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# The Solution of Nonlinear Time-Fractional Differential Equations: An Approximate Analytical Approach

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**Abstract:** The nonlinear time fractional order coupled differential equations are considered in the present investigation. In particular, homogeneous advection equation coupled Burger's equations, and coupled Schrodinger-Kdv equations are taken care of for the several fraction orders. The novelty of the current investigation is the explicit and analytical solution of these equations by employing a new approach called "Reduced Differential Transform Method" (RDTM) in association with the "Adomian Decomposition Method" (ADM). Finally, the method's efficiency and convergence are obtained by comparing the fractional order's exact solution through particular examples. These are presented via surface and contour plots.
**Keywords:** Fractional advection equation, Burger's equations, Schrodinger equations, ADM, RDTM.

## 1 Introduction

The most generalized form of standard calculus is fractional that deals with the various forms of differential and integrals of order in fractional form. Now a day it is vastly used due to its wide applications in engineering i.e. chemistry, ecology, biology, solid-state physics, signal processing, finance related to stochastic process, economics, control theory, and many other. In all these areas the problem is transformed converted into models in mathematical form via fractional orders employing various operators of fractional differentials.

Fractional calculus theory is originally developed by "*Leibniz*, *Liouville*, *Riemann*, *Grunwald*, and *Letnikov*", and many others [1-6]. For the sake of physical and engineering interests many mathematicians has given main attention for the solution of these differential equations either ordinary or partial, and integral equations (IEs). However, it is quite hard to explore the closed form resolution of the fractional differential equations; therefore, researchers are always trying to find analytical and numerical methods to find their approximation solutions. Whereas many researchers have adopted various concepts of the solutions and methods that derived for the fractional differential equations. Some classical methods are, "*Fourier transform technique* [7], *special methods for fractional DEs* [8], *Laplace transform technique* [9] and *the operational calculus method* [10]".

Now a days to solve the fractional DEs several mathematical methods have been used for example the approximate analytical method so called "*Adomian decomposition method* (ADM) [11], *differential transform method* (DTM) [12], classical matrix method known as *operational matrix method* [13], *variational iteration method* (VIM) [14], *Homotopy perturbation method* (HPM) [15], *Homotopy analysis method* (HAM) [16] and the *residual power series method* (RPSM) [17-22]". Employing these methods, not only the governing problems are transformed into simple but also the solution either in closed form or the simple power series forms are obtained to get the convergence of the solutions. Moreover, analytical technique using RPSM used for both fractional and non-fractional DEs [17-22] in which the power series coefficients are obtained. The crux of the methodology is used to get the approximate or the closed form solutions various linear and nonlinear equations. By RPS method we can successfully solve many types of "*fractional ordinary or partial DEs*, *neutrons diffusion equations* [19] used

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in multi-energy groups, *fractional KDV-Burgers equation* [20], *fractional Schrödinger equations* [21] and *fractional multi-pantograph system* [22]".
One of the traditional techniques is "*Laplace transform method*" but is still proposed to handle fractional differential equations. Here the original expression is distorted from one space to another, and solution is obtained purely algebraic process and taken back by using inverse Laplace transform. Unfortunately, for a small class of Des this method is applicable, as finding inverse Laplace transform for higher differential equations leads to non-integrable functions. The electric locomotive process forms a particular form of model called pantograph equations [23]. Various physical phenomena were described due to their application in electrodynamics [24-25]. For the nanoscale flow phenomena Liu et al. [26] presented an application of fractional calculus. Recently, Anjum and Ain [27], Wang and Yao [28] have employed differential approach and particular transformation for the fractional Camssa-Holm and evolution equations respectively. Further, Wang et al. [29] imposed the same concept for the Snow's thermal insulation properties. Moreover, Ahmad et al. [30] employed "*local meshless method*" for the solution of multi-term time fractional PDEs. Exact solution for the class of stochastic Benjamin–Bona–Mahony equation is obtained by Agarwal et al. [31]. Cesarano [32] and Assante et al. [33] have presented a generalized special function for the fractional diffusive equation. Further, numerical treatment on the different types of differential equations was presented in several papers [34-39]. For more about fractional calculus and its developments, one can be referred to [40-48].
In the present work, an effective and new analytical technique is introduced to get solutions of time-fractional and non-fractional DEs employing Reduced Differential Transformation Method in association with Adomian Decomposition Method. For the efficiency of the method, we have considered various types of coupled nonlinear equations such as homogeneous advection equation, Burger's equations, and Schrodinger equations etc. Validation with non-fractional order is exhibited with their closed form solutions accordingly. Finally, for different fractional order within the domain of [0, 1] the surface plots and the contour plot are presented graphically and discussed.

## 2 Basic idea of fractional calculus

To know about the fractional calculus, it is essential to start by considering a simple example such as $y = x$. The above expression is presented as $y^{(0)} = x$ (zero-order) and the next differentiation of this function is $y^{(1)} = 1$.
A simple question can arise, what is $y^{(\alpha)}$?, where $\alpha \in (0,1)$. It represents the gradient of $y$ (Fig.1). The function may be continuous but not differentiable anywhere. "A fractal medium has certainly a fractal boundary, however a differential equation cannot be established within that" (He [49])

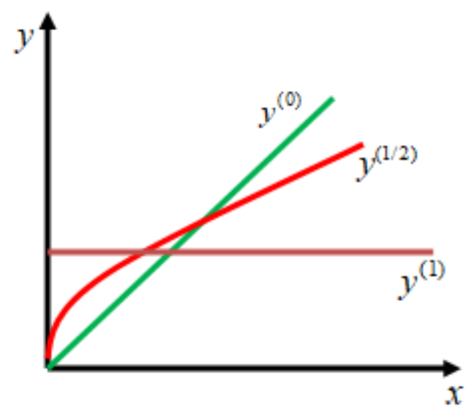



**Fig.1:** Geometrical interpretation of $y^{(\alpha)}$ for $\alpha \in (0,1)$.

### *2.1 Fractional derivative*

From various definitions on fractional derivatives however here few are described as follows.
Consider the nth order linear equation, $u^{(n)} = f(\tau)$.
The iteration formula employing variational iteration method (VIM), is expressed as

$$u_{i+1}(\tau) = u_i(\tau) + \int_{\tau_0}^{\tau} \lambda\left(u_i^{(n)}(s) - f_i(s)\right) ds$$

After identifying the multiplier,

$$u_{i+1}(\tau) = u_i(\tau) + (-1)^n \int_{\tau_0}^{\tau} \frac{1}{(n-1)!} (s-\tau)^{n-1} \left(u_i^{(n)}(s) - f_i(s)\right) ds$$

Since the equation is linear its solution becomes

$$u(\tau) = u_0(\tau) + (-1)^n \int_{\tau_0}^{\tau} \frac{1}{(n-1)!} (s-\tau)^{n-1} \left(u_0^{(n)}(s) - f(s)\right) ds$$

Here, $u_0(\tau)$ is a guess solution, obtained using specified initial condition. Introducing an operator $I^n$ defined by

$$I^n f = \int_{\tau_0}^{\tau} \frac{1}{(n-1)!} (s-\tau)^{n-1} \left(u_0^{(n)}(s) - f(s)\right) ds$$

$$I^n f = \frac{1}{\Gamma(n)} \int_{\tau_0}^{\tau} (s-\tau)^{n-1} \left(u_0^{(n)}(s) - f(s)\right) ds \text{ with } f_0(\tau) = u_0^{(n)}(\tau).$$

**Definition 2.1:** The standard form of the fractional derivative is,

$$D_t^\alpha f = D_t^\alpha \frac{d^n}{dt^n}(I^n f) = \frac{d^n}{dt^n}(I^{n-\alpha} f) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} (s-t)^{n-\alpha-1} \left(f_0(s) - f(s)\right) ds$$

**Definition 2.2:** The "*Caputo-fractional derivative*" is

$$D_x^\alpha f = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n}{dt^n} f(t) dt$$

**Definition 2.3:** The Riemann-Lioville *fractional derivative* is

$$D_x^\alpha f = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t) dt$$

## 3 Methodology (reduced differential transform method)

Class of fractional differential equations for the function $f(x,t)$ consisting of both time and space variables and combination of two distinct functions of single variables i.e. $f(x,t) = u(x)v(t)$ are solved using a new approach of *Reduced Differential Transform Method* (RDTM) in association with "*Adomian Decomposition Method*" (ADM) is described in this section. The equation can be articulated as

$$f(x,t) = \sum_{k=0}^{\infty} F_k(x) t^{k\alpha} \tag{1}$$

Where $\alpha$ indicates the Caputo time-fractional order derivative and $F_k(x)$ represents t-dimensional spectrum equation for the function $f(x,t)$.

**Definition 3.1:** Let us consider the continuously differentiable function $f(x,t)$ with respect to $t$ and $x$ in a certain domain then,

$$F_k(x) = \frac{1}{\Gamma(k\alpha+1)} \frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x,t) \bigg|_{t=0} \tag{2}$$

Here, $F_k(x)$ is the transformation of $f(x,t)$ and therefore $f(x,t)$ is the inverse transformed function.

Therefore, the inverse transformed function is expressed as

$$f(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x,t) \bigg|_{t=0} t^{k\alpha} \tag{3}$$

The nonlinear "*time-fractional differential equation*" of the form

$$L_\alpha\left(f(x,t)\right)+R\left(f(x,t)\right)+N\left(f(x,t)\right)=u(x,t) \tag{4}$$

Subject to

$$f(x,0)=u(x) \tag{5}$$

Here $L_\alpha$, the linear operator employed for the conformable derivative of order $\alpha$, $n<\alpha\le n+1$, $R$, the remaining part of the linear operator, $N$ indicates the nonlinear terms and $u$ is the non-homogenous function appears in the equation.

Here, the operator $L_\alpha$ is defined as

$$L_\alpha f(t)=\frac{\partial^\alpha f}{\partial t^\alpha}$$

Hence the transformed equation using RDTM is

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}F_{k+1}(x)=U_k(x)-R(F_k(x))-N(F_k(x)) \tag{6}$$

And the initial condition becomes

$$F_0(x)=u(x) \tag{7}$$

The approximate solution is obtained from the iterative formula by using several values of $k$ from $0$ to $n$ i.e.

$$f(x,t)=\sum_{k=0}^{n}F_k(x)t^{k\alpha} \tag{8}$$

Therefore, exact solution is approximated by

$$f(x,t)=\lim_{n\to\infty}\sum_{k=0}^{n}F_k(x)t^{k\alpha} \tag{9}$$

## 4 Applications using various problems

The proposed technique effectively used in several models known as homogeneous advection equation, coupled Berger's equations, and coupled Schrodinger equations are described and their properties have significant role in engineering Mathematics and physics.

**Problem-4.1:**

***Time-fractional nonlinear homogeneous advection equation***

Consider the homogeneous advection equation

$$\frac{\partial^\alpha}{\partial t^\alpha}\left[f(x,t)\right]=f(x,t)\frac{\partial}{\partial x}\left[f(x,t)\right],\quad f(x,0)=x \tag{10}$$

Introducing RDTM defined earlier the equation (10) can be expressed as

$$\frac{\Gamma(k.\alpha+\alpha+1)}{\Gamma(k.\alpha+1)}F_{k+1}(x)=ff_x \tag{11}$$

Assuming nonlinear operator $N(f,f_x)=ff_x$ in Adomian polynomial given by

$$N(f,f_x)=\sum_{n=0}^{\infty}A_n \tag{12}$$

Where $A_n's$ are called Adomian polynomials and calculated as

$$A_n = \frac{1}{n!}\frac{d}{d\lambda^n}\left(N\left(\sum_{n=0}^{\infty}\lambda^n u_n\right)\right)_{\lambda=0} \tag{13}$$

Using above expression, we get

$$\begin{aligned} A_0 &= f_0 f_{0x}, \\ A_1 &= f_1 f_{0x} + f_0 f_{1x}, \\ A_2 &= f_2 f_{0x} + f_1 f_{1x} + f_0 f_{2x}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{14}$$

Using the Adomian polynomial the iterative expression (14) is represented as

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}F_{k+1}(x) = A_k \text{ with } f_0(x) \approx F_0(x) = x \tag{15}$$

For various values of $k = 0, 1, 2....$,

$$F_1(x) = \frac{1}{\Gamma(\alpha+1)}x,$$

$$F_2(x) = \frac{2}{\Gamma(2\alpha+1)}x, \text{and so on.}$$

Therefore, the approximate solution is

$$\begin{aligned} f(x,t) &= \sum_{k=0}^{\infty} F_k(x) t^{k\alpha} \\ &= \left(1 + \frac{1}{\Gamma(\alpha+1)}t^{\alpha} + \frac{2}{\Gamma(2\alpha+1)}t^{2\alpha} + \ldots\ldots\ldots\ldots\right)x \end{aligned} \tag{16}$$

For the fractional order $\alpha = 1$, the exact solution for the homogenous advection equation is

$$f(x,t) = \left(\sum_{n=0}^{\infty} t^n\right)x = x/(1-t) \tag{17}$$

.

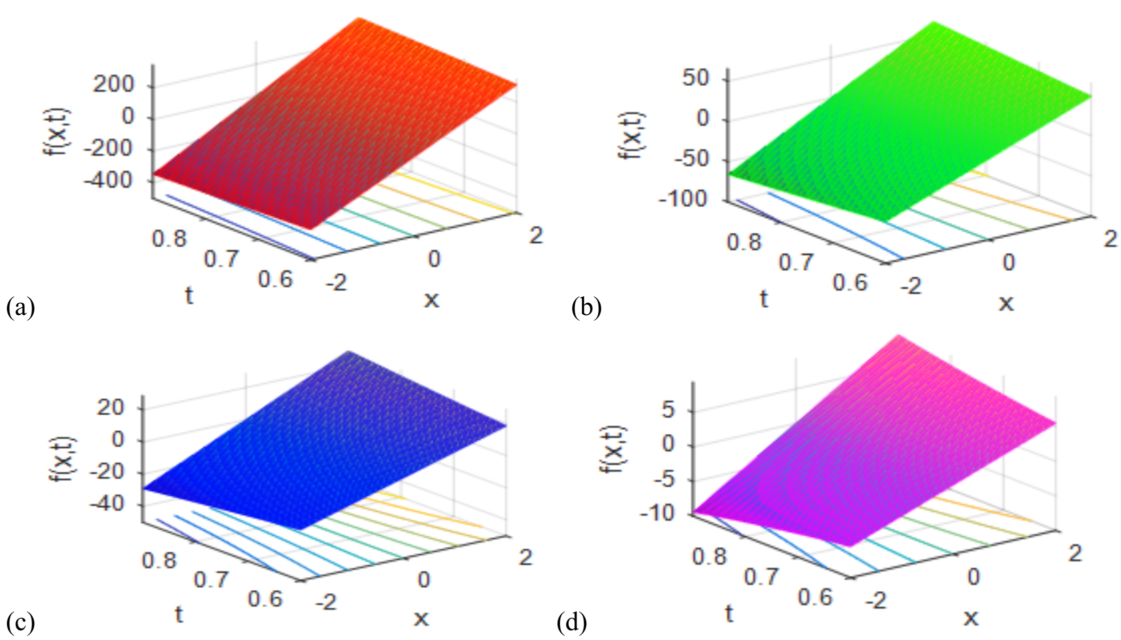



**Fig. 2** Surface plot of the solution using RDTM for the orders $\alpha$ (a) $0.1$ (b) $0.3$ (c) $0.5$ (d) $0.9$

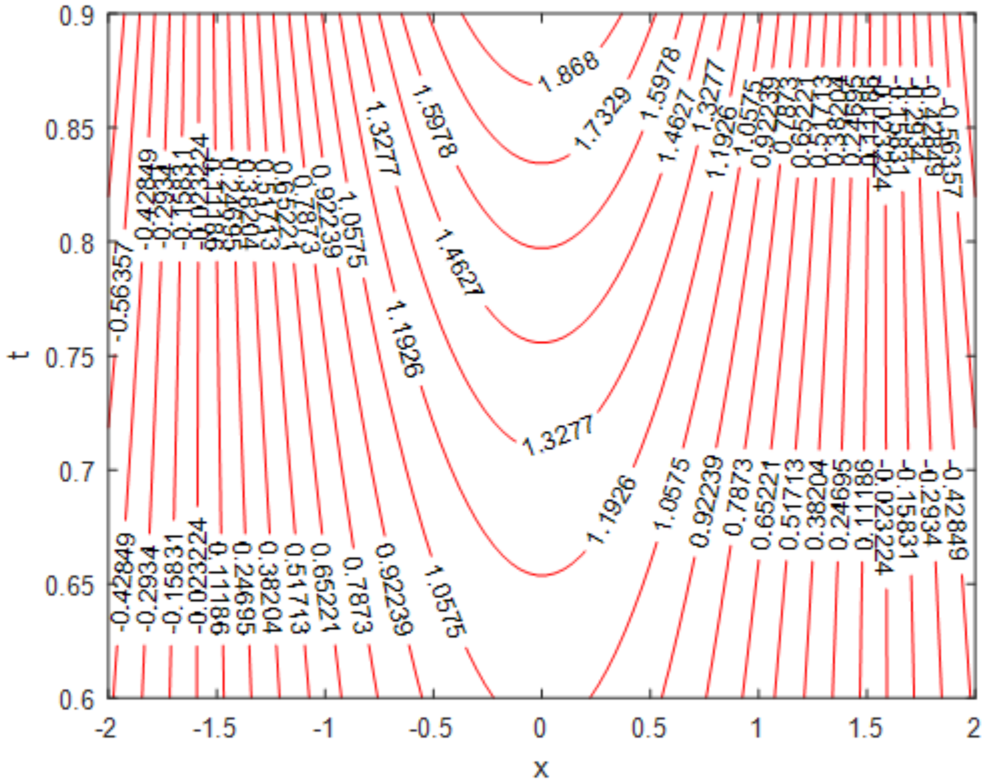



**Fig. 3** Contour for the exact result of Advection equation for $\alpha = 1$.

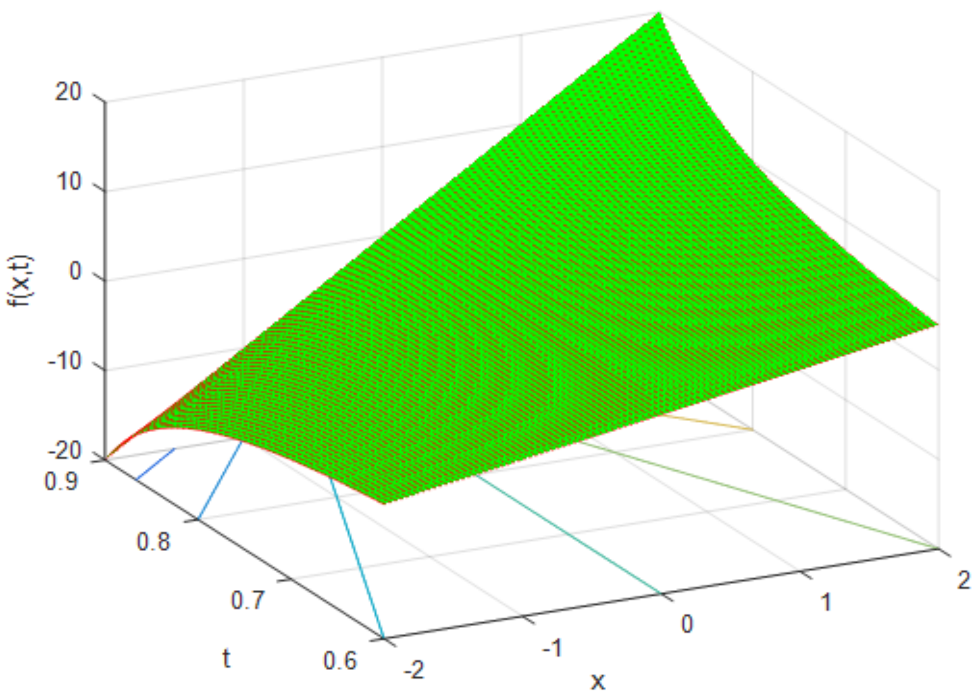



**Fig.4:** Surface plot for the exact result of $f(x,t) = \dfrac{x}{1-t}$ when $\alpha = 1$.

**Problem-4.2: Time-fractional coupled nonlinear Burger's equations**

Consider the Burger's equations:

$$\frac{\partial^\alpha}{\partial t^\alpha}\left[f(x,t)\right]-\frac{\partial^2}{\partial x^2}\left[f(x,t)\right]-2f(x,t)\frac{\partial}{\partial x}\left[f(x,t)\right]+\frac{\partial}{\partial x}\left[f(x,t)g(x,t)\right]=0, \tag{18}$$

$$\frac{\partial^\alpha}{\partial t^\alpha}\left[g(x,t)\right]-\frac{\partial^2}{\partial x^2}\left[g(x,t)\right]-2g(x,t)\frac{\partial}{\partial x}\left[g(x,t)\right]+\frac{\partial}{\partial x}\left[f(x,t)g(x,t)\right]=0, \tag{19}$$

The corresponding initial conditions are

$$f(x,0)=\cos x, \qquad g(x,0)=\cos x. \tag{20}$$

Introducing RDTM defined earlier the system of equations (18) and (19) can be expressed as

$$\frac{\Gamma(k.\alpha+\alpha+1)}{\Gamma(k.\alpha+1)}F_{k+1}(x)=f_{xx}+2ff_x-(fg)_x \tag{21}$$

$$\frac{\Gamma(k.\alpha+\alpha+1)}{\Gamma(k.\alpha+1)}G_{k+1}(x)=g_{xx}+2gg_x-(fg)_x \tag{22}$$

considering the nonlinear operator $N(f,f_x)=ff_x$, $M(f,g)=\left(fg\right)_x$ and $K(g,g_x)=gg_x$ the Adomian polynomials given by

$$N(f,f_x)=\sum_{n=0}^{\infty}A_n\,,M(f,g)=\sum_{n=0}^{\infty}B_n\,,\text{ and } K(g,g_x)=\sum_{n=0}^{\infty}C_n\,. \tag{23}$$

The Adomian polynomials are calculated and presented as

$$\begin{aligned}
&A_0=f_0f_{0x},\ B_0=\left(f_0g_0\right)_x,\ C_0=g_0g_{0x},\\
&A_1=f_1f_{0x}+f_0f_{1x},\ B_1=\left(f_0g_1+f_1g_0\right)_x,\ C_1=g_1g_{0x}+g_0g_{1x},\\
&A_2=f_2f_{0x}+f_1f_{1x}+f_0f_{2x},\ B_2=\left(f_0g_2+f_1g_1+f_2g_0\right)_x,\ C_2=g_2g_{0x}+g_1g_{1x}+g_0g_{2x},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned} \tag{24}$$

Using the Adomian polynomials the iterative expressions (21) and (22) are represented as

$$\frac{\Gamma(k.\alpha+\alpha+1)}{\Gamma(k.\alpha+1)}F_{k+1}(x)=f_{xx}+2A_k-B_k\,, \tag{25}$$

$$\frac{\Gamma(k.\alpha+\alpha+1)}{\Gamma(k.\alpha+1)}G_{k+1}(x)=g_{xx}+2C_k-B_k\,, \tag{26}$$

$$\text{with } f_0(x)\approx F_0(x)=\cos x=g_0(x)\approx G_0(x), \tag{27}$$

For various values of $k=0,1,2....$, we get

$$F_1(x)=-\frac{\cos x}{\Gamma(\alpha+1)}=G_1(x),$$

$$F_2(x)=\frac{\cos x}{\Gamma(2\alpha+1)}=G_2(x),\text{and so on.}$$

Therefore, the approximate solutions are

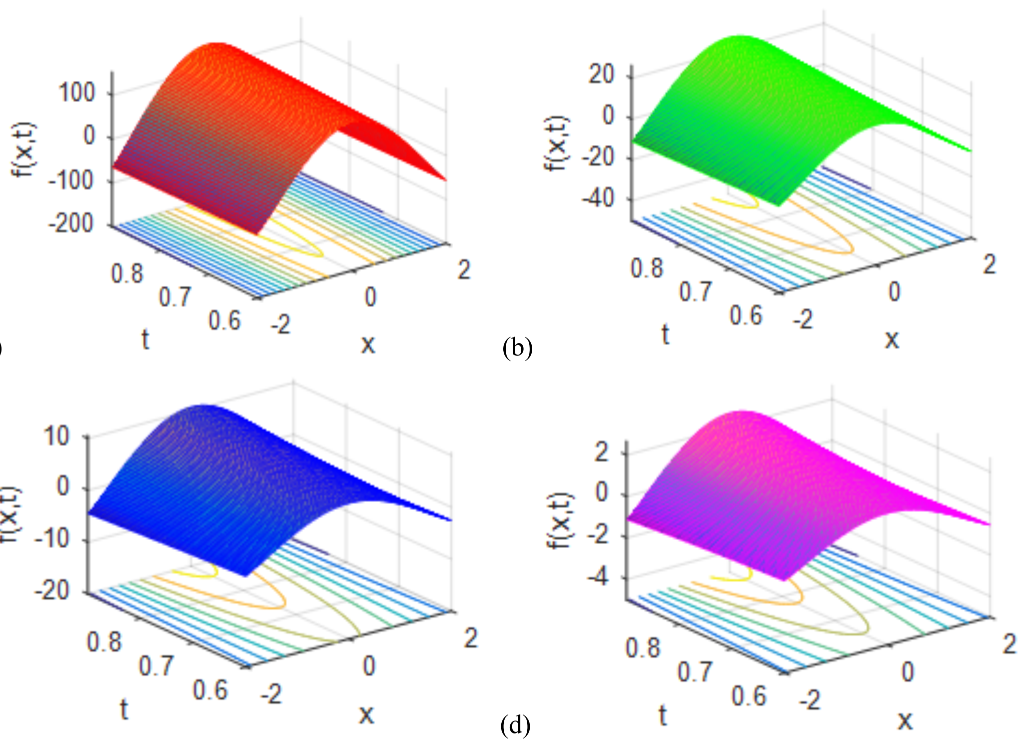
$$
\begin{aligned}
f(x,t) &= \sum_{k=0}^{\infty} F_k(x) t^{k\alpha} \\
&= \cos x \left( 1 + \sum_{k=1}^{\infty} (-1)^k \frac{1}{\Gamma(k\alpha+1)} t^{k\alpha} \right) \\
&= \left( 1 - \frac{1}{\Gamma(\alpha+1)} t^{\alpha} + \frac{1}{\Gamma(2\alpha+1)} t^{2\alpha} + .................. \right) \cos x,
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
g(x,t) &= \sum_{k=0}^{\infty} G_k(x) t^{k.\alpha} \\
&= \left( 1 + \sum_{k=1}^{\infty} (-1)^k \frac{1}{\Gamma(k.\alpha+1)} t^{k.\alpha} \right) \cos x \\
&= \left( 1 - \frac{1}{\Gamma(\alpha+1)} t^{\alpha} + \frac{1}{\Gamma(2\alpha+1)} t^{2\alpha} + ---- \right) \cos x.
\end{aligned}
\tag{29}
$$

For the fractional order $\alpha = 1$, the closed-form solutions for the coupled nonlinear homogenous Burger's equations are

$$
f(x,t) = \left( \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \right) \cos x \approx e^{-t} \cos x, \tag{30}
$$

$$
g(x,t) = \left( \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \right) \cos x \approx e^{-t} \cos x. \tag{31}
$$




(a) (b) (c) (d)

**Fig. 5** Surface plot of the solution of Burger's equation using RDTM for the fractional orders of $\alpha$ (a) $0.1$ (b) $0.3$ (c) $0.5$ (d) $0.9$.

**Problem-4.3:** ***Schrodinger equations (nonlinear coupled Time-fractional)***

Considering the time-fractional Schrodinger equations:

$$i\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(f(x,t)\right)+\frac{1}{2}\frac{\partial^{2}}{\partial x^{2}}\left(f(x,t)\right)+\left|f(x,t)\right|^{2}f(x,t)=0, t>0, 0<\alpha\leq 1 \tag{32}$$

The corresponding initial condition is

$$f(x,0)=\exp(ix). \tag{33}$$

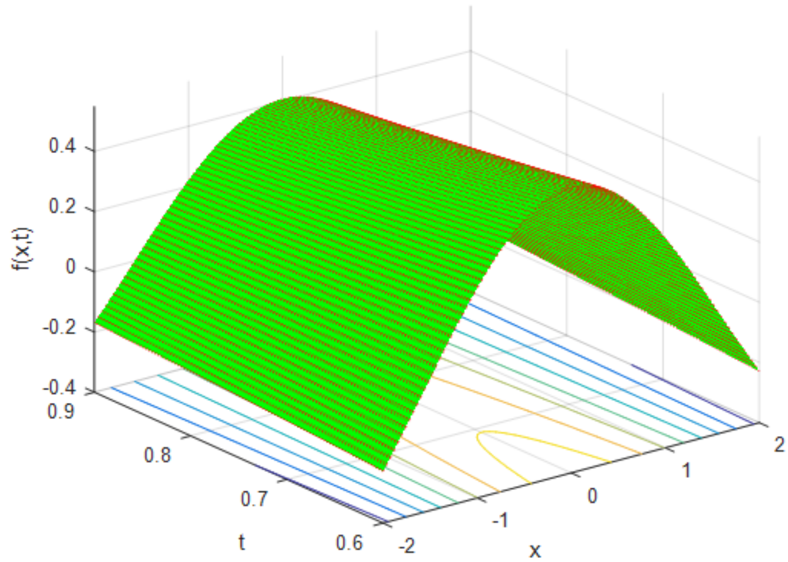



**Fig.6:** Surface plot for $f(x,t)=e^{-t}\cos x$ when $\alpha=1$.

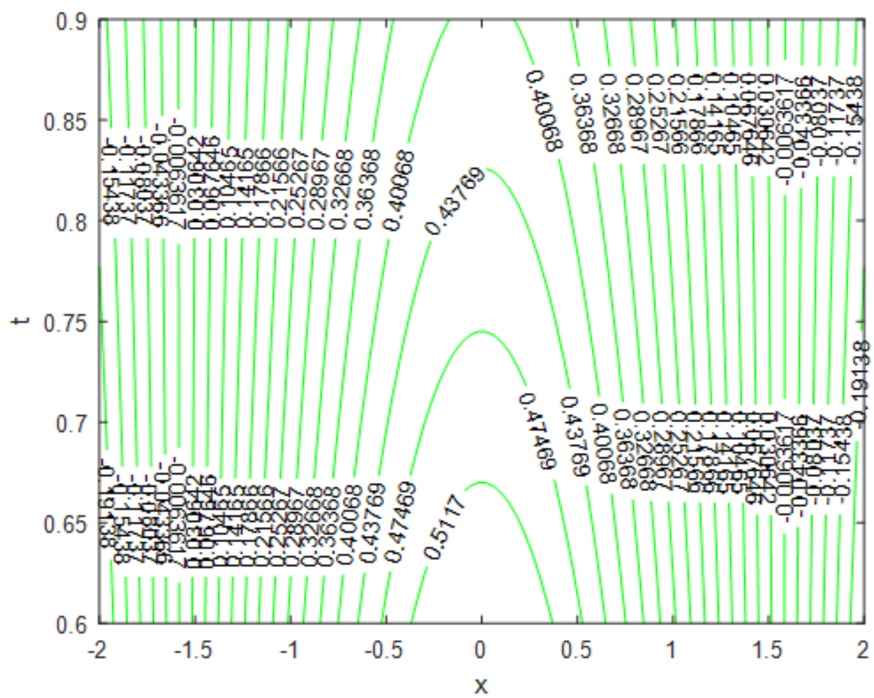



**Fig.7:** Contour for the exact solution of Burger's equation for $\alpha=1$.

Introducing RDTM defined earlier the equation (32) can be expressed as

$$\frac{\Gamma(k.\alpha+\alpha+1)}{\Gamma(k.\alpha+1)}F_{k+1}(x)=i\left(\frac{1}{2}f_{xx}+|f|^2 f\right) \tag{34}$$

Let us consider the nonlinear operator $N(f,f_x)=|f|^2 f$, used in Adomian polynomials given by

$$N(f,f_x)=\sum_{n=0}^{\infty}A_n \tag{35}$$
.

Where $A_n^{'}s$ called Adomain polynomials and the expression is

$$\begin{aligned}
&A_0=|f_0|^2 f_0,\\
&A_1=2|f_0|^2 f_1+f_0^{\,2}\overline{f_1},\\
&A_2=2|f_0|^2 f_2+2|f_1|^2 f_0+|f_1|^2\overline{f_0}+|f_0|^2\overline{f_2}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{36}$$

Using the Adomian polynomials the iterative expression (34) is represented as

$$\frac{\Gamma(k.\alpha+\alpha+1)}{\Gamma(k.\alpha+1)}F_{k+1}(x)=i\left(\frac{1}{2}f_{kxx}+A_k\right)$$
$$\text{With } f_0(x)\approx F_0(x)=\exp(ix) \tag{37}$$

For various values of $k=0,1,2....$, we get

$$F_1(x)=\frac{1}{2\Gamma(\alpha+1)}i\exp(ix),$$
$$F_2(x)=\frac{1}{4\Gamma(2\alpha+1)}i^2\exp(ix),$$
$$F_3(x)=\frac{5}{8\Gamma(3\alpha+1)}i^3\exp(ix)+\frac{\Gamma(2\alpha+1)}{4\Gamma(3\alpha+1)\Gamma(\alpha+1)^2}i\exp(ix),\text{etc.}$$

Hence, the approximate result is

$$\begin{aligned}
f(x,t)&=\sum_{k=0}^{\infty}F_k(x)t^{k.\alpha}\\
&=\left(\begin{aligned}&1+\frac{1}{2}\frac{1}{\Gamma(\alpha+1)}it^{\alpha}+\frac{1}{4}\frac{1}{\Gamma(2\alpha+1)}i^2t^{2\alpha}+\\&\left(\frac{5}{8}\frac{1}{\Gamma(3\alpha+1)}i^3+\frac{1}{4}\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)\Gamma(\alpha+1)^2}i\right)t^{3\alpha}+\ldots\ldots\ldots\end{aligned}\right)\exp(ix).
\end{aligned} \tag{38}$$

In particular for the fractional order $\alpha=1$, the closed-form solutions for the nonlinear Schrodinger equation is

$$f(x,t)=\exp(ix)\exp\left(\frac{it}{2}\right) \tag{39}$$
.

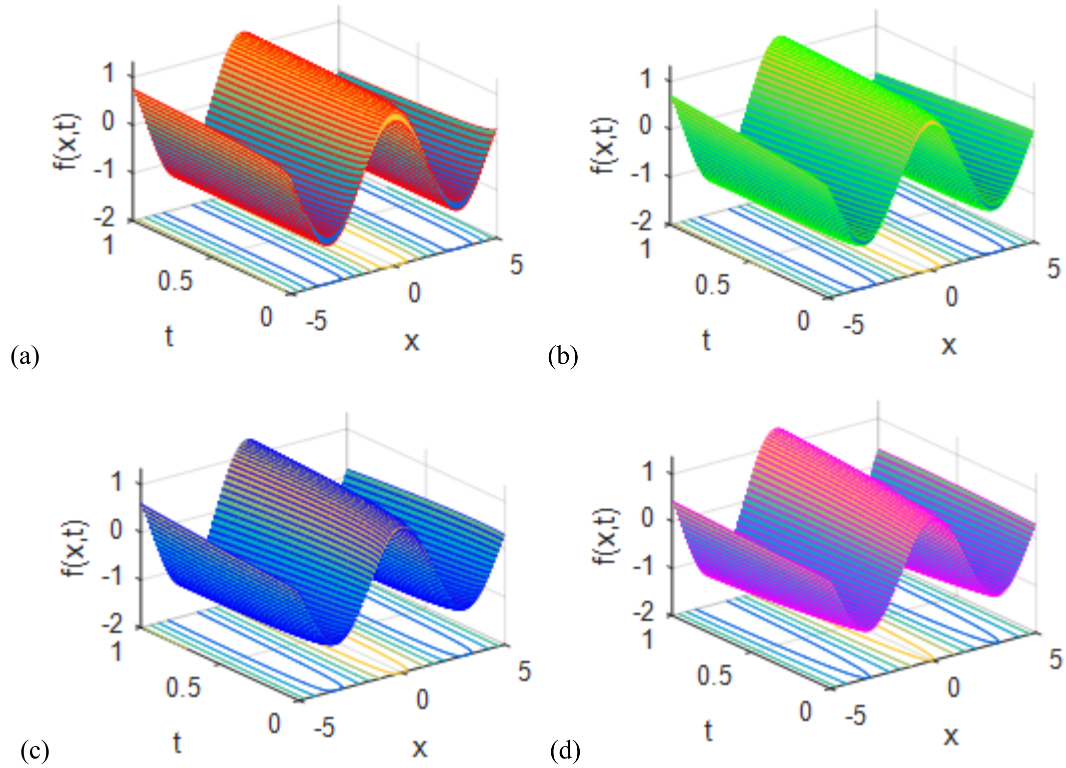



**Fig.8:** Surface plot of the solution of nonlinear Schrodinger equation using RDTM for the fractional orders of $\alpha$ (a) $0.1$ (b) $0.3$ (c) $0.5$ (d) $0.9$.

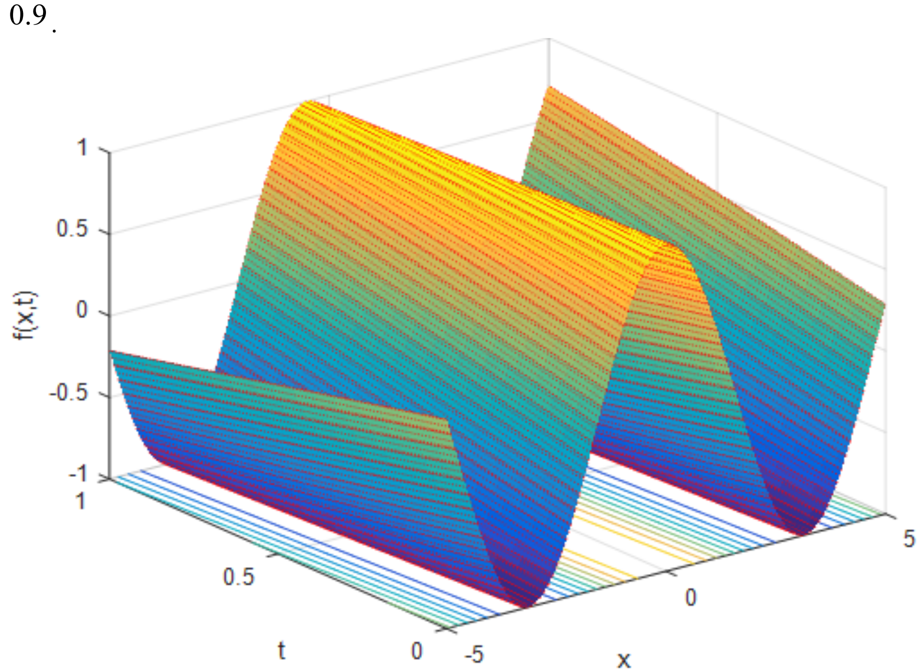



**Fig.9:** Surface plot for the exact solution $f(x,t)=\exp(ix)\exp(it/2)$ when $\alpha=1$.

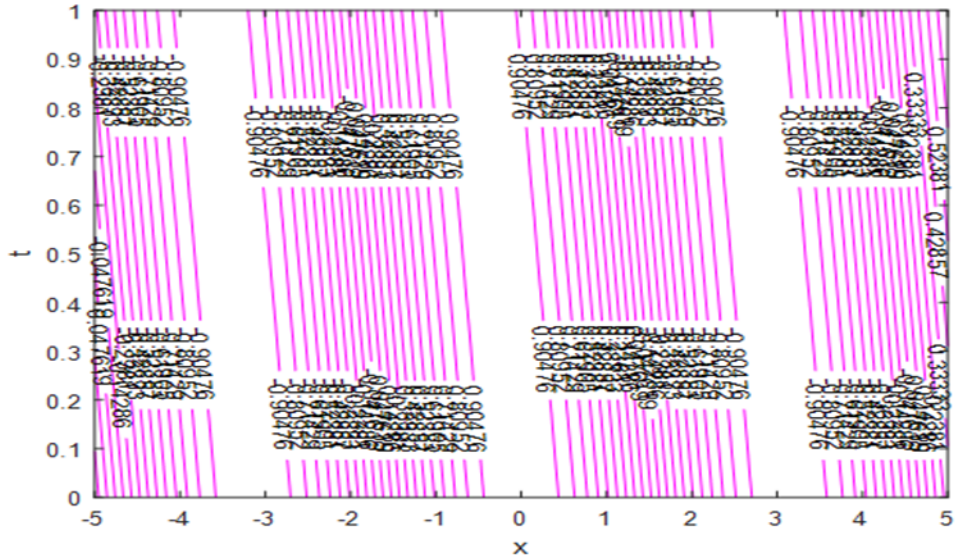



**Fig.10:** Contour for the Schrodinger equation with $\alpha = 1$.

## 5 Description and conclusion

Attempt has been made to analyze the behavior of approximate analytical solution as well as the exact result of nonlinear equations with time-fractional derivative. It is quite complicated to handle this equation using analytical methods in comparison to linear differential equations. Earlier various approximate methods like the perturbation method, asymptotic methods were employed for the solution of these types of weakly nonlinear problems. There are various perturbation parameters are used depending upon the application to various fields of engineering and the ranges of these parameter must be very small. Now a day because of several engineering applications the models have been developed comprised of nonlinear time-fractional differential equations. In the present analysis, we have considered homogeneous nonlinear advection equation, coupled nonlinear Burgers' equations and coupled nonlinear Schrödinger equations. Solutions of these equations are obtained analytically using Reduced Differential Transformation Method accompanied with Adomian Decomposition Method for various fractional orders. The basic steps followed by definition of the RDTM is described elaborately in articles-2 and 3 respectively. Fig.2 exhibits the surface plot for the solution obtained in Example-1, for the advection equations for the fractional orders $\alpha = 0.1, 0.3, 0.5$ and $0.9$ respectively within the domain $-2 \leq x \leq 2$ and $0 \leq t \leq 1$. Fig.3 exhibits the closed form solution in particular case of fractional order $\alpha = 1$ and Fig.4 presents the contour plot of the same solution. However, Fig.5 portrays the surface plots of the coupled nonlinear Burger's equations for the fractional orders $\alpha = 0.1, 0.3, 0.5$ and $0.9$. The mesh grid for both $x$ and $t$ are similar to Fig.2. Moreover Figs. 6 and 7 display the surface and the contour scheme of the accurate solution in the particular fraction order $\alpha = 1$ respectively. Finally, Fig. 8 portrays the surface design of the result of Schrodinger equation for a variety of values of time-fractional orders $\alpha = 0.1, 0.3, 0.5$ and $0.9$ within the ranges $-5 \leq x \leq 5$ and $0 \leq t \leq 1$. The surface and the contour plots of the exact analytical solution in particular case of $\alpha = 1$ for the Schrodinger equation is exhibited in Figs. 9 and 10 respectively. Further, we conclude that these nonlinear equations can be solved by using Laplace transformation technique followed by several fractional methods like fractional power series method.

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