_2) $\|y(t)\|_X \leq H\|y_t\|_{\mathfrak{B}_h}$, *where H is a positive constant.*
(C_3) $\|y_t\|_{\mathfrak{B}_h} \leq K(t)\sup\{\|y(s)\|_X : 0 \leq s \leq t\} + M(t)\|\phi\|_{\mathfrak{B}_h}, K, M : [0,\infty) \to [0,\infty)$, $K(\cdot)$ *are continuous,* $M(\cdot)$ *is locally bounded and* K, M *are independent of* $y(t)$.
(C_{4_ϕ}) *The map $t \to \phi_t$ is well defined and continuous from the set*

$$\mathfrak{R}(\rho^-) = \{\rho(s,\psi) : (s,\psi) \in J \times \mathfrak{B}_h\}$$

into $\mathfrak{B}_h$ and $\exists$ a continuous and bounded map $J^\phi : \mathfrak{R}(\rho^-) \to (0,\infty)$ s.t. $\|\phi_t\|_{\mathfrak{B}_h} \leq J^\phi(t)\|\phi\|_{\mathfrak{B}_h}$ for every $t \in \mathfrak{R}(\rho^-)$."

Lemma 2. *From the paper [6] "Let $y : (-\infty, T] \to X$ be map s.t. $y_0 = \phi, y \in \mathfrak{B}'_h$, and if (C_{4_ϕ}) hold, then*

$$\|y_s\|_{\mathfrak{B}_h} \leq (M_b + J^\phi)\|\phi\|_{\mathfrak{B}_h} + K_b\sup\{\|y(\theta)\|_X;\ \theta \in [0, \max\{0,s\}]\},\ s \in \mathfrak{R}(\rho^-) \cup J,$$

where $J^\phi = \sup_{t\in\mathfrak{R}(\rho^-)} J^\phi(t)$, $M_b = \sup_{s\in[0,T]} M(s)$ and $K_b = \sup_{s\in[0,T]} K(s)$."

Definition 1. *From the monograph [25]*"The Riemann-Liouville (R-L) Frac. integral operator of order $\alpha > 0$, for a map $g \in L^1_{loc}(\mathbb{R}^+, X)$ is defined by

$$J^0_t g(t) = g(t),\ J^\alpha_t g(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)ds, \quad t > 0,$$

where $\Gamma(\cdot)$ denotes the Euler-Gamma map."

Definition 2. *From the monograph [25]* " Caputo's Frac. derivative of order $\alpha > 0$ for a map $g \in C^n(\mathbb{R}^+, X)$ is defined by

$${}^C D^\alpha_t g(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} g^{(n)}(s)ds = J^{n-\alpha} g^{(n)}(t),$$

for $n-1 < \alpha < n$, $n \in N$. If $0 < \alpha < 1$, then

$${}^C D^\alpha_t g(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} g^{(1)}(s)ds.$$

It is cleat that, Caputo's Frac. derivative of a constant function is equal to zero."

To circumvent the reappearances of some definitions used in this paper we refer the researcher: such as Mittag–Lefller type function [25], sectorial operator [13], solution operator [14] and α-resolvent family [3].

Definition 3. *A function $y : (-\infty, T] \to X$ such that $y(t) \in \mathfrak{B}'_h$ is called the mild solution of the problem (1)-(3) if for any $u \in L^2(J,U)$ and $y(t) = \phi(t) - h(y_{\rho_1}, \ldots, y_{\rho_n})(t)$ for $t \in (-\infty, 0]$, and it satisfy the following integral equation*

$$y(t) = \begin{cases} S_\alpha(t)(\phi(0) - h(y_{\rho_1},\ldots,y_{\rho_n})(0) + p(0,\phi(0) - h(y_{\rho_1},\ldots,y_{\rho_n})(0)) - p(t, y_{\rho(t,y_t)}) \\ + \int_0^t T_\alpha(t-s)\{f(s, y_{\rho(s,y_s)}) + \int_0^s q(s-\xi)g(\xi, y_{\rho(\xi,y_\xi)})d\xi + Bu(s)\}ds, & t \in (0,t_1], \\ S_\alpha(t)(\phi(0) - h(y_{\rho_1},\ldots,y_{\rho_n})(0) + p(0,\phi(0) - h(y_{\rho_1},\ldots,y_{\rho_n})(0)) - p(t, y_{\rho(t,y_t)}) \\ + S_\alpha(t-t_1)\{I_i(y(t_1^-)) + p(t_1, y_{\rho(t_1, y(t_1^-)+I_1(y(t_i^-)))}) - p(t_1, y_{\rho(t_1,y_{t_1})})\} \\ + \int_0^t T_\alpha(t-s)\{f(s, y_{\rho(s,y_s)}) + \int_0^s q(s-\xi)g(\xi, y_{\rho(\xi,y_\xi)})d\xi + Bu(s)\}ds, & t \in (t_1,t_2], \\ \vdots \\ S_\alpha(t)(\phi(0) - h(y_{\rho_1},\ldots,y_{\rho_n})(0) + p(0,\phi(0) - h(y_{\rho_1},\ldots,y_{\rho_n})(0)) - p(t, y_{\rho(t,y_t)}) \\ + \sum_{i=1}^m S_\alpha(t-t_i)\{I_i(y(t_i^-)) + p(t_i, y_{\rho(t_i, y(t_i^-)+I_i(y(t_i^-)))}) - p(t_i, y_{\rho(t_i,y_{t_i})})\} \\ + \int_0^t T_\alpha(t-s)\{f(s, y_{\rho(s,y_s)}) + \int_0^s q(s-\xi)g(\xi, y_{\rho(\xi,y_\xi)})d\xi + Bu(s)\}ds, & t \in (t_m, T]. \end{cases} \quad (4)$$

where

$$S_\alpha(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha, A)d\lambda; T_\alpha(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda^\alpha, A)d\lambda,$$

are called analytic solutions operator, α-resolvent family and Γ is a suitable path lying on $\sum_{\theta,\omega}$.

*Remark.*The functions $S_\alpha(t); T_\alpha(t)$ are strongly continuous. If $\alpha \in (0,1)$ and $A \in \mathbb{A}^\alpha(\theta_0, \omega_0)$ then $\forall x \in X$ and $t > 0$ there $\exists\ \widetilde{M}$ such that $\|S_\alpha(t)\|_{L(X)} \leq \widetilde{M}; \|T_\alpha(t)\|_{L(X)} \leq \widetilde{M}$.

Note that, the mild solution (4) depends on control maps $u(\cdot)$. The solution of equations (1)-(3) under a control $u(\cdot)$, refereed as $y(\cdot;u)$, is said to the trajectory (state) map of (1) under $u(\cdot)$, set of all possible final states, refereed as

$$K_T(f) := \{y(T;u) \in X : u \in L^2(J;U)\},$$

is said to the approachable set of equation (1) at terminal time T. System (1)-(3) is called the controllable on J if $K_T(f) = X$.

3 Controllability Result

To prove our primary results we shall assume that the function $\rho : J \times \mathfrak{B}_h \to (-\infty, T]$ is continuous map and $\phi \in \mathfrak{B}_h$. If $y \in \mathfrak{B}_h$ we defined $\bar{y} : (-\infty, T) \to X$ as the denotation of y to $(-\infty, T]$ s.t. $\bar{y}(t) = \phi - g(y_{\rho_1}, \ldots, y_{\rho_p})(t)$. We defined $\widetilde{y} : (-\infty, T) \to X$ s.t. $\widetilde{y} = y + x$ where $x : (-\infty, T) \to X$ is the denotation of $\phi \in \mathfrak{B}_h$ s.t. $x(t) = S_\alpha(t)(\phi(0) - g(y_{\rho_1}, \ldots, y_{\rho_p})(0)$ for $t \in J$. In the continuation, we introduce the coming axioms:

(A_1) The function $f \in C(J \times \mathfrak{B}_h; X)$ and there $\exists\ \alpha_1 \in (0,1)$ and $L_f(t) \in L^{\frac{1}{\alpha_1}}(J,R)$ s.t.

$$\|f(t,\psi) - f(t,\chi)\|_X \leq L_f(t)\|\psi - \chi\|_{\mathfrak{B}_h},\ \psi,\chi \in \mathfrak{B}_h.$$

(A_2) The function $g \in C(J \times \mathfrak{B}_h; X)$ and there $\exists\ \alpha_2 \in (0,1)$ and $L_g(t) \in L^{\frac{1}{\alpha_2}}(J,R)$ s.t.

$$\|g(t,\psi) - g(t,\chi)\|_X \leq L_g(t)\|\psi - \chi\|_{\mathfrak{B}_h},\ \psi,\chi \in \mathfrak{B}_h.$$

(A_3) The function $p \in C(J \times \mathfrak{B}_h; X)$ and there $\exists\ \alpha_3 \in (0,1)$ and $L_p(t) \in L^{\frac{1}{\alpha_3}}(J,R)$ s.t.

$$\|p(t,\psi) - p(t,\chi)\|_X \leq L_p(t)\|\psi - \chi\|_{\mathfrak{B}_h},\ \psi,\chi \in \mathfrak{B}_h.$$

(A_4) The function $h \in C(J \times \mathfrak{B}_h^n; X)$ and there $\exists\ \alpha_4 \in (0,1)$ and $L_h(t) \in L^{\frac{1}{\alpha_4}}(J,R)$ s.t.

$$\|h(t,\psi^n) - h(t,\chi^n)\|_X \leq L_h(t)\|\psi - \chi\|_{\mathfrak{B}_h},\ \forall\ n\ \psi,\chi \in \mathfrak{B}_h.$$

(A_5) The functions $I_k \in C(X;X)$ and there $\exists\ \alpha_5 \in (0,1)$ and $L_I(t) \in L^{\frac{1}{\alpha_5}}(J,R)$ s.t.

$$\|I_k(x) - I_k(y)\|_X \leq L_I(t)\|x - y\|_X,\ x,y \in X.$$

(A_6) The linear operators $W_k : L^2([t_{k-1}, t_k]; U) \to X$ defined by

$$W_k u = \int_0^k T_\alpha(t_k - s)Bu(s)ds$$

has an invertible operator W_k^{-1} taking values in $L^2([t_{k-1}, t_k]; U) \setminus Ker(W_k)$ and there $\exists$ a $M > 0$ s.t. $\|BW_k^{-1}\| \leq M,\ \forall\ k$.

Now, we are in a situation to state the existence of theorem based on Contraction principal.

Theorem 1. Let the assumptions (A_1)-(A_6) hold and there $\exists$ a constant

$$\delta = \begin{cases} \widetilde{M}K_b\|L_h\|_{L^{\frac{1}{\alpha_4}}(J,R)}(1 + \|L_p\|_{L^{\frac{1}{\alpha_3}}(J,R)}) + K_b\|L_p\|_{L^{\frac{1}{\alpha_3}}(J,R)} + m\widetilde{M}(\|L_I\|_{L^{\frac{1}{\alpha_5}}(J,R)} + 2K_b\|L_p\|_{L^{\frac{1}{\alpha_3}}(J,R)}) \\ +T\widetilde{M}(K_b\|L_f\|_{L^{\frac{1}{\alpha_1}}(J,R)} + q^*K_b\|L_g\|_{L^{\frac{1}{\alpha_2}}(J,R)} + T\widetilde{M}C^* \end{cases} < 1,$$

where $q^* = \sup_{t\in J}\int_0^t \|q(t-s)\|ds$. Then there $\exists$ a control of the system (1)-(3).

*Proof.*Let $z \in \mathfrak{B}'_h$ be any arbitrary function, now to transfer the system (1)-(3) from the primary state to finial state $z(T)$, we consider the control function

$$u(t) = \begin{cases} W_1^{-1}[z(t_1) - S_\alpha(t_1)[\phi(0) - h(y_{\rho_1}, \dots, y_{\rho_n})(0) + p(0, \phi(0) - h(y_{\rho_1}, \dots, y_{\rho_n})(0)] + p(t_1, y_{\rho(t_1, y_{t_1})}) \\ - \int_0^{t_1} T_\alpha(t_1 - s)\{f(s, y_{\rho(s,y_s)}) + \int_0^s q(s-\xi) g(\xi, y_{\rho(\xi, y_\xi)}) d\xi\} ds](t), & t \in (0, t_1], \\ W_2^{-1}[z(t_2) - S_\alpha(t_2)[\phi(0) - h(y_{\rho_1}, \dots, y_{\rho_n})(0) + p(0, \phi(0) - h(y_{\rho_1}, \dots, y_{\rho_n})(0)] + p(t_2, y_{\rho(t_2, y_{t_2})}) \\ -S_\alpha(t_2 - t_1)\{I_1(y(t_1^-)) + p(t_1, y_{\rho(t_1, y(t_1^-) + I_1(y(t_1^-)))}) - p(t_1, y_{\rho(t_1, y_{t_1})})\} \\ - \int_0^{t_2} T_\alpha(t_2 - s)\{f(s, y_{\rho(s,y_s)}) + \int_0^s q(s-\xi) g(\xi, y_{\rho(\xi, y_\xi)}) d\xi\} ds](t), & t \in (t_1, t_2], \\ \vdots \\ W_m^{-1}[z(t_m) - S_\alpha(t_m)[\phi(0) - h(y_{\rho_1}, \dots, y_{\rho_n})(0) + p(0, \phi(0) - h(y_{\rho_1}, \dots, y_{\rho_n})(0)] + p(t_m, y_{\rho(t_m, y_{t_m})}) \\ \sum_{i=1}^m S_\alpha(t_m - t_i)\{I_i(y(t_i^-)) + p(t_i, y_{\rho(t_i, y(t_i^-) + I_i(y(t_i^-)))}) - p(t_i, y_{\rho(t_i, y_{t_i})})\} \\ - \int_0^{t_m} T_\alpha(t_m - s)\{f(s, y_{\rho(s,y_s)}) + \int_0^s q(s-\xi) g(\xi, y_{\rho(\xi, y_\xi)}) d\xi\} ds](t), & t \in (t_m, T], \end{cases}$$

Let $\overline{\phi} : (-\infty, T) \to X$ be the extension of ϕ to $(-\infty, T]$ such that $\overline{\phi}(t) = \phi(0)$ on J. Consider the space $\mathfrak{B}''_h = \{y \in \mathfrak{B}'_h : y(0) = \phi(0) - h(\overline{y}_{\rho_1}, \dots, \overline{y}_{\rho_n})(0)\}$ and $y(t) = \phi(t) - h(\overline{y}_{\rho_1}, \dots, \overline{y}_{\rho_n})(t)),$for $t \in (-\infty, 0]$ having the uniform convergence topology. Now, let us define an operator $P : \mathfrak{B}''_h \to \mathfrak{B}''_h$ by

$$P(y(t)) = \begin{cases} S_\alpha(t)(\phi(0) - h(\overline{y}_{\rho_1}, \dots, \overline{y}_{\rho_n})(0) + p(0, \phi(0) - h(\overline{y}_{\rho_1}, \dots, \overline{y}_{\rho_n})(0)) - p(t, \overline{y}_{\rho(t, \overline{y}_t)}) \\ + \int_0^t T_\alpha(t-s)\{f(s, \overline{y}_{\rho(s, \overline{y}_s)}) + \int_0^s q(s-\xi) g(\xi, \overline{y}_{\rho(\xi, \overline{y}_\xi)}) d\xi + B\overline{u}(s)\} ds, & t \in (0, t_1], \\ S_\alpha(t)(\phi(0) - h(\overline{y}_{\rho_1}, \dots, \overline{y}_{\rho_n})(0) + p(0, \phi(0) - h(\overline{y}_{\rho_1}, \dots, \overline{y}_{\rho_n})(0)) - p(t, \overline{y}_{\rho(t, \overline{y}_t)}) \\ S_\alpha(t - t_1)\{I_1(y(t_1^-)) + p(t_1, \overline{y}_{\rho(t_1, \overline{y}(t_1^-) + I_1(y(t_1^-)))}) - p(t_1, \overline{y}_{\rho(t_1, \overline{y}_{t_1})})\} \\ + \int_0^t T_\alpha(t-s)\{f(s, \overline{y}_{\rho(s, \overline{y}_s)}) + \int_0^s q(s-\xi) g(\xi, \overline{y}_{\rho(\xi, \overline{y}_\xi)}) d\xi + B\overline{u}(s)\} ds, & t \in (t_1, t_2], \\ \vdots \\ S_\alpha(t)(\phi(0) - h(\overline{y}_{\rho_1}, \dots, \overline{y}_{\rho_n})(0) + p(0, \phi(0) - h(\overline{y}_{\rho_1}, \dots, \overline{y}_{\rho_n})(0)) - p(t, \overline{y}_{\rho(t, \overline{y}_t)}) \\ \sum_{i=1}^m S_\alpha(t - t_i)\{I_i(y(t_i^-)) + p(t_i, \overline{y}_{\rho(t_i, \overline{y}(t_i^-) + I_i(y(t_i^-)))}) - p(t_i, \overline{y}_{\rho(t_i, \overline{y}_{t_i})})\} \\ + \int_0^t T_\alpha(t-s)\{f(s, \overline{y}_{\rho(s, \overline{y}_s)}) + \int_0^s q(s-\xi) g(\xi, \overline{y}_{\rho(\xi, \overline{y}_\xi)}) d\xi + B\overline{u}(s)\} ds, & t \in (t_m, T], \end{cases}$$

where $\overline{y} : (-\infty, T] \to X$ is such that $\overline{y}(0) = \phi(0) - h(\overline{y}_{\rho_1}, \dots, \overline{y}_{\rho_n})(0)$ and $\overline{y} = y$ on J. This is obvious that operator P is well specified. We will express that the operator $P : \mathfrak{B}''_h \to \mathfrak{B}''_h$ has a fixed point. Without loss of generality, we prove the result for the interval, $t \in (t_k, t_{k+1}]$. For convenience, let us take

$$P(y) = \begin{cases} S_\alpha(t)(\phi(0) - h(\overline{y}_{\rho_1}, \dots, \overline{y}_{\rho_n})(0) + p(0, \phi(0) - h(\overline{y}_{\rho_1}, \dots, \overline{y}_{\rho_n})(0)) - p(t, \overline{y}_{\rho(t, \overline{y}_t)}) \\ + \sum_{i=1}^k S_\alpha(t - t_i)\{I_i(y(t_i^-)) + p(t_i, \overline{y}_{\rho(t_i, \overline{y}(t_i^-) + I_i(y(t_i^-)))}) - p(t_i, y_{\rho(t_i, y_{t_i})})\} \\ + \int_0^t T_\alpha(t-s)\{f(s, \overline{y}_{\rho(s, \overline{y}_s)}) + \int_0^s q(s-\xi) g(\xi, \overline{y}_{\rho(\xi, \overline{y}_\xi)}) d\xi\} ds, + \int_0^t T_\alpha(t-s) D_j(s, y) ds, \end{cases}$$

where

$$D_j(s,y) = \begin{cases} BW_j^{-1}[z(t_j) - S_\alpha(t_j)[\phi(0) - h(\overline{y}_{\rho_1}, \dots, \overline{y}_{\rho_n})(0) + p(0, \phi(0) - h(\overline{y}_{\rho_1}, \dots, \overline{y}_{\rho_n})(0)] + p(t_j, \overline{y}_{\rho(t_j, \overline{y}_{t_j})}) \\ - \sum_{i=1}^k S_\alpha(t_j - t_i)\{I_i(y(t_i^-)) + p(t_i, \overline{y}_{\rho(t_i, \overline{y}(t_i^-) + I_i(y(t_i^-)))}) - p(t_i, \overline{y}_{\rho(t_i, \overline{y}_{t_i})})\} \\ - \int_0^{t_j} T_\alpha(t_j - s)\{f(s, \overline{y}_{\rho(s, \overline{y}_s)}) + \int_0^s q(s-\xi) g(\xi, \overline{y}_{\rho(\xi, \overline{y}_\xi)}) d\xi\} ds](s), \end{cases}$$

for $j = 1, 2, \dots, m$ and using the given assumptions, we have

$$\begin{aligned} \|D_j(s,y) - D_j(s,y^*)\|_X \le\ & M[\widetilde{M} K_b \|L_h\|_{L^{\frac{1}{\alpha_4}}(J,R)} (1 + \|L_p\|_{L^{\frac{1}{\alpha_3}}(J,R)}) + K_b \|L_p\|_{L^{\frac{1}{\alpha_3}}(J,R)} \\ & + m\widetilde{M}(\|L_I\|_{L^{\frac{1}{\alpha_5}}(J,R)} + 2K_b \|L_p\|_{L^{\frac{1}{\alpha_3}}(J,R)}) + T\widetilde{M}(K_b \|L_f\|_{L^{\frac{1}{\alpha_1}}(J,R)} + q^* K_b \|L_g\|_{L^{\frac{1}{\alpha_2}}(J,R)})] \|y - y^*\|_X \\ \le\ & C^* \|y - y^*\|_X. \end{aligned}$$

To show P has a fixed point, let us consider $y, y^* \in \mathfrak{B}_h''$ then

$$\begin{aligned}
\|P(y)-P(y^*)\|_X \le\ & \|S_\alpha(t)\|_{L(X)}\|(h(\bar{y}_{\rho_1},\dots,\bar{y}_{\rho_n})-h(\bar{y}^*_{\rho_1},\dots,\bar{y}^*_{\rho_n})\|_X \\
&+\|p(0,\phi(0)-h(\bar{y}_{\rho_1},\dots,\bar{y}_{\rho_n}))-p(0,\phi(0)-h(\bar{y}^*_{\rho_1},\dots,\bar{y}^*_{\rho_n}))\|_X \\
&+\|p(t,\bar{y}_{\rho(t,\bar{y}_t)})-p(t,\bar{y}^*_{\rho(t,\bar{y}^*_t)})\|_X+\sum_{i=1}^{k}\|S_\alpha(t-t_i)\|_{L(X)}\{\|I_i(y(t_i^-))-I_i(y^*(t_i^-))\|_X \\
&+\|p(t_i,y_{\rho(t_i,\bar{y}(t_i^-)+I_i(y(t_i^-)))})-p(t_i,\bar{y}^*_{\rho(t_i,\bar{y}^*(t_i^-)+I_i(y^*(t_i^-)))})\|_X \\
&+\|p(t_i,\bar{y}_{\rho(t_i,\bar{y}_{t_i})})-p(t_i,\bar{y}^*_{\rho(t_i,\bar{y}^*_{t_i})})\|_X\}+\int_0^t\|T_\alpha(t-s)\|_(L(X)) \\
&\times\{\|f(s,\bar{y}_{\rho(s,\bar{y}_s)})-f(s,\bar{y}^*_{\rho(s,\bar{y}^*_s)})\|_X+\int_0^s\|q(s-\xi)\|_X g(\xi,\bar{y}_{\rho(\xi,\bar{y}_\xi)})-g(\xi,\bar{y}^*_{\rho(\xi,\bar{y}^*_\xi)})\|_X d\xi\}ds \\
&+\int_0^t\|T_\alpha(t-s)\|_{L(X)}\|D_j(s,y)-D_j(s,y^*)\|_X ds \\
\le\ & [\widetilde{M}K_b\|L_h\|_{L^{\frac{1}{\alpha_4}}(J,R)}(1+\|L_p\|_{L^{\frac{1}{\alpha_3}}(J,R)})+K_b\|L_p\|_{L^{\frac{1}{\alpha_3}}(J,R)}+m\widetilde{M}(\|L_I\|_{L^{\frac{1}{\alpha_5}}(J,R)}+2K_b\|L_p\|_{L^{\frac{1}{\alpha_3}}(J,R)}) \\
&+T\widetilde{M}(K_b\|L_f\|_{L^{\frac{1}{\alpha_1}}(J,R)}+q^*K_b\|L_g\|_{L^{\frac{1}{\alpha_2}}(J,R)}+T\widetilde{M}C^*)]\|y-y^*\|_X.
\end{aligned}$$

Since $\delta<1$, its implies that P is contraction and has a unique fixed point $y\in\mathfrak{B}_h''$. Hence the system of equations (1)-(3) are controllable on interval J. This completes the proof.

4 Application

In this section, we look at an example to prove our result.

$$\begin{aligned}
&\frac{\partial^\alpha}{\partial t^\alpha}[z(t,y)+\frac{e^{-t}}{(e^t+e^{-t})}\int_{-\infty}^t e^{2(s-t)}\frac{z(s-\sigma_1(s)\sigma_2(\|z\|),y)}{49}ds] \\
&=\frac{\partial^2}{\partial y^2}[z(t,y)+\frac{e^{-t}}{(e^t+e^{-t})}\int_{-\infty}^t e^{2(s-t)}\frac{z(s-\sigma_1(s)\sigma_2(\|z\|),y)}{49}ds] \\
&+\frac{e^{-t}}{(1+t)(e^t+e^{-t})}\frac{1}{9}\int_{-\infty}^t e^{2(s-t)}\frac{z(s-\sigma_1(s)\sigma_2(\|z\|),y)}{(1+z(s-\sigma_1(s)\sigma_2(\|z\|),y))}ds \\
&+\frac{e^{-t}}{(1+t)(1+e^t)}\int_0^t\cos(t-s)\frac{z(t-\sigma_1(t)\sigma_2(\|z\|),y)}{25}ds+Bu(t,y),\ t\neq\frac{1}{2}, \qquad (5)\\
&z(t,0)=0=z(t,\pi),\ t\ge 0, \qquad (6)\\
&z(t,y)+\frac{e^{-t}}{(1+e^t)}\int_0^\pi\sin(1+|z(s,y)|)ds=\phi(t),\ t\in(-\infty,0],\ y\in[0,\pi], \qquad (7)\\
&\Delta u|_{t=\frac{1}{2}}=\frac{e^{-t}}{(1+e^{-t})}\frac{u(y,\frac{1}{2}^-)}{16+u(y,\frac{1}{2}^-)}, \qquad (8)
\end{aligned}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is Caputo's fractional derivative of order $\alpha\in(0,1)$ $0<t_1<t_2<\dots<t_n<T$ are prefixed numbers and $\phi\in\mathfrak{B}_h$. From paper [14]. "Let $X=L^2[0,\pi]$ and define the operator $A: D(A)\subset X\to X$ which is the infinitesimal generator of a solution operator $\{S_\alpha(t)\}_{t\ge0}$, such that $\|S_\alpha(t)\|_{L(X)}\le M$ for $t\in(0,T]$. Let $h(s)=e^{2s},\ s<0$ then $l=\int_{-\infty}^0 h(s)ds=\frac{1}{2}<\infty$, for $t\in(-\infty,0]$ and define

$$\|\phi\|_{\mathfrak{B}_h}=\int_{-\infty}^0 h(s)\sup_{\theta\in[s,0]}|\phi(\theta)\|_{L^2}ds.$$

Hence for $(t,\phi)\in[0,T]\times\mathfrak{B}_h$, where $\phi(\theta)(y)=\phi(\theta,y),\ (\theta,y)\in(-\infty,0]\times[0,\pi]$."

Set $z(t)(y) = z(t,y)$, and $\rho(t,\phi) = \rho_1(t)\rho_2(\|\phi(0)\|)$ we have

$$f(t,\phi)(y) = \frac{e^{-t}}{(1+t)(e^t+e^{-t})}\frac{1}{9}\int_{-\infty}^{0} e^{2(s)}\frac{\phi}{(1+\phi)}ds;\ g(t,\phi)(y) = \frac{e^{-t}}{(1+t)(1+e^t)}\frac{\phi}{25},$$
$$p(t,\phi)(y) = \frac{e^{-t}}{(e^t+e^{-t})}\int_{-\infty}^{0} e^{2(s)}\frac{\phi}{49}ds;\ I_i(\phi)(y) = \frac{e^{-t}}{1+e^t)}\frac{\phi}{16+\phi},$$
$$h(t,\phi)(y) = \frac{e^{-t}}{1+e^{-t})}\int_0^{\pi}\sin(1+\phi)d\theta.$$

Then, with these above settings, the system (5)-(8) can be written in the abstract pattern of the system (1)-(3). To treat this system, we take that $\rho_i : [0,\infty) \to [0,\infty),\ i = 1,2$, are continuous functions. Now, let us see that for $(t,\phi),(t,\psi) \in J \times \mathfrak{B}_h$, we have

$$\begin{aligned}
\|f(t,\phi) - f(t,\psi)\|_{L^2} &= \left[\int_0^{\pi}\left\{\frac{1}{9}\frac{e^{-t}}{(1+t)(e^t+e^{-t})}\|\int_{-\infty}^{0} e^{2(s)}\frac{\phi}{(1+\phi)}ds - \int_{-\infty}^{0} e^{2(s)}\frac{\psi}{(1+\psi)}ds\|\right\}^2 dy\right]^{1/2} \\
&\le \left[\int_0^{\pi}\left\{\frac{1}{9}\frac{e^{-t}}{(1+t)(e^t+e^{-t})}\int_{-\infty}^{0} e^{2(s)}\|\frac{\phi}{1+\phi} - \frac{\psi}{1+\psi}\|ds\right\}^2 dy\right]^{1/2} \\
&\le \left[\int_0^{\pi}\left\{\frac{1}{9}\frac{e^{-t}}{(1+t)(e^t+e^{-t})}\int_{-\infty}^{0} e^{2(s)}\frac{\|\phi-\psi\|}{(1+\phi)(1+\psi)}ds\right\}^2 dy\right]^{1/2} \\
&\le \left[\int_0^{\pi}\left\{\frac{1}{9}\frac{e^{-t}}{(1+t)(e^t+e^{-t})}\int_{-\infty}^{0} e^{2(s)}\sup\|\phi-\psi\|ds\right\}^2 dy\right]^{1/2} \\
&\le \frac{\sqrt{\pi}}{9}\frac{e^{-t}}{(1+t)(e^t+e^{-t})}\|\phi-\psi\|_{\mathfrak{B}_h}.
\end{aligned}$$

Hence function f satisfies (A_1). Similarly, we can show that the functions $g,\ p,\ I_i,\ h$ satisfy (A_2), (A_3), (A_4) respectively. Hence, all the conditions of the Theorem 1 have been attained, so, we derived that the system (5)-(8) has a control on J.

References

[1] S. Abbas and M. Benchohra, Impulsive partial hyperbolic functional differential equations of fractional order with state-dependent delay, *Fract. Calc. Appl. Anal.*,**13**(3), 225– 244 (2010).

[2] R. P. Agarwal and B. de Andrade, On fractional integro-differential equations with state-dependent delay, *Comput. Math. Appl.* **62**, 1143–1149 (2011).

[3] D. Araya and C. Lizama, Almost automorphic mild solutions to fractional differential equations, *Nonlinear Anal.TMA* , **69**(11), 3692-3705 (2009).

[4] M. M. Arjunan and V. Kavitha, Existence results for impulsive neutral functional differential equations with state-dependent delay, *Electr. J. Qual. Theor. Differ. Equ.*, **26**, 1–13 (2009).

[5] D. Bahuguna, Existence, uniqueness and regularity of solutions to semi-linear nonlocal functional diferential problems, *Nonlinear Anal.* **57**, 1021-1028 (2004).

[6] M. Benchohra and F. Berhoun, Impulsive fractional differential equations with state dependent delay, *Commun. Appl. Anal.*, **14**(2), 213–224 (2010).

[7] M. Benchohra, S. Litimein and G. N'Guerekata, On fractional integro-differential inclusions with state-dependent delay in Banach spaces, *Appl. Anal.* **2011**, 1–16 (2011).

[8] J. P. Carvalho dos Santos, C. Cuevas and B. de Andrade, Existence results for a fractional equation with state-dependent delay, *Adv. Differ. Equ.*, **2011**, Article ID 642013, (2011).

[9] J. P. Carvalho dos Santos and M. M. Arjunan, Existence results for fractional neutral integro-differential equations with state-dependent delay, *Comput. Math. Appl.* **62**, 1275–1283 (2011).

[10] A. Chauhan and J. Dabas, Local and global existence of mild solution to an impulsive fractional functional integro-differential equation with nonlocal condition, *Commun. Nonlinear Sci.* **19**, 821–829 (2014).

[11] A. Chauhan, J. Dabas and M. Kumar, Integral boundary value problem for impulsive fractional functional differential equations with infinite delay, *Electr. J. Differ. Equ.* **2012**(229), 1–13 (2012).

[12] A. Chauhan and J. Dabas, Existence of mild solutions for impulsive fractional order semilinear evolution equations with nonlocal conditions, *Electr. J. Differ. Equ.* **2011**(107), 110 pages (2011).

[13] J. Dabas, A. Chauhan and M. Kumar, Existence of mild solution for impulsive fractional equation with infinity delay, *Int. J. Differ. Equ.*, **2011**, Art ID 793023, (2011).
[14] J. Dabas and A. Chauhan, Existence and uniqueness of mild solution for an impulsive neutral fractional integro-differential equations with infinity delay, *Math. Comput. Model.* **57**, 754–763 (2013).
[15] M. A. Darwish and S. K. Ntouyas, Functional differential equations of fractional order with state-dependent delay, *Dyn. Syst. Appl.*, **18**, 539–550 (2009).
[16] M. A. Darwish and S. K. Ntouyas, Semilinear functional differential equations of fractional order with state-dependent delay, *Electr. J. Differ. Equ.*, **2009**, 1–10 (2009).
[17] R. Ganesh, R. Sakthivel, Y. Ren, S. M. Anthoni and N. I. Mahmudov, Controllability of neutral fractional functional equations with impulses and infinite delay, *Abstr. Appl. Anal.*, **2013**, Article ID 901625, 12 pages (2013).
[18] G. R. Gautam and J. Dabas, Mild solution for fractional functional integro-differential equation with not instantaneous impulse, *Mal. J. Math.*, **2**(3), 428–437 (2014).
[19] G. R. Gautam and J. Dabas, Results of local and global mild solution for impulsive fractional differential equation with state dependent delay, *Differ. Equ. Appl.* **6**(3), 429–440 (2014).
[20] G. R. Gautam, A. Chauhan and J. Dabas, Existence and uniqueness of mild solution for nonlocal impulsive integro-differential equation with state dependent delay, *Fract. Differ. Calc.*, **4**(2), 137–150 (2014).
[21] G. R. Gautam and J. Dabas, Mild solutions for class of neutral fractional functional differential equations with not instantaneous impulses, *Appl. Math. Comput.* **259**, 480–489 (2015).
[22] E. Hernndez, D. ORegan and K. Balachandran, Comments on some recent results on controllability of abstract differential problems, *J. Optim. Theor. Appl.* **159**, 292–295 (2013).
[23] V. Kavitha, P. Wang and R. Murugesu, Existence results for neutral functional fractional differential equations with state dependent delay, *Mal. J. Mat.* **1**(1), 50–61 (2012).
[24] F. Li, Nonlocal Cauchy problem for delay fractional integro-differential equations of neutral type, *Adv. Differ. Equ.*, **47**, (2012).
[25] I. Podlubny, *Fractional differential equations*, Acadmic Press, New York, 1999.
[26] H. Qin, X. Zuo and J. Liu, Existence and controllability results for fractional impulsive integro-differential systems in Banach Spaces, *Abstr. Appl. Anal.*, **2013**, Article ID 295837, 12 pages (2013).
[27] Z. Tai and S. Lun, On controllability of fractional impulsive neutral infinite delay evolution integrodifferential systems in Banach spaces, *Appl. Math. Lett.*, **25**, 104–110 (2012).
[28] Z. Tai and X. Wang, Controllability of fractional-order impulsive neutral functional infinite delay integrodifferential systems in Banach spaces, *Appl. Math. Lett.*, **22**, 1760–1765 (2009).
[29] N. K. Tomar and J. Dabas, Controllability of impulsive fractional order semilinear evoluation equations with nonlocal conditions, *J. Nonlin. Evol. Equ. Appl.*, **5**, 57–67 (2012).
[30] H. Zhang, J. Cao and W. Jiang, Controllability criteria for linear fractional differential systems with state delay and impulses, *J. Appl. Math.* **2013**, Article ID 146010(2013), 9 pages (2013).