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A Review on Local and Non-Local Fractal Calculus

Alireza Khalili Golmankhaneh [1,*], *Cemil Tunç* [2], *Safa Measoomy Nia* [3] *and Ali Khalili Golmankhaneh* [4]

[1]Department of Physics, Urmia Branch, Islamic Azad University, Urmia, Iran
[2]Department of Mathematics, Faculty of Sciences, Van Yuzuncu Yil University, 65080, Turkey
[3]Department of Mathematics, Urmia Branch Islamic Azad University, Urmia, Iran
[4]Young Researchers and Elite Club, Urmia Branch, Islamic Azad University, Urmia, Iran

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Abstract: In this paper, we review local and non-local fractal derivatives. Fractal calculus is formulated for the cases of functions supported on fractal Cantor-like sets, fractal curves, and fractal Cantor tartan. Scale properties of the suggested fractal functions and derivatives are given. Some examples and graphs are given for more details.

Keywords: Local fractal derivatives, non-local fractal derivative, fractal calculus, F^{α}-calculus.

1 Introduction

Fractal geometry includes objects and shapes with self-similarity, fractional dimension, and scale-invariant properties [1, 2,3,4,5,6,7,8,9]. Analysis on fractals was suggested by many researchers using probability, fractional calculus, harmonic analysis, and fractional spaces [10,11,12,13,14,15,16,17,18,19,20,21]. Fractal calculus was formulated in seminal papers by Gangal; it is a generalization of standard calculus which is algorithmic and simple for use in applications [22,23,24,25,26,27]. Fractal calculus has found many applications in physics [28,29,30,31,32,33,34,35, 36,37,38]. In this paper, we summarize fractal calculus and its applications in the last decades.

2 Fractal calculus

In this section, we give a brief summary of fractal calculus which involves functions defined on thin Cantor-like sets, fractal curves, and fractal Cantor tartan.

2.1 Fractal calculus for thin fractal Cantor sets

We present the steps which construct middle-$\mathfrak{a}$ Cantor set [39]. First, remove interval of length $0<\mathfrak{a}<1$ from the middle of the $J=[0,1]$. By doing the same procedures we find middle-$\mathfrak{a}$ Cantor set as follows:
Step 1.

$$C_1^{\mathfrak{a}}=[0,\frac{1}{2}(1-\mathfrak{a})]\cup[\frac{1}{2}(1+\mathfrak{a}),1].,\tag{1}$$

Step 2.

$$C_2^{\mathfrak{a}}=[0,\frac{1}{4}(1-\mathfrak{a})^2]\cup[\frac{1}{4}(1-\mathfrak{a}^2),\frac{1}{2}(1-\mathfrak{a})]\cup[\frac{1}{2}(1+\mathfrak{a})+\frac{1}{2}((1+\mathfrak{a})+\frac{1}{2}(1-\mathfrak{a})^2)]\cup[\frac{1}{2}(1+\mathfrak{a})(1+\frac{1}{2}(1-\mathfrak{a})),1].$$

* Corresponding author e-mail: alirezakhalili2002@yahoo.co.in or (a.khalili@iaurmia.ac.ir)

Then, we have after m steps

$$C^{\mathfrak{a}}=\bigcap_{m=1}^{\infty} C_m^{\mathfrak{a}}, \tag{2}$$

which is called middle-$\mathfrak{a}$ Cantor set (thin Cantor-like set).
The Hausdorff dimension of middle-$\mathfrak{a}$ Cantor set is defined by

$$\mathfrak{D}_H(C^{\mathfrak{a}})=\frac{\log(2)}{\log(2)-\log(1-\mathfrak{a})}, \tag{3}$$

which is base of Hausdorff measure [2,39].
In Figure 1, we show the processes that established the middle-$\mathfrak{a}$ Cantor set.
The flag function is defined by [22,23],

$$\mathscr{F}(C^{\mathfrak{a}},\mathbf{J})=\begin{cases}1 & \text{if} \quad C^{\mathfrak{a}}\cap \mathbf{J}\neq \emptyset\\ 0 & \text{otherwise.,}\end{cases} \tag{4}$$

where $\mathbf{J}=[b_1,b_2]$. Let $\mathbf{Q}_{[b_1,b_2]}=\{b_1=t_0,t_1,t_2,\ldots,t_n=b_2\}$ be a subdivision of $\mathbf{J}$. Then, $\Upsilon^{\alpha}[C^{\mathfrak{a}},\mathbf{Q}]$ is defined in [28,22,23] by

$$\Upsilon^{\alpha}[C^{\mathfrak{a}},\mathbf{Q}]=\sum_{i=1}^{n}\Gamma(\alpha+1)(t_i-t_{i-1})^{\alpha}\Lambda(C^{\mathfrak{a}},[t_{i-1},t_i]), \tag{5}$$

where $0<\alpha\leq 1$.
The mass function $\mathscr{M}^{\alpha}(C^{\mathfrak{a}},b_1,b_2)$ is given in [22,23] by

$$\mathscr{M}^{\alpha}(C^{\mathfrak{a}},b_1,b_2)=\lim_{\delta\to 0}\mathscr{M}^{\alpha}_{\delta}(C^{\mathfrak{a}},b_1,b_2)=\lim_{\delta\to 0}\left(\inf_{\mathbf{Q}_{[b_1,b_2]}:|Q|\leq\delta}\Upsilon^{\alpha}[C^{\mathfrak{a}},\mathbf{Q}]\right), \tag{6}$$

here, it is taken infimum over all subdivisions $\mathbf{Q}$ of $[b_1,b_2]$ satisfying $|\mathbf{Q}|:=\max_{1\leq i\leq n}(t_i-t_{i-1})\leq\delta$.
The integral staircase function of the fractal sets is defined in [22,23] by

$$S^{\alpha}_{C^{\mathfrak{a}}}(t)=\begin{cases}\mathscr{M}^{\alpha}(C^{\mathfrak{a}},t_0,t) & \text{if} \quad t\geq t_0\\ -\mathscr{M}^{\alpha}(C^{\mathfrak{a}},t_0,t) & \text{otherwise,}\end{cases} \tag{7}$$

where t_0 is an arbitrary real and fixed number.
In Figure 2, we plot Eq.(20) middle-$\mathfrak{a}$ Cantor set by letting $\mathfrak{a}=1/2$.
The γ-dimension of $C^{\mathfrak{a}}\cap[b_1,b_2]$ is

$$\begin{aligned}\dim_{\gamma}(C^{\mathfrak{a}}\cap[b_1,b_2])&=\inf\{\alpha:\mathscr{M}^{\alpha}(C^{\mathfrak{a}},b_1,b_2)=0\}\\&=\sup\{\alpha:\mathscr{M}^{\alpha}(C^{\mathfrak{a}},b_1,b_2)=\infty\}.\end{aligned} \tag{8}$$

In Figure 3, we obtain γ-dimension in view of Eq.(8).
For a middle-$\mathfrak{a}$ Cantor set fractal, the characteristic function is defined by

$$\chi_{C^{\mathfrak{a}}}(\alpha,t)=\begin{cases}\frac{1}{\Gamma(\alpha+1)}, & t\in C^{\mathfrak{a}};\\ 0, & Otherwise\end{cases} \tag{9}$$

In Figure 4, we have plotted characteristic function for the middle-$\mathfrak{a}$ Cantor set by choosing $\mathfrak{a}=1/2$.
C^{α}-limit of $h(t):C^{\mathfrak{a}}\to\Re$ is defined by

$$t,z\in C^{\mathfrak{a}} \quad \text{and} \quad |z-t|<\delta\Rightarrow|h(z)-l|<\varepsilon, \tag{10}$$

if l exists, namely

$$l=\mathrm{C}^{\alpha}\text{-}\lim_{z\to t}h(t). \tag{11}$$

C^{α}-continuity of $h(t)$ is defined by

$$h(z)=\mathrm{C}^{\alpha}\text{-}\lim_{z\to t}h(t). \tag{12}$$

Fig. 1: Cantor-like set with $a = 1/2$

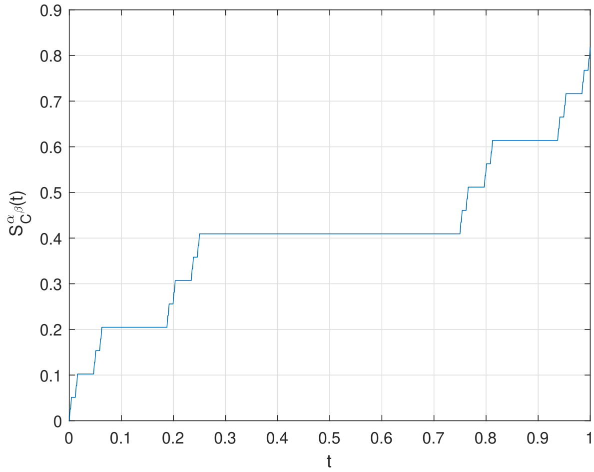

Fig. 2: Staircase function corresponding to Cantor-like set with $a = 1/2$

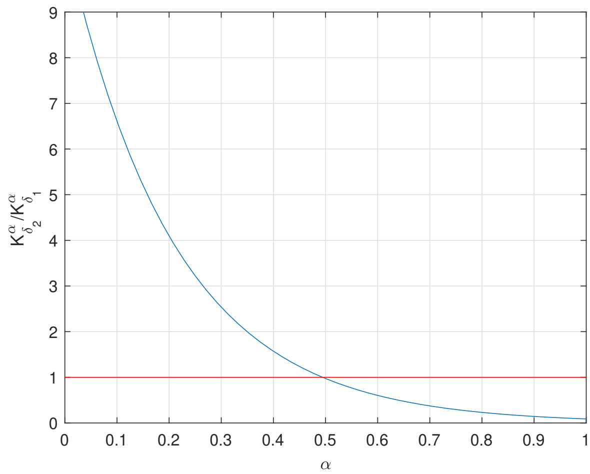

Fig. 3: The γ-dimension gives $\alpha = 0.5$ to Cantor-like set with $a = 1/2$

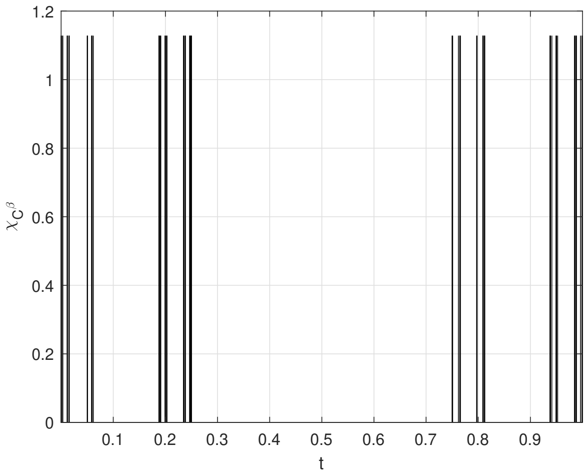

Fig. 4: Characteristic function for Cantor-like set with $\alpha = 1/2$

C^{α}-Differentiation of $h(t)$ on α-perfect set, is defined by [22,23,28],

$$D^{\alpha}_{C^{\alpha}}h(t) = \begin{cases} C^{\alpha}\text{-}\lim\limits_{z\to t} \frac{h(z)-h(t)}{S^{\alpha}_{C^{\alpha}}(z)-S^{\alpha}_{C^{\alpha}}(t)}, & \text{if } z \in C^{\alpha}, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

C^{α}-integral of $h(t)$ on $[b_1,b_2]$ is denoted by $\int_{b_1}^{b_2} h(t)d^{\alpha}_{C^{\alpha}}t$ and approximately given by [22,23,28]

$$\int_{b_1}^{b_2} h(t)d^{\alpha}_{C^{\alpha}}t \approx \sum_{i=1}^{n} h_i(t)(S^{\alpha}_{C^{\alpha}}(t_j) - S^{\alpha}_{C^{\alpha}}(t_{j-1})). \tag{14}$$

2.2 Fractal calculus for Cantor tartan fractals

In this subsection, we review calculus on the Cantor tartan, which is defined by [29]

$$\mathfrak{T} = T_1 \cup T_2 \subset [0,1]^2,$$
$$T_1 = C^{\alpha} \times [0,1], \quad T_2 = [0,1] \times C^{\alpha},$$

where $C^{\alpha} \subset [0,1]$ is a middle-α Cantor set. We consider intersections of the Cantor tartan $\mathfrak{T}$ with a box $I = [b_1,b_2] \times [c_1,c_2]$, b_1, b_2, c_1, $c_2 \in \Re$ [29]. In Figures 5 and 6, we show pre-fractal Cantor tartan with different dimensions.

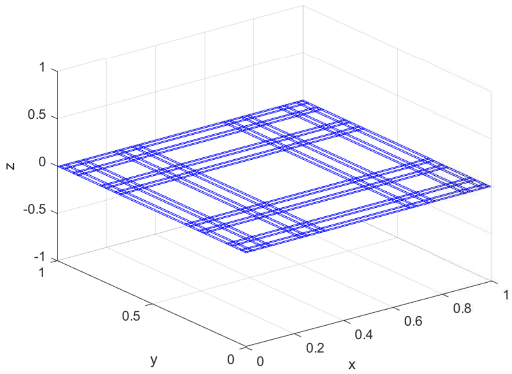

Fig. 5: Cantor tartan with dimension 1.63

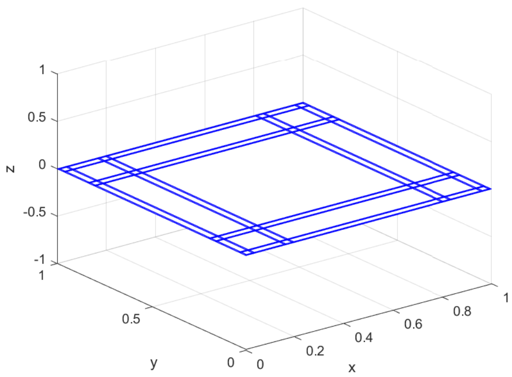

Fig. 6: Cantor tartan with dimension 1.43

The flag function for the Cantor tartan is denoted by $\Omega(\mathfrak{T}, I)$, and defined by

$$\Omega(\mathfrak{T}, I) = \begin{cases} 1, & \text{if } \mathfrak{T} \cap I \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

A subdivision of the box $I = [b_1, b_2] \times [c_1, c_2]$ is defined by

$$P_{[b_1,b_2]\times[c_1,c_2]} = \{b_1 = x_0, x_1, x_2, ..., x_n = b_2\} \\ \times \{c_1 = y_0, y_1, y_2, ..., \ y_m = c_2\}, \tag{16}$$

where $\times$ denotes the Cartesian product [29].
Mass functions for $\mathfrak{T}$ and a subdivision $P_{[b_1,b_2]\times[c_1,c_2]}$ as above such that, and for any $0 < \alpha < 1, 0 < \beta < 1$ are defined by

$$\pi^{\alpha,\beta}\left(\mathfrak{T}, P_{[b_1,b_2]\times[c_1,c_2]}\right) = \sum_{j=1}^{m}\sum_{i=1}^{n}\Gamma(\alpha+1)\Gamma(\beta+1)(x_i - x_{i-1})^{\alpha}(y_j - y_{j-1})^{\beta}\ \Theta(\mathfrak{T}, [x_{i-1}, x_i] \times [y_{j-1}, y_j]). \tag{17}$$

In view of the mass function Eq.(17) and Cantor tartan $\mathfrak{T}$ one can define

$$\lambda^{\alpha,\beta}(\mathfrak{T}, b_1, b_1, c_1, c_2) = \lim_{\delta\to 0}\left[\inf_{P_{[b_1,b_2]\times[c_1,c_2]}:|P|\leq\delta} \sigma^{\alpha,\beta}\left(\mathfrak{T}, P_{[b_1,b_2]\times[c_1,c_2]}\right)\right], \tag{18}$$

where $|P|$ is defined by

$$|P| = \max_{1\leq i\leq n,\ 1\leq j\leq m}(x_i - x_{i-1}) \times (y_j - y_{j-1}). \tag{19}$$

The integral staircase function $S_{\mathfrak{T}}^{\alpha,\beta}(x,y)$ for $\mathfrak{T}$ is defined by

$$S_{\mathfrak{T}}^{\alpha,\beta}(x,y) = \begin{cases} \lambda^{\alpha,\beta}(\mathfrak{T},b_0,c_0,x,y), & \text{if} \quad x \geq b_0,\ \ y \geq c_0; \\ -\lambda^{\alpha,\beta}(\mathfrak{T},b_0,c_0,x,y), & \text{otherwise}, \end{cases} \tag{20}$$

where $a_0,\ c_0$ are real numbers and fixed numbers [29]. In Figures 7 and 8, we have plotted integral staircase functions for different $\mathfrak{T}$ [29].

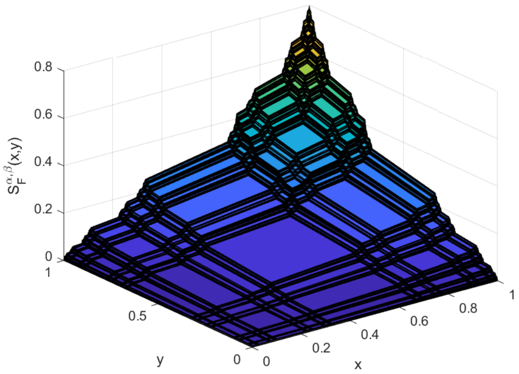

Fig. 7: For Cantor tartan with dimension 1.63

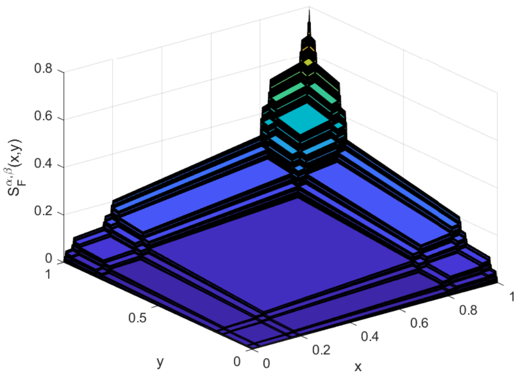

Fig. 8: For Cantor tartan with dimension 1.43

The integral staircase function of $\mathfrak{T}$ is continuous and monotonically increasing in term of each variables x, y [29]. The γ_2-dimension of $\mathfrak{T} \cap ([b_1,b_2] \times [c_1,c_2])$ is defined by

$$\begin{aligned} \dim_{\gamma_2}(\mathfrak{T} \cap ([b_1,b_2] \times [c_1,c_2])) &= \inf\{\max\{\alpha,\beta\} : \lambda^{\alpha,\beta}(\mathfrak{T},b_1,b_2,c_1,c_2) = 0\} \\ &= \sup\{\max\{\alpha,\beta\} : \lambda^{\alpha,\beta}(\mathfrak{T},b_1,b_2,c_1,c_2) = \infty\}. \end{aligned} \tag{21}$$

The Hausdorff dimension for standard fractals is equal to γ_2-dimension, namely

$$\dim_H(\mathfrak{T}) = \max\{\dim_H(T_1), \dim_H(T_2)\} = 1 + \dim_H(C^{\alpha}),$$

where $\dim_H$ is Hausdorff dimension [29].
If $g(x,y)$ be a bounded function on $\mathfrak{T}$, then one can define

$$\mathbf{M}[g,\mathfrak{T},I] = \sup_{(x,y)\in\mathfrak{T}\cap I} g(x,y), \quad \text{if} \quad \mathfrak{T}\cap I \neq \emptyset. = 0, \quad \text{otherwise}, \tag{22}$$

and similarly

$$\mathbf{m}[g,\mathfrak{T},I] = \inf_{(x,y)\in\mathfrak{T}\cap I} g(x,y), \quad \text{if} \quad \mathfrak{T}\cap I \neq \emptyset. = 0, \quad \text{otherwise}. \tag{23}$$

The upper $\mathbf{U}^{\alpha,\beta}$-sum and lower $\mathbf{L}^{\alpha,\beta}$-sum for $g(x,y)$ over the subdivision $\mathbf{P}$ are defined by

$$\begin{aligned}
\mathbf{U}^{\alpha,\beta}[g,\mathfrak{T},\mathbf{P}] &= \sum_{j=1}^{m}\sum_{i=1}^{n} \mathbf{M}[g,\mathfrak{T},[x_{i-1},x_i]\times[y_{j-1},y_j]] \left(S_{\mathfrak{T}}^{\alpha,\beta}(x_i,y_j) - S_{\mathfrak{T}}^{\alpha,\beta}(x_{i-1},y_{j-1})\right),\\
\mathbf{L}^{\alpha,\beta}[g,\mathfrak{T},\mathbf{P}] &= \sum_{j=1}^{m}\sum_{i=1}^{n} \mathbf{m}[g,\mathfrak{T},[x_{i-1},x_i]\times[y_{j-1},y_j]] \left(S_{\mathfrak{T}}^{\alpha,\beta}(x_i,y_j) - S_{\mathfrak{T}}^{\alpha,\beta}(x_{i-1},y_{j-1})\right).
\end{aligned} \tag{24}$$

The function $g(x,y)$ is called $F^{\alpha,\beta}$-integrable on the Cantor tartan $\mathfrak{T}$ if we have

$$\underline{\int_{(b_1,b_2)}^{(c_1,c_2)}} g(x,y)d_F^{\alpha}x d_F^{\beta}y \approx \sum_{i=1}^{n} g(x,y)\left(S_{\mathfrak{T}}^{\alpha,\beta}(x_i,y_j) - S_{\mathfrak{T}}^{\alpha,\beta}(x_{i-1},y_{j-1})\right). \tag{25}$$

2.3 Fractal calculus for fractal curves

The fractal $\mathbf{C}$ is called parameterized, if there exists a function $\mathbf{v}(t) : [a_0,d_0] \to \mathbf{C}$ which is continuous one to one and onto $\mathbf{C}$ [26,35].
The mass function for $\mathbf{C}$ is defined by

$$\lambda^{\alpha}(\mathbf{C},a,d) = \lim_{\delta\to 0}\lambda_{\delta}^{\alpha} = \lim_{\delta\to 0}\inf_{\{\mathbf{P}_{[c,d]}:|\mathbf{P}|\leq\delta\}}\sum_{i=0}^{n-1}\Gamma(\alpha+1)|\mathbf{v}(t_{i+1}) - \mathbf{v}(t_i)|^{\alpha}, \tag{26}$$

where $\mathbf{P}_{[a,d]}\{a = t_0,...,t_n = d\} \subset [a_0,d_0]$ is subdivision and $|.|$ is the Euclidean norm on R^3 [26,35].
The staircase function for fractal curves $\mathbf{C}$ is defined by

$$S_{\mathbf{C}}^{\alpha}(t) = \begin{cases} \gamma^{\alpha}(\mathbf{C},q_0,t) & t \geq q_0,\\ -\gamma^{\alpha}(\mathbf{C},t,q_0) & t < q_0, \end{cases} \tag{27}$$

where $q_0 \in [a_0,d_0]$ is arbitrary and fixed point.
In Figures 9 and 10, we plot staircase functions for the Koch and the Cesáro curves [26,35].
The γ-dimension of fractal curve is defined by

$$\dim_{\gamma}(\mathbf{C}) = \inf\{\alpha : \lambda^{\alpha}(\mathbf{C},a,d) = 0\} = \sup\{\alpha :\ \lambda^{\alpha}(\mathbf{C},a,d) = \infty\}. \tag{28}$$

The similarity dimension of fractal Koch curves is given by

$$D_s = \frac{\log(4)}{\log[2(1+\cos\vartheta)]}, \tag{29}$$

where ϑ is angle of the iteration for generating Koch curves [26].
Fractal derivative of function $g(\theta)$ at $\theta \in \mathbf{C}$ is defined by

$$D_C^{\alpha}g(\theta) = C - \lim_{\theta'\to\theta}\frac{g(\theta') - g(\theta)}{K(\theta') - K(\theta)}, \tag{30}$$

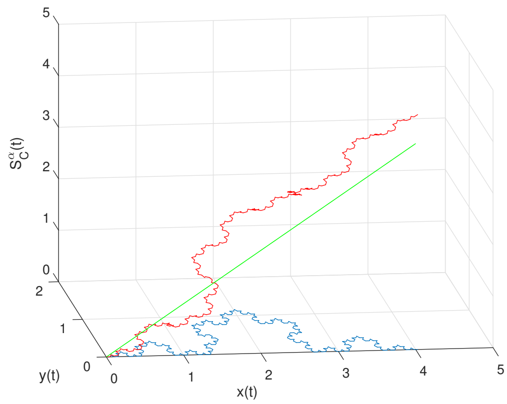

Fig. 9: Koch curve with fractal dimension 1.2 and corresponding staircase functions in red

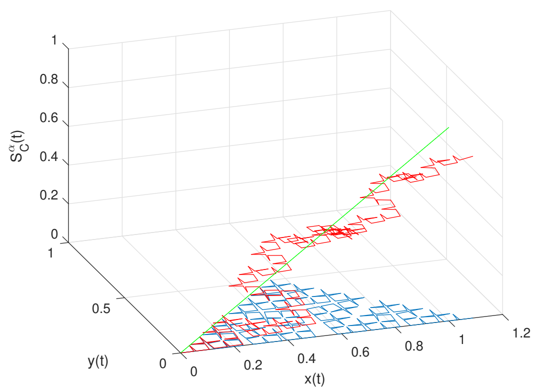

Fig. 10: Cesáro curve with fractal dimension 1.79 in blue and corresponding staircase functions in red

where $K(\theta) = S_C^{\alpha}(\mathbf{v}^{-1}(\theta)), \theta \in C$.
F-limit of g is defined by

$$\theta' \in \mathbf{C} \ \text{ and } \ |\theta' - \theta| < \delta \Rightarrow |g(\theta') - l| < \varepsilon, \tag{31}$$

if l exists [26,35]. Then we can write

$$l = C - \lim_{\theta' \to \theta} g(\theta'). \tag{32}$$

A segment $B(t_1,t_2)$ of fractal curve is defined by

$$B(t_1,t_2) = \{\mathbf{v}(t') : t' \in [t_1,t_2]\}. \tag{33}$$

It follows that

$$\mathbf{M}[g, B(t_1,t_2)] = \sup_{\theta \in B(t_1,t_2)} g(\theta), \tag{34}$$

$$\mathbf{m}[g, B(t_1,t_2)] = \inf_{\theta \in B(t_1,t_2)} g(\theta). \tag{35}$$

The upper and the lower fractal sum for the function g over the subdivision $\mathbf{P}$ are defined by

$$\mathbf{U}^{\alpha}[g,\mathbf{C},\mathbf{P}] = \sum_{i=0}^{n-1} \mathbf{M}[g, B(t_i,t_{i+1})][S_{\mathbf{C}}^{\alpha}(t_{i+1}) - S_{\mathbf{C}}^{\alpha}(t_i)], \tag{36}$$

$$\mathbf{L}^{\alpha}[g,\mathbf{C},\mathbf{P}] = \sum_{i=0}^{n-1} \mathbf{m}[g,B(t_i,t_{i+1})][S_{\mathbf{C}}^{\alpha}(t_{i+1}) - S_{\mathbf{C}}^{\alpha}(t_i)]. \tag{37}$$

Integral of the function g on fractal curve is defined by

$$\int_{B(a,b)} g(\theta)d_{\mathbf{C}}^{\alpha}\theta = \underline{\int_{B(a,b)}} g(\theta)d_{\mathbf{C}}^{\alpha}\theta = \sup_{\mathbf{P}_{[a,b]}} \mathbf{L}^{\alpha}[g,\mathbf{C},\mathbf{P}] = \overline{\int_{B(a,b)}} g(\theta)d_{\mathbf{C}}^{\alpha}\theta = \inf_{\mathbf{P}_{[a,b]}} \mathbf{U}^{\alpha}[g,\mathbf{C},\mathbf{P}]. \tag{38}$$

3 Nonlocal fractal calculus for thin Cantor fractal sets

In this section, we present a review on nonlocal fractal [31].
A function $h(S_F^{\alpha}(x)) \in \mathscr{C}_{C^{\mathfrak{a}},\rho},\ \rho \in \Re$ if we have

$$\begin{aligned} &\exists\ p > \rho \Rightarrow \\ &h(S_{C^{\mathfrak{a}}}^{\alpha}(x)) = S_{C^{\mathfrak{a}}}^{\alpha}(x)^p D_{C^{\mathfrak{a}}}^{\alpha} h(S_{C^{\mathfrak{a}}}^{\alpha}(x)) = S_{C^{\mathfrak{a}}}^{\alpha}(x)^p h_1(S_{C^{\mathfrak{a}}}^{\alpha}(x)), \end{aligned} \tag{39}$$

where

$$h_1(S_{C^{\mathfrak{a}}}^{\alpha}(x)) \in \mathscr{C}_{C^{\mathfrak{a}}}^{\alpha}[a,b].$$

Then,

$$h(S_{C^{\mathfrak{a}}}^{\alpha}(x)) \in \mathscr{C}_{C^{\mathfrak{a}},\rho}^{n\alpha}[a,b],$$

if and only if

$$(D_{C^{\mathfrak{a}}}^{\alpha})^n h(S_{C^{\mathfrak{a}}}^{\alpha}(x)) \in \mathscr{C}_{C^{\mathfrak{a}},\rho}, \quad n \in N. \tag{40}$$

Subsequently, the fractal left-sided Riemann-Liouville integral order ε is defined by

$${}_a\mathscr{I}_x^{\varepsilon} h(x) := \frac{1}{\Gamma_{C^{\mathfrak{a}}}^{\alpha}(\varepsilon)} \int_a^x \frac{h(t)}{(S_{C^{\mathfrak{a}}}^{\alpha}(x) - S_{C^{\mathfrak{a}}}^{\alpha}(t))^{\alpha-\varepsilon}} d_{C^{\mathfrak{a}}}^{\alpha} t, \tag{41}$$

where $S_{C^{\mathfrak{a}}}^{\alpha}(x) > S_{C^{\mathfrak{a}}}^{\alpha}(a)$.
Remark 1. In Eq.(41) by choosing $\varepsilon = \alpha$, one can obtain fractal integral of $h(x)$.
The fractal left-sided Riemann-Liouville derivative of order ε is defined by

$$\begin{aligned} &{}_a\mathscr{D}_x^{\varepsilon} h(x) \\ &:= \frac{1}{\Gamma_{C^{\mathfrak{a}}}^{\alpha}(n-\varepsilon)} (D_{C^{\mathfrak{a}}}^{\alpha})^n \int_a^x \frac{h(t)}{(S_{C^{\mathfrak{a}}}^{\alpha}(x) - S_{C^{\mathfrak{a}}}^{\alpha}(t))^{-n\alpha+\varepsilon+\alpha}} d_{C^{\mathfrak{a}}}^{\alpha} t. \end{aligned} \tag{42}$$

For a $h(x) \in C^{\alpha n}[a,b],\ n\alpha - \alpha \leq \varepsilon < \alpha n$, the fractal left-sided Caputo derivative of order ε is defined by

$$\begin{aligned} &{}_a^C\mathscr{D}_x^{\varepsilon} h(x) \\ &:= \frac{1}{\Gamma_{C^{\mathfrak{a}}}^{\alpha}(n-\varepsilon)} \int_a^x \frac{(D_{C^{\mathfrak{a}}}^{\alpha})^n h(t)}{(S_{C^{\mathfrak{a}}}^{\alpha}(x) - S_{C^{\mathfrak{a}}}^{\alpha}(t))^{-n\alpha+\varepsilon+\alpha}} d_{C^{\mathfrak{a}}}^{\alpha} t. \end{aligned} \tag{43}$$

Remark 2. We can obtain the standard fractional derivatives by choosing $\alpha = 1$ [31].

3.1 Scale properties of the fractal functions

A function $h(S_{C^{\mathfrak{a}}}^{\alpha}(x))$ is fractal homogenous of degree-$n\alpha$ or invariant under fractal rescalings if we have

$$h(S_{C^{\mathfrak{a}}}^{\alpha}(\mu x)) = \mu^{n\alpha} h(S_{C^{\mathfrak{a}}}^{\alpha}(x)),\ \exists\ n,\ \forall\ \mu. \tag{44}$$

For the function $h(S_{C^{\mathfrak{a}}}^{\alpha}(x))$ on triadic Cantor set ($\mathfrak{a} = 1/3$) if we choose $m = 1$ and $\mu = 1/3^n, n = 1,2,...,$ then

$$h(S_{C^{\mathfrak{a}}}^{\alpha}(\frac{1}{3^n}x)) = (\frac{1}{3^n})^{\alpha} h(S_{C^{\mathfrak{a}}}^{\alpha}(x)). \tag{45}$$

Scale change of the local fractal derivative and fractal homogenous function $h(S_{C^a}^{\alpha}(x))$ are given by

$$D_{C^a}^{\alpha}h(S_{C^a}^{\alpha}(\mu x)) = \mu^{n\alpha-\alpha}D_F^{\alpha}h(S_{C^a}^{\alpha}(x)). \tag{46}$$

Scale change for the staircase function $S_{C^a}^{\alpha}(x)$, implies that gives

$$x \to \mu x \Rightarrow S_{C^a}^{\alpha}(\mu x) = \mu^{\alpha}S_{C^a}^{\alpha}(x). \tag{47}$$

In view of Eq. (42), $a = 0$, and scale change $x \to \mu x$, we get

$${}_0\mathscr{D}_x^{\varepsilon}(h(S_{C^a}^{\alpha}(\mu x))) = \mu^{\varepsilon\alpha}\,{}_0\mathscr{D}_{\mu x}^{\mu}(h(S_{C^a}^{\alpha}(\mu x))), \tag{48}$$

which is called scale change on the non-local fractal derivatives [31].
Some important formulas:

$${}_x\mathscr{I}_b^{\varepsilon}\,{}_x\mathscr{D}_b^{\varepsilon}h(x) = h(x) - \sum_{j=1}^{n}\frac{({}_x\mathscr{D}_b^{\varepsilon-j}h(x))|_{(S_{C^a}^{\alpha}(b))}}{\Gamma(\varepsilon+\alpha-j)}(S_{C^a}^{\alpha}(b) - S_{C^a}^{\alpha}(x))^{\varepsilon-j}, \tag{49}$$

$${}_a\mathscr{I}_x^{\varepsilon}\,{}_a^C\mathscr{D}_x^{\varepsilon}h(x) = h(x) - \sum_{j=1}^{n}\frac{((D_{C^a}^{\alpha})^j h(x))|_{(S_{C^a}^{\alpha}(a))}}{\Gamma(j+\alpha)}(S_{C^a}^{\alpha}(x) - S_{C^a}^{\alpha}(a))^{j}, \tag{50}$$

$${}_x\mathscr{I}_b^{\varepsilon}\,{}_x^C\mathscr{D}_b^{\varepsilon}h(x) = h(x) - \sum_{j=1}^{n}\frac{((D_{C^a}^{\alpha})^j h(x))|_{(S_F^{\alpha}(b))}}{\Gamma(j+\alpha)}(S_{C^a}^{\alpha}(b) - S_{C^a}^{\alpha}(x))^{j}. \tag{51}$$

Example 1. Consider function $h(x) = x^2$ and $f(x) = S_{C^a}^{\alpha}(x)$ *not* $S_F^{\alpha}(x)$. Then we have

$${}_0D_x^{0.5}h(x) = \frac{\Gamma(3)}{\Gamma(2.5)}x^{1.5}, \tag{52}$$

where ${}_0D_x^{0.5}$ is fractional Riemann-Liouville derivative.

$${}_0\mathscr{D}_x^{0.5}f(x) = \frac{\Gamma_{C^a}^{\alpha}(3)}{\Gamma_{C^a}^{\alpha}(2.5)}S_{C^a}^{\alpha}(x)^{1.5}, \tag{53}$$

where ${}_0\mathscr{D}_x^{0.5}$ is called fractal Riemann-Liouville derivative. In Figure 11, we plot Eq.(53) and Eq.(52).
Example 2. Let us consider function $h(x) = x^2$ and $f(x) = S_{C^a}^{\alpha}(x)$ *not* $S_F^{\alpha}(x)$. Then we have

$${}_0I_x^{0.5}h(x) = \frac{\Gamma(3)}{\Gamma(3.5)}x^{2.5}, \tag{54}$$

where ${}_0I_x^{0.5}$ is fractional Riemann-Liouville integral.

$${}_0\mathscr{I}_x^{0.5}f(x) = \frac{\Gamma_{C^a}^{\alpha}(3)}{\Gamma_{C^a}^{\alpha}(3.5)}S_{C^a}^{\alpha}(x)^{2.5}, \tag{55}$$

where ${}_0\mathscr{I}_x^{0.5}$ is called fractal Riemann-Liouville integral. In Figure 12, we plot Eq.(55) and Eq.(54).
Example 3. Consider fractal differential equation on the fractal Cesàro curve as follows:

$$D_{\mathbf{C}}^{\alpha}g(x) = 3, \quad \alpha = 1.79. \tag{56}$$

Using the fractal integral, we obtain

$$g(x) = 3S_{\mathbf{C}}^{\alpha}(x), \tag{57}$$

which is the solution of Eq.(56).
In Figure 13, we plot Eq.(57).
Remark 3. The green lines in Figures 9 and 10 indicate the solution for the case of standard calculus, namely $\alpha = 1$.

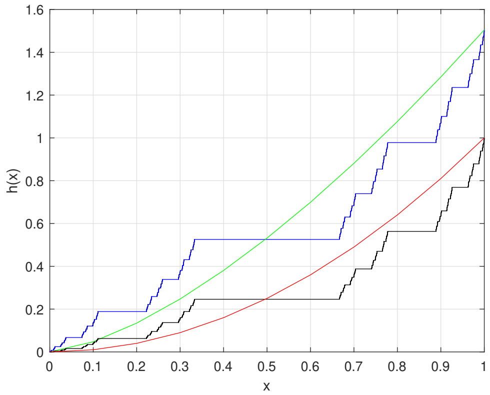

Fig. 11: Graph of $y(x) = x^2$ in (red), $f(x) = S^{\alpha}_{C^a}(x)$ *not* $S^{\alpha}_{F}(x)$ in (black), ${}_0D^{0.5}_x y(x)$ in (green), and ${}_0\mathscr{D}^{0.5}_x f(x)$ in (blue).

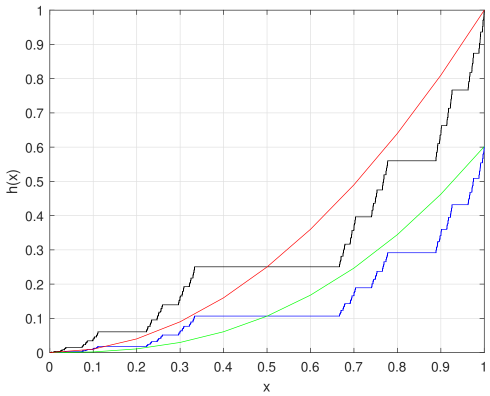

Fig. 12: Graph of $g(x) = x^2$ in (red), $f(x) = S^{\alpha}_{C^a}(x)$ *not* $S^{\alpha}_{F}(x)$ in (black), ${}_0I^{0.5}_x g(x)$ in (green) and ${}_0\mathscr{I}^{0.5}_x f(x)$, in (blue)

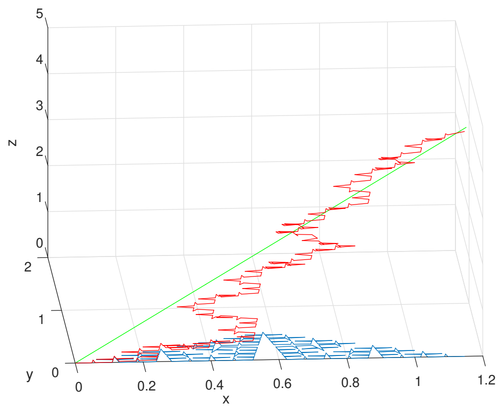

Fig. 13: Graph of the solution Eq.(56)

4 Conclusion

In this paper, we give a summary of fractal calculus. Fractal calculus has been adapted to fractal sets, fractal curves, and Cantor tartan. Non-local fractal calculus was formulated which leads to local fractal calculus. The given results enable the recovery of results in standard calculus. We believe that research in this direction will find many applications in physics, chemistry, and engineering.

Conflict of Interests

There is no conflict of interests by authors regarding the publication of this manuscript.

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