Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.12785/pfda/010204

On Some Hermite-Hadamard Type Inequalities for Convex Functions via Hadamard Fractional Integrals

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Received: 20 Nov. 2014, Revised: 20 Dec. 2014, Accepted: 23 Dec. 2014
Published online: 1 Apr. 2015

Abstract: We explore a new Hadamard type inequality for Hadamard fractional integrals and derive a new fractional integral identity. We use the new established fractional integral identity we obtain new Hadamard fractional version Hermite-Hadamard inequalities for once and twice convex functions. Then, we derive new inequality results by applying these identities.

Keywords: Hermite-Hadamard type inequality, Convex function, Hadamard fractional integrals

1 Introduction

The well-known Hermite-Hadamard integral inequality was established by Hermite at the end of 19th century (see [1]). There are many recent contributions to improve this inequality, please refer to [2,3,4,5,6] and references therein. It is worth noting that there are some interesting results about Hermite-Hadamard inequalities via fractional integrals according to the corresponding integral equalities involving a given class of differential convex functions. For more details on such new, important and interesting mathematical branch, one can refer to papers [7,8,9,10,11,12,13,14].

For a well defined convex function g on $[c,d]$, the left (right) Hadamard fractional integral of order $v>0$ is defined by (see [15])

$$({}_HJ^{v}_{c+}g)(z)=\frac{1}{\Gamma(v)}\int_c^z\left(\ln\frac{z}{s}\right)^{v-1}g(s)\frac{ds}{s},$$

and

$$({}_HJ^{v}_{d-}g)(z)=\frac{1}{\Gamma(v)}\int_z^d\left(\ln\frac{s}{z}\right)^{v-1}g(s)\frac{ds}{s}.$$

Throughout this paper, we denote

$$\begin{aligned}&I_f(z,\mu,v,c,d)\\&=(1-\mu)g(z)[(d-z)^v+(z-c)^v]+\mu\left[g(c)(z-c)^v+g(d)(d-z)^v\right]-\Gamma(v+1)\left[{}_HJ^{v}_{(e^z)^+}(g\circ\ln)(e^d)+{}_HJ^{v}_{(e^z)^-}(g\circ\ln)(e^c)\right],\end{aligned}$$

where $\mu\in[0,1]$ and $\mu>0$.

In [10, Theorem 2.1], the authors obtained an interesting Hadamard type inequality for Hadamard fractional integrals via nondecreasing and convex function. Here, we establish a new Hadamard type inequality for Hadamard fractional integrals via convex function (see Theorem 2.1):

$$f\left(\frac{c+d}{2}\right)\le\frac{\Gamma(v+1)}{2(d-c)^v}\left[{}_HJ^{v}_{(e^c)^+}(g\circ\ln)(e^d)+{}_HJ^{v}_{(e^d)^-}(g\circ\ln)(e^c)\right]\le\frac{g(c)+g(d)}{2}.$$

In [10, Lemma 3.1], the authors obtained an Hadamard fractional integrals identity involving once differentiable mapping. Here, we give a new Hadamard fractional integrals identity involving once differentiable mapping as follows.

$$I_g(z,\mu,v;c,d)=(z-c)^{v+1}\int_0^1(s^v-\mu)g'(sz+(1-s)c)ds-(d-z)^{v+1}\int_0^1(s^v-\mu)g'(sz+(1-s)d)ds.$$

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Moreover, we also establish new Hadamard fractional integrals identity involving twice differentiable mapping (see Theorem 3.6).

The next aim of this paper is to establish some new Hermite-Hadamard type inequalities for once and twice convex functions via Hadamard fractional integrals (see Theorems 3.3,3.8) which improves [10, Theorems 3.3]. These results have some relationships with [10], however, we point that our current results are different from the previous results in [10] and generalize [10] in some sense.

2 A new Hadamard type inequalities

Theorem 2.1. Assume that $v > 0$ and the function $g : [c,d] \to R$ is convex. Then we have

$$f\left(\frac{c+d}{2}\right) \leq \frac{\Gamma(v+1)}{2(d-c)^v}\left[{}_HJ^v_{(e^c)+}(g\circ \ln)(e^d) + {}_HJ^v_{(e^d)-}(g\circ \ln)(e^c)\right] \leq \frac{g(c)+g(d)}{2}. \quad (1)$$

Proof. It follows from the convexity of the function f that

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}.$$

For $0 \leq s \leq 1$, let $y = sc+(1-s)d$, $z = sd+(1-s)c$ and multiply by s^{v-1} in each side, then

$$2t^{v-1}f\left(\frac{c+d}{2}\right) \leq t^{v-1}[f(sc+(1-s)d)+g(sd+(1-s)c)]. \quad (2)$$

Integrating (2) over $[0,1]$, we obtain:

$$\begin{aligned}\frac{2}{v}g\left(\frac{c+d}{2}\right) &\leq \int_0^1 s^{v-1}g(sc+(1-s)d)ds + \int_0^1 s^{v-1}g(sd+(1-s)c)ds \\ &= \frac{1}{d-c}\int_{e^c}^{e^d}\left(\frac{d-\ln t}{d-c}\right)^{v-1}g(\ln t)\frac{dt}{t} + \frac{1}{d-c}\int_{e^c}^{e^d}\left(\frac{\ln t-c}{d-c}\right)^{v-1}g(\ln t)\frac{dt}{t} \\ &= \frac{\Gamma(v)}{(d-c)^v}[{}_HJ^v_{(e^c)+}(g\circ \ln)(e^d) + {}_HJ^v_{(e^d)-}(g\circ \ln)(e^c)],\end{aligned}$$

which implies that

$$\frac{2}{v}g\left(\frac{c+d}{2}\right) \leq \frac{\Gamma(v)}{(d-c)^v}[{}_HJ^v_{(e^c)+}(g\circ \ln)(e^d) + {}_HJ^v_{(e^d)-}(g\circ \ln)(e^c)].$$

Now using the convexity of g again, for $s \in [0,1]$, we have

$$\begin{aligned}g(sc+(1-s)d) &\leq sg(c)+(1-s)g(d),\\ g(sd+(1-s)c) &\leq sg(d)+(1-s)g(c),\end{aligned}$$

which yields:

$$s^{v-1}[g(sc+(1-s)d)+g(sd+(1-s)c)] \leq s^{v-1}[g(c)+g(d)]. \quad (3)$$

Integrating (3) over $[0,1]$, we have

$$\int_0^1 s^{v-1}f(sv+(1-s)d))ds + \int_0^1 s^{v-1}f(sd+(1-s)c))ds \leq \int_0^1 [g(c)+g(d)]s^{v-1}ds.$$

So we can get the following result

$$\frac{\Gamma(v)}{(d-c)^v}\left[{}_HJ^v_{(e^c)+}(g\circ \ln)(e^d) + {}_HJ^v_{(e^d)-}(g\circ \ln)(e^c)\right] \leq \frac{[g(c)+g(d)]}{v}.$$

Then the proof is well completed.

3 New Hadamard fractional Hermite-Hadamard type inequalities

We begin to establish a new fractional integral identity which will be used in what follows.

Lemma 3.1. Assume that $g:[c,d]\to R$ and $g'\in L[c,d]$, g is a convex function, then we have

$$I_g(z,\mu,v;c,d) = (z-c)^{v+1}\int_0^1 (s^v-\mu)g'(sz+(1-s)c)ds-(d-z)^{v+1}\int_0^1 (s^v-\mu)g'(sz+(1-s)d)ds, \tag{4}$$

for all $z\in[c,d]$, $v>0$ and $\mu\in[0,1]$.

Proof. For $z\neq c$ we have

$$\begin{aligned}
&(z-c)\int_0^1 (s^v-\mu)g'(sz+(1-s)c)ds\\
&=(s^v-\mu)g(sz+(1-s)c)\Big|_0^1 - v\int_0^1 s^{v-1}g(sz+(1-s)c)ds\\
&=(1-\mu)g(z)+\mu g(c)-\frac{v}{(z-c)^v}\int_{e^c}^{e^z}(\ln t-c)^{v-1}g(\ln t)\frac{dt}{t}\\
&=(1-\mu)g(z)+\mu g(c)-\frac{\Gamma(v+1)}{(z-c)^v}\,{}_HJ^v_{(e^z)^-}(g\circ\ln)(e^c).
\end{aligned} \tag{5}$$

For $z\neq d$, we get

$$\begin{aligned}
&-(d-z)\int_0^1 (s^v-\mu)g'(sz+(1-s)d)ds\\
&=(s^v-\mu)g(sz+(1-s)d)\Big|_0^1 - v\int_0^1 s^{v-1}g(sz+(1-s)d)ds\\
&=(1-\mu)g(z)+\mu g(d)-\frac{v}{(d-z)^v}\int_{e^z}^{e^d}(d-\ln t)^{v-1}g(\ln t)\frac{dt}{t}\\
&=(1-\mu)g(z)+\mu g(d)-\frac{\Gamma(v+1)}{(d-z)^v}\,{}_HJ^v_{(e^z)^+}(g\circ\ln)(e^d).
\end{aligned} \tag{6}$$

Then multiplying both sides of (5) and (6) by $(z-c)^v$ and $(d-z)^v$, respectively, we obtain the desired result immediately.

Concerning (4), if we set $z=d$ and $z=c$, one has

$$I_g(d,\mu,v;c,d)=(d-c)^{v+1}\int_0^1 (s^v-\mu)g'(sd+(1-s)c)ds,$$

and

$$I_g(c,\mu,v;c,d)=-(d-c)^{v+1}\int_0^1 (s^v-\mu)g'(sc+(1-s)d)ds.$$

Using the right-sided inequality result of (4), we have the following conclusion.

Remark 3.2. Assume that $v>0$ and the function $g:[c,d]\to R$ is convex. Then

$$\begin{aligned}
&\frac{1}{2}[I_g(d,\mu,v;c,d)+I_g(c,\mu,v;c,d)]\\
&=\frac{g(c)+g(d)}{2}-\frac{\Gamma(v+1)}{2(d-c)^v}[\,{}_HJ^v_{(e^d)^-}(g\circ\ln)(e^c)+\,{}_HJ^v_{(e^c)^+}(g\circ\ln)(e^d)]\\
&=\frac{(d-c)^{v+1}}{2}\int_0^1 (s^v-\mu)[g'(sd+(1-s)c)-g'(sc+(1-s)d)]ds.
\end{aligned} \tag{7}$$

Theorem 3.3. Assume that $g:[c,d]\to R$ and $g'\in L[c,d]$. Suppose that $|g'|^p$ is a convex function for some fixed $p\geq 1$, then

$$\begin{aligned}
|I_g(z,\mu,v;c,d)| \leq A_1^{1-\frac{1}{p}}(v,\mu)\Bigg\{&(z-c)^{v+1}\Big[A_2(v,\mu)|g'(z)|^p+A_3(v,\mu)|g'(c)|^p\Big]^{\frac{1}{p}}\\
&+(d-z)^{v+1}\Big[A_2(v,\mu)|g'(z)|^p+A_3(v,\mu)|g'(d)|^p\Big]^{\frac{1}{p}}\Bigg\},
\end{aligned} \tag{8}$$

for all $z \in [c,d]$, $\mu \in [0,1]$ and $v > 0$, where

$$A_1(v,\mu) = \frac{2v\mu^{1+\frac{1}{v}}+1}{v+1} - \mu,$$

$$A_2(v,\mu) = \frac{v}{v+1}\mu^{1+\frac{2}{v}} + \frac{1}{v+2} - \frac{\mu}{2},$$

$$A_3(v,\mu) = \frac{2v}{v+1}\mu^{1+\frac{1}{v}} - \frac{2}{v+2}\mu^{1+\frac{2}{v}} + \frac{1}{(v+2)(v+1)} - \frac{\mu}{2}.$$

Proof. Using Lemma 3.1, we obtain

$$|I_g(z,\mu,v;c,d)| \le I_{g_1}(z,\mu,v;c,d) + I_{g_1}(z,\mu,v;c,d), \tag{9}$$

where

$$I_{g_1}(z,\mu,v;c,d) := (z-c)^{v+1}\int_0^1 |s^v-\mu||g'(sz+(1-s)c)|ds,$$

$$I_{g_2}(z,\mu,v;c,d) := (d-z)^{v+1}\int_0^1 |s^v-\mu||g'(sz+(1-s)d)|ds.$$

Following Hölder inequality, we derive

$$\begin{aligned}
I_{g_1}(z,\mu,v;c,d) \le {} & (z-c)^{v+1}\left(\int_0^1 |s^v-\mu|dt\right)^{1-\frac{1}{p}} \\
& \left\{\int_0^{\mu^{\frac{1}{v}}} (\mu - s^v)\left[s|g'(z)|^p + (1-s)|g'(c)|^p\right]ds \right. \\
& \left. + \int_{\mu^{\frac{1}{v}}}^1 (s^v-\mu)\left[s|g'(z)|^p + (1-s)|g'(c)|^p ds\right]\right\}^{\frac{1}{p}} \\
= {} & (z-c)^{v+1}\left(\frac{2v\mu^{1+\frac{1}{v}}+1}{v+1} - \mu\right)^{1-\frac{1}{p}}\left[\left(\frac{v}{v+1}\mu^{1+\frac{2}{v}} + \frac{1}{v+2} - \frac{\mu}{2}\right)|g'(z)|^p \right. \\
& \left. + \left(\frac{2v}{v+1}\mu^{1+\frac{1}{v}} - \frac{2}{v+2}\mu^{1+\frac{2}{v}} + \frac{1}{(v+2)(v+1)} - \frac{\mu}{2}\right)|g'(c)|^p\right]^{\frac{1}{p}}.
\end{aligned} \tag{10}$$

Similarly, we obtain

$$\begin{aligned}
I_{g_2}(z,\mu,v;c,d) \le {} & (d-z)^{v+1}\left(\frac{2v\mu^{1+\frac{1}{v}}+1}{v+1} - \mu\right)^{1-\frac{1}{p}}\left[\left(\frac{v}{v+1}\mu^{1+\frac{2}{v}} + \frac{1}{v+2} - \frac{\mu}{2}\right)|g'(z)|^p \right. \\
& \left. + \left(\frac{2v}{v+1}\mu^{1+\frac{1}{v}} - \frac{2}{v+2}\mu^{1+\frac{2}{v}} + \frac{1}{(v+2)(v+1)} - \frac{\mu}{2}\right)|g'(d)|^p\right]^{\frac{1}{p}}.
\end{aligned} \tag{11}$$

From above, one can submitting (10) and (11) to (9) to derive the result.

Remark 3.4. In Theorem 3.3, we set $z = \frac{c+d}{2}$, $\mu = 0$. It follows the inequality (8) that

$$\begin{aligned}
& \left|I_g\left(\frac{c+d}{2}, 0, v; c, d\right)\right| \\
\le & \left(\frac{1}{v+1}\right)^{1-\frac{1}{p}}\left[\frac{1}{v+2}\left(\frac{d-c}{2}\right)^{v+1}\right]^{\frac{1}{p}}\left\{\left[\left|g'\left(\frac{c+d}{2}\right)\right|^p + \frac{1}{v+1}|g'(c)|^p\right]^{\frac{1}{p}} + \left[\left|g'\left(\frac{c+d}{2}\right)\right|^p + \frac{1}{v+1}|g'(d)|^p\right]^{\frac{1}{p}}\right\}.
\end{aligned}$$

Remark 3.5. Let $p = 1$. Then for all $z \in [c,d]$, $\mu \in [0,1]$ and $v > 0$,

$$\left|\frac{1}{2}[I_g(c,\mu,v;c,d) + I_g(d,\mu,v;c,d)]\right| \le \frac{1}{2}(d-c)^{v+1}[|g'(c)| + |g'(d)|][A_2(v,\mu) + A_3(v,\mu)]$$

In what follows, we give the corresponding results for twice differential functions.

Theorem 3.6. Assume that $g'' \in L[c,d]$. Then

$$I_g(z,\mu,v;c,d) = \left(\frac{1}{v+1}-\mu\right)g'(z)[(z-c)^{v+1}-(d-z)^{v+1}]$$
$$-\left[(z-c)^{v+2}\int_0^1 \frac{s^{v+1}-\mu(v+1)s}{v+1}g''(sz+(1-s)c)ds\right.$$
$$\left.+(d-z)^{v+2}\int_0^1 \frac{s^{v+1}-\mu(v+1)s}{v+1}g''(sz+(1-s)d)ds\right]. \tag{12}$$

Proof. Note that

$$\int_0^1 \frac{s^{v+1}-\mu(v+1)s}{v+1}dg'(sz+(1-s)c)$$
$$=(z-c)\int_0^1 \frac{s^{v+1}-\mu(v+1)s}{v+1}g''(sz+(1-s)c)ds$$
$$=\frac{s^{v+1}-\mu(v+1)s}{v+1}g'(sz+(1-s)c)\Big|_0^1-\int_0^1 g'(sz+(1-s)c)(s^v-\mu)ds$$
$$=\left(\frac{1}{v+1}-\mu\right)g'(z)-\int_0^1 g'(sz+(1-s)c)(s^v-\mu)ds, \tag{13}$$

and

$$\int_0^1 \frac{s^{v+1}-\mu(v+1)s}{v+1}dg'(sz+(1-s)d)$$
$$=-(d-z)\int_0^1 \frac{s^{v+1}-\mu(v+1)s}{v+1}g''(sz+(1-s)d)ds$$
$$=\frac{s^{v+1}-\mu(v+1)s}{v+1}g'(sz+(1-s)d)\Big|_0^1-\int_0^1 g'(sz+(1-s)d)(s^v-\mu)ds$$
$$=\left(\frac{1}{v+1}-\mu\right)g'(z)-\int_0^1 g'(sz+(1-s)d)(s^v-\mu)ds, \tag{14}$$

Multiplying both sides of (13) and (14) by $(z-c)^{v+1}$ and $-(d-z)^{v+1}$, respectively, we have

$$(z-c)^{v+1}\int_0^1 g'(sz+(1-s)c)(s^v-\mu)ds]$$
$$=(z-c)^{v+1}\left(\frac{1}{v+1}-\mu\right)g'(z)-(z-c)^{v+2}\int_0^1 \frac{s^{v+1}-\mu(v+1)s}{v+1}g''(sz+(1-s)c)ds. \tag{15}$$

and

$$-(d-z)^{v+1}\int_0^1 g'(sz+(1-s)d)(s^v-\mu)ds]$$
$$=-(d-z)^{v+1}\left(\frac{1}{v+1}-\mu\right)g'(z)-(d-z)^{v+2}\int_0^1 \frac{s^{v+1}-\mu(v+1)s}{v+1}g''(sz+(1-s)d)ds. \tag{16}$$

By adding the results of (15) and (16), we complete the proof.

Further, we have

Remark 3.7.

$$\frac{1}{2}[I_g(d,\mu,v;c,d)+I_g(c,\mu,v;c,d)]$$
$$=\frac{(d-c)^{v+1}}{2}\left[\left(\frac{1}{v+1}-\mu\right)[g'(d)-g'(c)]\right.$$
$$\left.-\frac{(d-c))^{v+2}}{v+1}\int_0^1(s^{v+1}-\lambda(v+1)a)[g''(sz+(1-s)c)+g''(sc+(1-s)d)]ds\right].$$

The conditions are just like Theorem 3.6.

Theorem 3.8. Assume that $g'' \in L[c,d]$ and the function $|g''|^p$ is convex for some fixed $p \geq 1$ on $[c,d]$, then

$$|I_g(z,\mu,\nu;c,d)| \leq \left|\left(\frac{1}{\nu+1}-\mu\right)g'(z)[(z-c)^{\nu+1}-(d-z)^{\nu+1}]\right| + A_4^{1-\frac{1}{p}}(\nu,\mu)\left\{\frac{(z-c)^{\nu+2}}{\nu+1}\left[A_5(\nu,\mu)|g''(z)|^p + A_6(\nu,\mu)|g''(c)|^p\right]^{\frac{1}{p}} + \frac{(d-z)^{\nu+2}}{\nu+1}\left[A_5(\nu,\mu)|g''(z)|^p + A_6(\nu,\mu)|g''(d)|^p\right]^{\frac{1}{p}}\right\}, \quad (17)$$

where

$$A_4(\nu,\mu) = \frac{\nu}{\nu+2}[\mu(\nu+1)]^{1+\frac{2}{\nu}} - \frac{\mu(\nu+1)}{2} + \frac{1}{\nu+2},$$

$$A_5(\nu,\mu) = \frac{2\nu}{3(\alpha+3)}[\mu(\nu+1)]^{1+\frac{3}{\nu}} + \frac{1}{\nu+3} - \frac{\mu(\nu+1)}{3},$$

$$A_6(\nu,\mu) = \frac{\nu}{\nu+2}[\mu(1+\nu)]^{1+\frac{2}{\nu}} - \frac{2\nu}{3(\nu+3)}[\mu(1+\nu)]^{1+\frac{3}{\nu}} + \frac{1}{(\nu+2)(\nu+3)} - \frac{\mu(\nu+1)}{6}.$$

Proof. It follows from (12) that:

$$|I_g(z,\mu,\nu;c,d)| \leq \left|\left(\frac{1}{\nu+1}-\mu\right)g'(z)[(z-c)^{\nu+1}-(d-z)^{\nu+1}]\right| + \frac{(z-c)^{\nu+2}}{\nu+1}|J_{g_1}(z,\mu,\nu;c,d)| + \frac{(d-z)^{\nu+2}}{\nu+1}|J_{g_2}(z,\mu,\nu;c,d)|, \quad (18)$$

where

$$J_{g_1}(z,\mu,\nu;c,d)| = \int_0^1 [s^{\nu+1} - \mu(\nu+1)s]g''(sz+(1-s)c)ds,$$

$$J_{g_2}(z,\mu,\nu;c,d)| = \int_0^1 [s^{\nu+1} - \mu(\nu+1)s]g''(sz+(1-s)d)ds.$$

By using the Hölder inequality we can get

$$\begin{aligned}
&|J_{f_1}(z,\mu,\nu;c,d)| \\
&\leq \int_0^1 |s^{\nu+1} - \mu(\nu+1)s||g''(sz+(1-s)c)|ds \\
&\leq \left(\int_0^1 |s^{\nu+1} - \mu(\nu+1)s|ds\right)^{1-\frac{1}{q}} \\
&\times\left\{\int_0^{[\mu(\nu+1)]^{\frac{1}{\nu}}} [\mu(\nu+1)s - s^{\nu+1}](s|g''(z)|^p + (1-s)|g''(c)|^p)ds \right. \\
&\left. + \int_{[\mu(\nu+1)]^{\frac{1}{\nu}}}^1 [s^{\nu+1} - \mu(\nu+1)s](s|g''(z)|^p + (1-s)|g''(c)|^p)ds\right\}^{\frac{1}{p}} \\
&\leq \left(\int_0^1 |s^{\nu+1} - \mu(\nu+1)s|ds\right)^{1-\frac{1}{p}} \times \left\{|g''(z)|^p\left[\int_0^{[\mu(\nu+1)]^{\frac{1}{\nu}}}[\mu(\nu+1)s - s^{\nu+2}]ds + \int_{[\mu(\nu+1)]^{\frac{1}{\nu}}}^1 [s^{\nu+2} - \mu(\nu+1)s^2]ds\right]\right. \\
&\left. + |g''(c)|^p\left[\int_0^{[\mu(\nu+1)]^{\frac{1}{\nu}}}[\mu(\nu+1)s - \mu(\nu+1)s^2 - s^{\nu+1} + s^{\nu+2}]ds + \int_{[\mu(\nu+1)]^{\frac{1}{\nu}}}^1 [s^{\nu+1} - s^{\nu+2} - \mu(\nu+1)s + \mu(\nu+1)s^2]ds\right]\right\}^{\frac{1}{p}} \\
&= A_4^{1-\frac{1}{p}}(\nu,\mu)\left[A_5(\nu,\mu)|g''(z)|^p + A_6(\nu,\mu)|g''(c)|^p\right]^{\frac{1}{p}}. \quad (19)
\end{aligned}$$

Using the same method, we get

$$|J_{f_2}(z,\mu,\nu;c,d)| \leq A_4^{1-\frac{1}{p}}(\nu,\mu)\left[A_5(\nu,\mu)|g''(z)|^p + A_6(\nu,\mu)|g''(d)|^p\right]^{\frac{1}{p}}. \quad (20)$$

On the other hand,

$$\int_0^1 |s^{v+1}-\mu(v+1)s|ds = \int_0^{[\mu(v+1)]^{\frac{1}{v}}} [\mu(v+1)s-s^{v+1}]ds + \int_{[\mu(v+1)]^{\frac{1}{v}}}^1 [s^{v+1}-\mu(v+1)s]ds$$
$$= \frac{v}{v+2}[\mu(v+1)]^{1+\frac{2}{v}} - \frac{\mu(v+1)}{2} + \frac{1}{v+2}. \qquad (21)$$

By submitting (19), (20) and (21) into (18), the proof is completed.

Remark 3.9. If $q=1$, then

$$\left|\frac{1}{2}[I_g(c,\mu,v;c,d)+I_g(d,\mu,v;c,d)]\right| \le \frac{(d-c)^{v+1}}{2}\left|\left(\frac{1}{v+1}-\mu\right)[g'(d)-g'(c)]\right| + \frac{(d-c)^{v+2}}{2(v+1)}[|g''(c)|+|g''(d)|][A_5(v,\mu)+A_6(v,\mu)]$$

for all $z\in[c,d]$, $\mu\in[0,1]$ and $v>0$.

Remark 3.10. In Theorem 3.8, we set $z=\frac{c+d}{2}$, $\mu=0$. It follows the inequality (17) that

$$\left|I_g\left(\frac{c+d}{2},0,v;c,d\right)\right|$$
$$\le \frac{1}{v+1}\left(\frac{1}{v+2}\right)^{1-\frac{1}{p}}\left(\frac{1}{v+3}\right)^{\frac{1}{p}}\left(\frac{d-c}{2}\right)^{v+2}\left\{\left[\left|g''\left(\frac{c+d}{2}\right)\right|^p+\frac{1}{v+2}|g''(c)|^p\right]^{\frac{1}{p}}+\left[\left|g''\left(\frac{c+d}{2}\right)\right|^p+\frac{1}{v+2}|g''(d)|^p\right]^{\frac{1}{p}}\right\}.$$

Remark 3.11. Assume that $|g''(z)|\le M$, $z\in[c,d]$ and $M>0$. Then

$$\left|\frac{1}{2}[I_g(d,\mu,v;c,d)+I_g(c,\mu,v;c,d)]\right| \le \frac{vM(d-c)^{v+2}}{2(v+1)(v+2)}.$$

Proof. By using mean value theorem for g', we have

$$\left|\frac{(d-c)^{v+1}}{2}\int_0^1 (s^v-\mu)[g'(sd+(1-s)c)-g'(sc+(1-s)d)]ds\right|$$
$$= \frac{(d-c)^{v+1}}{2}\left|\int_0^1 (s^v-\mu)[g'(sd+(1-s)c)-g'(sc+(1-s)d)]ds\right|$$
$$\le \frac{(d-c)^{v+1}}{2}\left|\int_0^1 (s^v-\mu)M\big[(sd+(1-s)c)-(sc+(1-s)d)\big]ds\right|$$
$$= \frac{(d-c)^{v+1}}{2}\left|\int_0^1 (s^v-\mu)(2s-1)ds\right|$$
$$= \frac{vM(d-c)^{v+2}}{2(v+1)(v+2)}.$$

The proof is ok.

4 Applications

For $v,\omega, v\neq\omega$, we consider special means of real numbers as follows:

$$H(v,\omega)=\frac{2}{\frac{1}{v}+\frac{1}{\omega}}, v,\omega\in R\setminus\{0\}$$

$$A(v,\omega)=\frac{v+\omega}{2}, v,\omega\in R,$$

$$L(v,\omega)=\frac{\omega-v}{\ln|\omega|-\ln|v|}, |\omega|\neq|v|, v\omega\neq 0.$$

$$L_n(v,\omega)=\left[\frac{\omega^{n+1}-v^{n+1}}{(n+1)(\omega-v)}\right]^{\frac{1}{n}}, n\in Z\setminus\{-1,0\}, v,\omega\in R, v\neq\omega$$

Let $c,d\in R^+$ and $c<d$.

Proposition 1.

$$\left|A(e^c,e^d)-L(e^c,e^d)\right| \le \frac{1}{6}(d-c)^2(e^c+e^d).$$

Proof. Choose $g(\ln z)=z$, $\mu=1$ and $\nu=1$. One can apply Remark 3.5 to complete the proof.

Proposition 2.

$$\left|H^{-1}(e^c,e^d)-L(e^{-c},e^{-d})\right| \le \frac{1}{6}(d-c)^2(e^{-c}+e^{-d}).$$

Proof. Choose $c^{-1}>d^{-1}$, $f(\ln z)=\frac{1}{z}$, $\mu=1$ and $\nu=1$, one can apply Remark 3.5 to obtain the results.

Proposition 3.

$$\left|A(e^{c(n+1)},e^{d(n+1)})-\frac{e^d-e^c}{d-c}L_n^n(e^c,e^d)\right| \le \frac{1}{6}(d-c)^2(n+1)[e^{c(n+1)}+e^{d(n+1)}].$$

Proof. Choose $f(\ln z)=z^{n+1}$, $\mu=1$ and $\nu=1$, one can apply Remark 3.5 to obtain the results.

Acknowledgement

The authors acknowledge the comments from the referees and the Editors. This work is supported by NNSF of China (11201091) and Outstanding Scientic and Technological Innovation Talent Award of Education Department of Guizhou Province ([2014]240).

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