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Energy of Some Wheel Related Graphs

Muhammad Khurram Zafar[1], *Abdul Qudair Baig*[2], *Muhammad Imran*[3,*] *and Andrea Semaničová–Feňovčíková*[4]

[1] Department of Mathematics, Air University, Islamabad, Pakistan
[2] Department of Mathematics, COMSATS Institute of Information Technology, Attock, Pakistan
[3] Department of Mathematics, School of Natural Science, National University of Sciences and Technology, Islamabad, Pakistan
[4] Department of Appl. Mathematics, Technical University Letná 9, 042 00 Košice, Slovakia

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Abstract: The energy of a graph is the sum of the eigenvalues of its adjacency matrix. In this paper, we are interested in how the energy of a graph changes when edges in a graph are added or removed. We prove that the energy of a graph is increased or decreased. The energy of the graph depend upon edges due to edge addition or deletion. In this paper we are dealing with various families of wheel related graphs such as *Jahangir graphs*, *wheels* and *fans*.

Keywords: eigenvalues, energy of a graph, Jahangir graph, wheel, fan

1 Energy of a graph

Let G be a finite, undirected graph without loops and multiple edges with p vertices and q edges, and let $A=[a_{ij}]$, $i,j\in\{1,2,\dots,p\}$ be the adjacency matrix of G. The eigenvalues $\lambda_1,\lambda_2,\dots,\lambda_{p-1},\lambda_p$ of A, assumed in non-increasing order, are the *eigenvalues of the graph G*. Since A is a symmetric matrix with zero trace, the eigenvalues are real and their sum is equal to zero. Thus

$$\begin{aligned}\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_{p-1}\geq\lambda_p,\\ \lambda_1+\lambda_2+\cdots+\lambda_{p-1}+\lambda_p=0.\end{aligned}\tag{1}$$

If we consider the coefficient of x^{p-2} in the characteristic polynomial

$$f(x)=det(xI_p-A),$$

we get

$$\sum_{1\leq i\leq j\leq p}\lambda_i\lambda_j=-q.$$

We also have

$$2q=trace(A^2)=\sum_{i=1}^{p}\lambda_i^2.$$

The *energy of a graph G*, denoted by $\mathscr{E}(G)$, is defined as the sum of the absolute values of its eigenvalues, i.e.

$$\mathscr{E}(G)=\sum_{i=1}^{p}|\lambda_i|.$$

This notion was introduced by Gutman [3]. The concept of graph energy arose in chemistry where certain numerical quantities, such as the heat of formation of a hydrocarbon, are related to total π-electron energy that can be calculated as the energy of appropriate "molecule", see [2], [3] and [4]. Recently, Nikiforov studied the energy of matrices in [7], [8]. Let A be a $p\times q$ real matrix, $2\leq p\leq q$. Then all the eigenvalues of AA^t are non-negative. Let λ_i^2 $(1\leq i\leq p)$ be the eigenvalues of AA^t. The energy of A is defined as

$$\mathscr{E}(A)=\mathscr{E}(G)=\sum_{i=1}^{p}|\lambda_i|.$$

Since the energy of a graph is not effected by isolated vertices, we assume throughout that the graphs have no isolated vertices. This implies that $q\geq\frac{p}{2}$. If a graph is not connected, its energy is the sum of the energies of its connected components. Thus, without loss of generality we can assume only connected graphs.

From (1) we have

$$\sum\{\lambda_i;\lambda_i>0\}=-\sum\{\lambda_i;\lambda_i<0\}=\frac{\mathscr{E}(G)}{2}.$$

There is an other formula for the energy of a graph due to Coulson [2]
$$\mathscr{E}(G)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\left(p-\frac{ixf'(ix)}{f(ix)}\right)\mathrm{d}x=\frac{1}{\pi}\int_{-\infty}^{+\infty}\left(p-x\frac{\mathrm{d}}{\mathrm{d}x}\log f(ix)\right)\mathrm{d}x,$$

* Corresponding author e-mail: mimran@sns.nust.edu.pk

where $f'(x)$ is the derivative of the characteristic polynomial $f(x)$. As a consequence of this formula, the difference of the energies of two graph with the same number of vertices we get

$$\mathscr{E}(G_1) - \mathscr{E}(G_2) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \log \frac{f_1(ix)}{f_2(ix)} dx.$$

If G is a tree or a disjoint union of trees, then the characteristic polynomial of G has the form

$$f(x) = x^p + \sum_{k\geq 1} (-1)^k q_k(G) x^{p-2k},$$

where $q_k(G)$ is the number of matchings in G of size k. Later on Gutman [3] gave a simple formula for the energy of the forest G

$$\mathscr{E}(G) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{x^2} \log\left(1 + \sum_{k\geq 1} q_k(G) x^{2k}\right) dx.$$

2 Known results

Let T_n, S_n and P_n be a tree, a star and a path on n vertices, respectively. In [1] it is proved

$$\mathscr{E}(S_n) < \mathscr{E}(T_n) < \mathscr{E}(P_n),$$

for $T_n \ncong S_n, P_n$. In the same paper Balakrishnan proved that if G is a bipartite graph with l positive eigenvalues (and so l negative eigenvalues), then

$$f(x) = x^p + \sum_{k=1}^{l} (-1)^k b_k(G) x^{p-2k},$$

where $b_k(G) > 0$ for $k = 0, 1, \ldots, l$. In this case we get

$$\mathscr{E}(G) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{x^2} log\left(1 + \sum_{k\geq 1} b_k(G) x^{2k}\right) dx.$$

Thus, the energy of a bipartite graph is a strictly increasing function of each of the parameters $b_k(G)$.

For k-regular graph G Balakrishnan [1] showed

$$\mathscr{E}(G) \leq k + \sqrt{k(n-1)(n-k)}$$

and he proved that this bound is sharp, the equality holds for complete graphs on n vertices K_n.

Rada and Tineo [9] investigated polygonal chains with maximum energy.

Using the arithmetic/geometric mean inequality and the nonnegativity of the variance of nonnegative numbers, McClelland [6] gave the following general bounds for the energy of a graph with p vertices and q edges with adjacency matrix A. Let $det(A) = \lambda_1\lambda_2\ldots\lambda_p$, then

$$\sqrt{2q + p(p-1)|det(A)|^{2/p}} \leq \mathscr{E}(G) \leq \sqrt{2pq}.$$

Let G be a graph with p vertices and q edges and let G has no isolated vertices, i.e. $2q \geq p$. Then for the maximal energy Koolen and Moulton [5] proved

$$\mathscr{E}(G) \leq \frac{2q}{p} + \sqrt{(p-1)\left(2q - (\frac{2q}{n})^2\right)}.$$

The equality holds if and only if G is isomorphic to $\frac{p}{2}K_2$ or K_p or G is a non-complete connected strongly regular graph.

3 New results

In this section we present our results. We start with some definitions.

Let n, k be positive integers, $n \geq 3$ and $n \equiv 0 \pmod{k}$. A *Jahangir graph* $J_{k,n}$, is a graph on $n+1$ vertices consisting of a cycle C_n with one additional vertex which is adjacent to $\frac{n}{k}$ vertices of C_n at distance k to each other on C_n.

A *wheel* W_n, $n \geq 3$, is a graph obtain by joining all vertices of cycle C_n to a further vertex called the *center*. Thus W_n contains $n+1$ vertices and $2n$ edges. The edges incident to the center we call spokes and the edges not incident to the central vertex we call rim edges.

A *fan* F_n, $n \geq 2$ is a graph obtained by joining all vertices of path on n vertices to a further vertex called the center. Alternatively, a fan F_n can be constructed from a wheel W_n by removing one rim edge.

In this paper we are dealing with Jahangir graph $J_{2,n}$. Note, that Jahangir graph $J_{2,n}$ can be obtained from the wheel W_n, $n \equiv 0 \pmod{2}$, by removing alternating spoke.

Bounds for the energies of some wheel related graphs

The Jahangir graph $J_{2,n}$ is a simple graph with $n+1$ vertices and $\frac{3n}{2}$ edges. We obtained

$$\frac{3n-4}{2} \leq \mathscr{E}(J_{2,n}) \leq \frac{3n+2}{2}$$

for all $n \geq 4$.

Example 31 $\mathscr{E}(J_{2,4}) = 4.898$.

Proof The roots of the characteristic polynomial of the $J_{2,4}$ are $\{0, 0, 0,$
$2.449, -2.449\}$. Thus the polynomial equation of $J_{2,4}$ is

$$f(z) = z^3(z+2.449)(z-2.449) = z^3(z^2 - 2.449^2) = z^5 - z^3 \cdot 2.449^2$$

and

$$f'(z) = 5z^4 - 3z^2 \cdot 2.449^2 = z^2(5z^2 - 3 \cdot 2.449^2).$$

There is an energy formula for the energy of a graph due to Coulson

$$\mathscr{E}(G) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\left(p - \frac{ix\acute{f}(ix)}{f(ix)}\right)\mathrm{d}x,$$

where p is the number of vertices in $J_{2,4}$, thus $p = 5$. Then

$$p - z\frac{f'(z)}{f(z)} = p - z\frac{z^2(5z^2 - 3\cdot 2.449^2)}{z^3(z^2 - 2.449^2)}$$

replacing the value of z by ix and putting the value of $p = 5$.

$$= 5 - (ix)\frac{(ix)^2(5(ix)^2 - 3\cdot 2.449^2)}{(ix)^3((ix)^2 - 2.449^2)}$$

after simplification we get

$$= \frac{2\cdot 2.449^2}{x^2 + 2.449^2}.$$

Applying Coulson integral

$$\mathscr{E}(G) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{2\cdot 2.449^2}{x^2+2.449^2}\mathrm{d}x = \frac{2\cdot 2.449^2}{\pi\cdot 2.449}\arctan(\frac{x}{2.449})\Big|_{-\infty}^{+\infty} \square$$

$$= \frac{2\cdot 2.449}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{2\cdot 2.449}{\pi}\pi = 4.898.$$

It is easy to see that the graph W_n can be obtained from $J_{2,n}$ by adding exactly $\frac{n}{2}$ new edges joining the central vertex of $J_{2,n}$ with the rim vertices of degree 2. In the next we show what happen on energy of a graph if some edges are added to the original graph.

As a wheel W_n is a simple graph with $n+1$ vertices and $2n$ edges we obtain

$$\left\lfloor\frac{3n-1}{2}\right\rfloor \le \mathscr{E}(W_n) \le \left\lceil\frac{3n+5}{2}\right\rceil.$$

for all $n \ge 3$.

It is easy to see that the fan F_n can be obtained from W_n by deleting exactly one edge joining the rim vertex of the cycle. In the next we show what happen on energy of a graph if one edge is deleted.

A fan F_n is a simple graph with $n+1$ vertices and $2n-1$ edges, then

$$\left\lfloor\frac{3n}{2}\right\rfloor - 3 \le \mathscr{E}(F_n) \le \left\lceil\frac{3n+1}{2}\right\rceil.$$

for all $n \ge 4$.

Example 32*As in Example 31, we are already calculated the energy of* $J_{2,4}$*, i.e.* $\mathscr{E}(J_{2,4}) = 4.898$.

The wheel W_4 is a graph on 5 vertices obtained from the graph $J_{2,4}$ by adding two edges, see Figure 1. The roots of W_4 are the numbers $\{0, 0, -1.236, -2, 3.236\}$. So,

$$\mathscr{E}(W_4) = (1.236 + 2 + 3.236) = 6.472.$$

Jahangir graph	$\mathscr{E}(J_{2,n})$	wheel	$\mathscr{E}(W_n)$	fan	$\mathscr{E}(F_n)$
$J_{2,4}$	4.898	W_4	6.472	F_4	5.124
		W_5	9.37	F_5	7.107
$J_{2,6}$	9.292	W_6	11.292	F_6	8.671
		W_7	12.034	F_7	10.098
$J_{2,8}$	11.312	W_8	13.656	F_8	11.87
		W_9	15.84	F_9	13.631
$J_{2,10}$	14.944	W_{10}	17.578	F_{10}	15.114
		W_{11}	18.984	F_{11}	16.818
$J_{2,12}$	17.252	W_{12}	20.14	F_{12}	18.225
		W_{13}	22.076	F_{13}	19.888
$J_{2,14}$	20.61	W_{14}	23.722	F_{14}	21.304

Table 1: Energies of the Jahangir graph $J_{2,n}$, a wheel W_n and a fan F_n.

Thus we have that $\mathscr{E}(J_{2,4}) < \mathscr{E}(W_4)$.

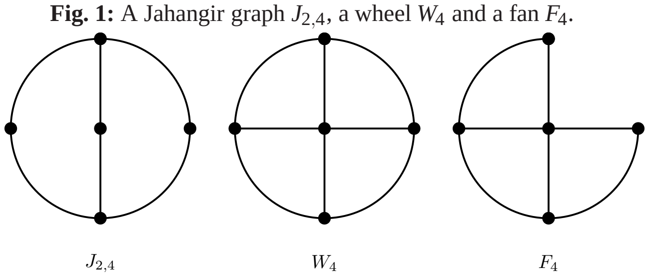
The fan F_4 is a graph obtained from W_4 by deleting one rim edge, see Figure 1. The roots of the characteristic polynomial of F_4 are $\{2.935, 0.618, -1.473, -0.463, -1.618\}$ and

$$\mathscr{E}(F_4) = (2.935 + 0.618 + 1.473 + 0.463 + 1.618) = 5.124.$$

It means

$$\mathscr{E}(W_4) > \mathscr{E}(F_4).$$

Fig. 1: A Jahangir graph $J_{2,4}$, a wheel W_4 and a fan F_4.



Analogously we can obtain the following results for energies of other graphs. The values are summarized in Table 1.

Thus, we can show the energies of the Jahangir graph $J_{2,n}$, a wheel W_n and a fan F_n in a similar way for every $n \ge 4$.

4 Conclusion

The energy of the graph changed when some edges are added or deleted. We see in above results, when some edges are added the energy of new graph increased relative to the original one

$$\mathscr{E}(J_{2,m}) < \mathscr{E}(W_n).$$

Also it is decreased when edge is deleted

$$\mathscr{E}(W_n) > \mathscr{E}(F_n).$$

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References

[1] R. Balakrishnan, The energy of a graph, *Linear Algebra Appl.*, **387** (2004), 287–395.
[2] C.A. Coulson, B. ÓLeary and R.B. Mallio, *Hückel Theory for organic chemists*, Acadmic Press, London, 1978.
[3] I. Gutman, The energy of graph: Old and new results, *Algebraic combinatorics and applications*, Springer, Berlin, 2001, 196–211.
[4] I. Gutman, The energy of a graph, *Ber. Math. Statist. Sekt. Forsch. Graz.*, **103** (1978), 1–22.
[5] J.H. Koolen and V. Moulton, Maximal energy graphs, *Adv. Appl. Math.*, **26** (2001), 47–52.
[6] B.J. McClelland, Properties of the latent roots of a matrix: The estimation of π-electron energies, *J. Chem. Phys.*, **54** (1971), 640–643.
[7] V. Nikiforov, Graphs and matrices with maximal energy, *J. Math. Anal. Appl.*, **327** (2007), 735–738.
[8] V. Nikiforov, The energy of graphs and matrices, *J. Math. Anal. Appl.*, **326** (2007), 1472–1475.
[9] J. Rada and A. Tineo, Polygonal chains with minimal energy, *Linear Algebra Appl.*, **372** (2003), 333–344.

M. K. Zafar is Assistant Professor of Mathematics at Air University, Isamabad, Pakistan. His research interests are in applied linear algebra and applied graph theory.

A. Q. Baig is an Assistant Professor at COMSATS Institute of Information Technology, Attock, Pakistan. His research interests are in the areas of graph labeling, metric graph theory and chemical graph theory. He is referee of several international journals of mathematics and informatics.

M. Imran is Assistant Professor of Mathematics at Department of Mathematics, School of Natural Sciences, National University of Sciences and Technology, Islamabad, Pakistan. His research interests are in the areas of metric graph theory, chemical graph theory, graph labeling and spectral graph theory. He has published regularly research articles in reputed journals of mathematics and informatics.

A. Semaničová–Feňovčíková is Associate Professor of Mathematics at Technical University of Košice, Slovakia. Her research interests are in graph labeling, metric graph theory and spectral graph theory.