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Linear Differential Equations of Fractional Order with Recurrence Relationship

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Abstract: This paper introduces the basic general theory for Linear Sequential Fractional Differential Equations which include a recurrence relationship, involving the Riemann-Liouville fractional operator. The presented equation is not a generalization of the known sequential linear fractional differential equations, but both are closely related.

Keywords: Fractional differential equation, linear difference equation.

1 Introduction, motivation and preliminaries

The application of differential equations for modeling in sciences is well known, so they are studied in many areas. The functions that verify the equation and establish its solutions (for example [1,2,3]) are of crucial interest. In recent years, several fractional calculus operators have been investigated and applied in various fields. The Mittag-Leffler function, and some of its generalizations, play a similar role in many differential equations involving fractional derivatives, (see, for example [4,5,6]). Due to the interest in the study of the behavior of different generalizations of the Mittag-Leffler function in the analysis of a broader field of fractional differential equations, different relations of recurrence involving fractional operators have been studied recently. They allow to establish recurrence relationships between different generalizations of the Mittag-Leffler function (see for example [7,8,9]).

The present work is motivated by the interest in the study of Mittag-Leffler-type functions as solutions of fractional differential equations, and the recurrence relationships that involve them.

This paper is structured as follows: In Section 1 we have compiled some basic fact. In Section 2 we introduce the notion of linear sequential fractional differential equations with recurrence relationships associated with Riemann-Liouville operator, and we develop a general theory for this differential equation. Finally, a direct method is also introduced to solve the homogeneous and non-homogeneous case with constant coefficients, and explicit expressions are obtained for the solutions in both cases.

1.1 Fractional operators

For development of this work we need to remember basic elements of fractional calculus as derivatives and integrals of arbitrary orders. It is well known that there are several definitions of fractional derivative, but we will consider the called Riemann-Liouville fractional derivative (see, for example [10,11,12]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function f is defined by

$$\left(I^{\alpha}_{a+}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}dt,\; x > a, \tag{1.1}$$

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where $f(x) \in L_1(a,b)$. If $n = [\alpha]+1$, the Riemann-Liouville fractional derivative of the function $f(x)$, $x \in [a,b]$, is defined by

$$(D^{\alpha}_{a+}f)(x) = \left(\frac{d}{dx}\right)\left(I^{n-\alpha}_{a+}f\right)(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}}dt. \tag{1.2}$$

To ensure the existence of (1.2), it will be enough that

$$\int_a^x \frac{f(t)}{(x-t)^{\{\alpha\}}}dt \in AC^{[\alpha]}([a,b]), \tag{1.3}$$

while the condition above is verified if $f(x) \in AC^{[\alpha]}([a,b])$. Moreover, if $\alpha, \beta \in \mathbb{R}$, $\beta > 0$, $\alpha \geq 0$, then

$$\left(D^{\alpha}_{a+}(t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1}. \tag{1.4}$$

The following Lemma, presented in [10], gives a rule for parametric derivation under the integral sign.

Lemma 1. *Let $0 < \alpha \leq 1$, $f(x)$ and $k(x)$ defined in $[a,b]$ such that*

$$f(x) \in C([a,b]) \;\; and \;\; L(x) = \int_0^x \tau^{-\alpha}k(x-\tau)d\tau \in C^1[a,b]. \tag{1.5}$$

Then, if $x \in [a,b]$, we have

$$D^{\alpha}_{a+}\left[\int_a^t k(t-\tau)f(u)du\right](x) = \int_a^x D^{\alpha}_{a+}[k(t-a)](u)f(x+a-u)du + f(x)\lim_{x\to a+} I^{1-\alpha}_{a+}[k(t-a)](x). \tag{1.6}$$

1.2 Mittag-Leffler type functions

The well known Mittag-Leffler function $E_{\alpha,\beta}(x)$ is defined (see, for instance [10]) by the following series:

$$E_{\alpha,\beta}(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(\alpha j+\beta)} \quad (x \in \mathbb{C}; \mathfrak{Re}(\alpha), \mathfrak{Re}(\beta) > 0), \tag{1.7}$$

where $\Gamma(x)$ is the classical Gamma function. The α-Exponential Function is defined by

$$e^{\lambda x}_{\alpha} = x^{\alpha-1}E_{\alpha,\alpha}(\lambda x^{\alpha}), \tag{1.8}$$

with $x \in \mathbb{C}\setminus\{0\}$, $\mathfrak{Re}(\alpha) > 0$, y $\lambda \in \mathbb{C}$; which satisfies the properties:

$$\left(\frac{\partial}{\partial x}\right)^n \left[e^{\lambda x}_{\alpha}\right] = x^{\alpha-n-1}E_{\alpha,\alpha-n}(\lambda x^{\alpha}), \tag{1.9}$$

$$\left(\frac{\partial}{\partial \lambda}\right)^n \left[e^{\lambda x}_{\alpha}\right] = n!x^{\alpha n+\alpha-1}E^{n+1}_{\alpha,\alpha n+\alpha}(\lambda x^{\alpha}), \;\; n \in \mathbb{N}, \; \lambda \in \mathbb{C}. \tag{1.10}$$

The following Mittag-Leffler type function will be considered

$$e^{\lambda x}_{\alpha,n} = \frac{1}{n!}\left(\frac{\partial}{\partial \lambda}\right)^n \left[e^{\lambda x}_{\alpha}\right] = x^{\alpha n+\alpha-1}E^{n+1}_{\alpha,\alpha n+\alpha}(\lambda x^{\alpha}). \tag{1.11}$$

where $x \in \mathbb{C}\setminus\{0\}$, $\mathfrak{Re}(\alpha) > 0, \lambda \in \mathbb{C}$, and $n \in \mathbb{N}_0$. In particular, when $n = 0$, we obtain from (1.11): $e^{\lambda x}_{\alpha,0} = e^{\lambda x}_{\alpha}$. We can easily see from (1.11) that

$$\mathfrak{Re}\left[e^{\lambda x}_{\alpha,n}\right] = \sum_{j=0}^{\infty}(-1)^j c^{2j}\frac{x^{2j\alpha}}{(2j)!}e^{bx}_{\alpha,n+2j} \quad and \quad \mathfrak{Im}\left[e^{\lambda x}_{\alpha,n}\right] = \sum_{j=0}^{\infty}(-1)^j c^{2j+1}\frac{x^{(2j+1)\alpha}}{(2j+1)!}e^{bx}_{\alpha,n+2j+1}. \tag{1.12}$$

with $x > 0$ y $\lambda = b + ic$ ($b, c \in \mathbb{R}$). In [13], Prabhakar introduces the Mittag-Leffler type function

$$E^{\gamma}_{\alpha,\beta}(x) = \sum_{j=0}^{\infty} \frac{(\gamma)_j x^j}{\Gamma(\alpha j + \beta) j!}, \tag{1.13}$$

with α, β, $\gamma \in \mathbb{C}$; $\mathfrak{Re}(\alpha)$, $\mathfrak{Re}(\beta) > 0$, and $x \in \mathbb{C}$; where $(\gamma)_j$ is the Pochhammer symbol (see, for example [10]) and verifies $E^1_{\alpha,\beta} = E_{\alpha,\beta}$. The following formula is obtained from (1.4):

$$\left(D^{\alpha}_{a+}(t-a)^{\beta-1}E_{\mu,\beta}\left[\lambda(t-a)^{\mu}\right]\right)(x) = (x-a)^{\alpha+\beta-1}E_{\mu,\alpha+\beta}\left[\lambda(x-a)^{\mu}\right] \tag{1.14}$$

with $\lambda \in \mathbb{C}$, $\alpha, \beta, \mu \in \mathbb{R}^+$. In particular, under certain restrictions, by mean (1.14), it can prove that

$$\left(D^{\alpha}_{a+} e^{\lambda(t-a)}_{\alpha}\right)(x) = \lambda e^{\lambda(x-a)}_{\alpha}, \tag{1.15}$$

when $\alpha > 0$, y $\lambda \in \mathbb{C}$ (see, [10]). Moreover, the α-Exponential Function satisfies the property

$$\lim_{x \to a^+} \left(I^{1-\alpha}_{a+} e^{\lambda(t-a)}_{\alpha}\right)(x) = 1, \quad (\alpha \geq 0). \tag{1.16}$$

1.3 Linear differential equation of fractional order

In [10, Chapter 7], the theory of the linear sequential fractional differential equation develops. In this section we highlight only some necessary aspects of the theory.

Definition 1. *Let $N \in \mathbb{N}$. We will call Linear Sequential Fractional Differential Equation (LFDE) of order $N\alpha$ the equation of the type*

$$\sum_{k=0}^{N} b_k(x) y^{(k\alpha)}(x) = f(x) \quad (a < x < b), \tag{1.17}$$

where $b_k(x)$ y $f(x)$ are known functions, $y^{(0)}(x) = y(x)$, and $y^{(k\alpha)} = \left(\mathscr{D}^{k\alpha}_{a+} y(x)\right)(x)$ ($k = 1, 2, ..., N$) represents a fractional sequential derivative introduced by Miller and Ross in [5]:

$$\begin{aligned} \mathscr{D}^{\alpha}_{a+} &= \mathbf{D}^{\alpha}_{a+} \quad (0 < \alpha \leq 1), \\ \mathscr{D}^{k\alpha}_{a+} &= \mathscr{D}^{\alpha}_{a+} \mathscr{D}^{(k-1)\alpha}_{a+}, \end{aligned} \tag{1.18}$$

where $\mathbf{D}^{\alpha}_{a+}$ is a fractional derivative, for example, the Riemann-Liouville fractional derivative: $\mathbf{D}^{\alpha}_{a+} = D^{\alpha}_{a+}$.

A Sequential Fractional Differential Equation of order $N\alpha$ is given by the following expression

$$F\left(x, y(x), \left(\mathscr{D}^{\alpha} y\right)(x), \left(\mathscr{D}^{2\alpha} y\right)(x), ..., \left(\mathscr{D}^{N\alpha} y\right)(x)\right) = f(x), \tag{1.19}$$

with $\alpha > 0$, $F(x, y_1, y_2, ..., y_N)$ and $f(x)$ are known functions (see [10]).

Let $b_N(x) \neq 0$, $\forall x \in [a, b]$; the equation (1.17) can be written in the following normalized form:

$$\left[\mathrm{L}_{N\alpha}(y)\right](x) = \left(\mathscr{D}^{N\alpha}_{a+} y\right)(x) + \sum_{j=0}^{N-1} b_k(x) \left(\mathscr{D}^{j\alpha}_{a+} y\right)(x) = f(x). \tag{1.20}$$

Definition 2. *A fundamental set of solution to the equation $[\mathbf{L}_{N\alpha}(y)](x) = f(x)$ in some interval $V \subset [a, b]$ is a set of N linearly independent functions in V, which are solutions to the equation.*

Proposition 1. *If $\{u_j(x)\}_{j=1}^{N}$ is a fundamental set of solutions to the equation $[\mathbf{L}_{N\alpha}(y)](x) = 0$ in some interval $V \subset (a, b]$, then the general solution to this differential equation is given by*

$$y_g(x) = \sum_{k=1}^{N} c_k u_k(x), \tag{1.21}$$

with $\{c_k\}_{k=1}^{N}$ arbitrary constants.

Proposition 2. *The set of solutions to $[\mathbf{L}_{N\alpha}(y)](x) = 0$, in some $V \subset (a, b]$, is a vector space of dimension N.*

Proposition 3. *If $y_p(x)$ is a particular solution to $[\mathbf{L}_{N\alpha}(y)](x) = f(x)$, then a general solution to this equation is*

$$y_g(x) = y_h(x) + y_p(x), \tag{1.22}$$

where $y_h(x)$ is the general solution to associated homogeneous equation, $[\mathbf{L}_{N\alpha}(y)](x) = 0$.

1.3.1 *Solution of linear sequential differential equations with constant coefficients*

Now, we address at the following equation

$$[\mathbf{L}_{N\alpha}(y)](x) = \left(\mathscr{D}_{a+}^{N\alpha}y\right)(x) + \sum_{j=1}^{N} a_{N-j}\left(\mathscr{D}_{a+}^{(N-j)\alpha}y\right)(x) = 0, \tag{1.23}$$

where $\{a_j\}_{j=0}^{N-1}$ are real constants. Let us mention an important property of the α-Exponential Function:

$$\left[\mathbf{L}_{N\alpha}\left(e_{\alpha}^{\lambda(t-a)}\right)\right](x) = P_N(\lambda)e_{\alpha}^{\lambda(x-a)}, \tag{1.24}$$

where

$$P_N(\lambda) = \lambda^N + \sum_{j=1}^{N} a_{N-j}\lambda^{N-j}. \tag{1.25}$$

is the characteristic polynomial associated with the equation $[\mathbf{L}_{N\alpha}(y)](x) = 0$.

Lemma 2. *If $\lambda \in \mathbb{C}$ is a root of the characteristic polynomial (1.25), then*

$$\frac{\partial}{\partial\lambda}\left[\mathbf{L}_{N\alpha}\left(e_{\alpha}^{\lambda(t-a)}\right)\right](x) = \left[\mathbf{L}_{N\alpha}\left(\frac{\partial}{\partial\lambda}e_{\alpha}^{\lambda(t-a)}\right)\right](x). \tag{1.26}$$

and

$$\frac{\partial^{\ell}}{\partial\lambda^{\ell}}e_{\alpha}^{\lambda(x-a)} = (x-a)^{\ell\alpha}e_{\alpha,\ell}^{\lambda(x-a)}. \tag{1.27}$$

Proposition 4. *If λ_1 is a root of multiplicity ℓ_1 of the caracteristic polinomial (1.25), then the functions $\{y_{1,j}(x)\}_{j=0}^{\ell_1-1}$:*

$$y_{1,j}(x) = (x-a)^{j\alpha}e_{\alpha,j}^{\lambda_1(x-a)}, \tag{1.28}$$

whit $e_{\alpha,j}^{\lambda_1(x-a)}$, defined by (1.11), are solutions of the equation $[\mathbf{L}_{N\alpha}(y)](x) = 0$.

Corollary 1. *Let $\{\lambda_j\}_{j=1}^{M}$ be M different roots of multiplicity $\{\ell_j\}_{j=1}^{M}$ of the characteristic polynomial (1.25). Then, the functions*

$$\bigcup_{k=1}^{M}\left\{(x-a)^{j\alpha}e_{\alpha,j}^{\lambda_k(x-a)}\right\}_{j=0}^{\ell_j-1} \tag{1.29}$$

are linearly independent solutions of the equations (1.23).

Proposition 5. *If λ_1 and $\overline{\lambda}_1$ ($\lambda_1 = b+ic$, $c \neq 0$) are two solutions of multiplicity ℓ_1 of the characteristic polynomial (1.25), then the functions*

$$\left\{\sum_{j=0}^{\infty}(-1)^j\frac{c^{2j}}{(2j)!}(x-a)^{(2j+k)\alpha}e_{\alpha,k+2j}^{b(x-a)}\right\}_{k=0}^{\ell_1-1} \quad \text{and} \quad \left\{\sum_{j=0}^{\infty}(-1)^j\frac{c^{2j+1}}{(2j+1)!}(x-a)^{(2j+k+1)\alpha}e_{\alpha,k+2j+1}^{b(x-a)}\right\}_{k=0}^{\ell_1-1} \tag{1.30}$$

determine $2\ell_1$ real linearly independent solutions of the equation $[\mathbf{L}_{N\alpha}(y)](x) = 0$.

Remark. Taking into account (1.12), (1.30) can be written, as follows:

$$\sum_{j=0}^{\infty}(-1)^j\frac{c^{2j}}{(2j)!}(x-a)^{(2j+k)\alpha}e_{\alpha,k+2j}^{b(x-a)} = \mathfrak{Re}\left[(x-a)^{k\alpha}e_{\alpha,k}^{\lambda_1(x-a)}\right], \tag{1.31}$$

and

$$\sum_{j=0}^{\infty}(-1)^j\frac{c^{2j+1}}{(2j+1)!}(x-a)^{(2j+k+1)\alpha}e_{\alpha,k+2j+1}^{b(x-a)} = \mathfrak{Im}\left[(x-a)^{k\alpha}e_{\alpha,k}^{\lambda_1(x-a)}\right]. \tag{1.32}$$

Corollary 2. *Let* $\left\{\lambda_m, \overline{\lambda}_m\right\}_{m=1}^{p}$, $\lambda_m = b_m + ic_m$ $(c_m \neq 0)$*, be all different pairs of complex conjugate solutions of multiplicity* $\{\sigma_m\}_{m=1}^{p}$ *of the characteristic polynomial (1.25) for the fractional differential equation (1.23). Then, the functions*

$$\bigcup_{m=1}^{p}\left\{\mathfrak{Re}\left[(x-a)^{k\alpha}e_{\alpha,k}^{\lambda_m(x-a)}\right]\right\}_{k=0}^{\sigma_m-1} \quad \text{and} \quad \bigcup_{m=1}^{p}\left\{\mathfrak{Im}\left[(x-a)^{k\alpha}e_{\alpha,k}^{\lambda_m(x-a)}\right]\right\}_{k=0}^{\sigma_m-1} \tag{1.33}$$

form a linearly independent set of solutions to the equations (1.23).

Theorem 1. *Let* $\{\lambda_j\}_{j=1}^{k}$ *be all real different roots of the characteristic polynomial (1.25) of multiplicity* $\{\ell_j\}_{j=1}^{k}$*, and let* $\{r_j, \overline{r_j}\}_{j=1}^{p}$ *(* $r_j = b_j + ic_j$ *) be the set of all distinct pairs of complex conjugate roots of (1.25) of multiplicity* $\{\sigma_j\}_{j=1}^{p}$ *such that* $\sum_{j=1}^{k}\ell_j + 2\sum_{j=1}^{p}\sigma_j = N$*. Then, the functions*

$$\bigcup_{m=1}^{k}\left\{(x-a)^{\ell\alpha}e_{\alpha,j}^{\lambda_m(x-a)}\right\}_{j=1}^{\ell_m-1}\ ;\ \bigcup_{m=1}^{p}\left\{\mathfrak{Re}\left[(x-a)^{k\alpha}e_{\alpha,k}^{r_m(x-a)}\right]\right\}_{k=1}^{\sigma_m-1} \text{ and } \bigcup_{m=1}^{p}\left\{\mathfrak{Im}\left[(x-a)^{k\alpha}e_{\alpha,k}^{r_m(x-a)}\right]\right\}_{k=1}^{\sigma_m-1} \tag{1.34}$$

form a fundamental set of solutions of the differential equation (1.23).

1.3.2 *Solution of Linear Sequential Differential Equations in the Non-Homogeneous Case.*

Proposition 6. *Let* $f(x) \in L_1(a,b) \cap C[(a,b)]$*. Then, the LFDE*

$$\left(\mathscr{D}_{a+}^{\alpha}y\right)(x) - \lambda y(x) = f(x)\ (x > a), \tag{1.35}$$

has the general solution

$$y_g(x) = ce_{\alpha}^{\lambda(x-a)} + y_p(x), \tag{1.36}$$

where

$$y_p(x) = \left(e_{\alpha}^{\lambda t} *^a f\right)(x) \tag{1.37}$$

is a particular solution to (1.35), being $*^a$ *the convolution:*

$$(g *^a f)(x) = \int_a^x g(x-t)f(t)dt. \tag{1.38}$$

In addition, if $f(x)$ *is continuous in* $[a,b]$*, then* $y_p(a+) = 0$*; while if* $f(x) \in \mathscr{C}_{1-\alpha}([a,b])$*, then* $\left(I_{a+}^{1-\alpha}y_p\right)(a+) = 0$.

Theorem 2. *Let* $\{\lambda_j\}_{j=1}^{k}$ *be the k different complex roots of multiplicity* $\{\sigma_j\}_{j=1}^{k}$ *of the characteristic polynomial (1.25) for the following non-homogeneous LFDE:*

$$[\mathbf{L}_{N\alpha}(y)](x) = \left(\prod_{j=1}^{k}\left(D_{a+}^{\alpha} - \lambda_j\right)^{\sigma_j} y\right)(x) = f(x)\ (x > a). \tag{1.39}$$

Then the particular solution to (1.39) is given by:

$$y_p(x) = (G_a *^a f_0)(x) \tag{1.40}$$

where

$$G_\alpha(x) = \prod_{j=1}^{k}{}^{*^a}\left(\prod_{\ell=1}^{\sigma_j}{}^{*^a} e_{\alpha}^{\lambda_j(t-a)}\right)(x). \tag{1.41}$$

Furthermore, if $f(x) \in \mathscr{C}_{1-\alpha}([a,b])$*, then* $\left(I_{a+}^{1-\alpha}y_p\right)(a+) = 0$*, while* $y_p(a+) = 0$ *when* $f(x)$ *is continuous* $[a,b]$*. Moreover,* $\left(I_{a+}^{1-\alpha}G_\alpha\right)(a+) = 0$.

2 General theory for sequential linear fractional differential equations with recurrence relationship

In this section our main results are proved.

Definition 3. *Let $N \in \mathbb{N}$ and $0 < \alpha \leq 1$. We will call Linear Sequential Fractional Differential Equations with Recurrence Relationship (LFDERR) of order $N\alpha$ to an equation of the type:*

$$[\mathbf{R}_{N\alpha}(y_n(t))_{n=0}^{\infty}](x) = \left(\mathscr{D}_{a+}^{N\alpha} y_n\right)(x) + \sum_{j=1}^{N} a_{N-j}(x) \left(\mathscr{D}_{a+}^{(N-j)\alpha} y_{n+j}\right)(x) = f_n(x) \quad (n \in \mathbb{N}_0,\ x > a), \tag{2.1}$$

where $\mathscr{D}_{a+}^{k\alpha}$ is defined by (1.18), $\{a_j(x)\}_{j=0}^{N-1}$ are real functions defined in $(a,b] \subset \mathbb{R}$, $a_0 \neq 0$, and $f_n(x) \in C((a,b])$, for each $n \in \mathbb{N}_0$. When $f_n \equiv 0$ the equation (2.1) we will call Homogeneous LFDERR (LFDERRH) asociated with (2.1):

$$\left(\mathscr{D}_{a+}^{N\alpha} y_n\right)(x) + \sum_{j=1}^{N} a_{N-j}(x) \left(\mathscr{D}_{a+}^{(N-j)\alpha} y_{n+j}\right)(x) = 0. \tag{2.2}$$

If $a_0, a_1, \ldots, a_{N-1}$ are constants, the equation (2.1) will be called equation to constant coefficients:

$$\left(\mathscr{D}_{a+}^{N\alpha} y_n\right)(x) + \sum_{j=1}^{N} a_{N-j} \left(\mathscr{D}_{a+}^{(N-j)\alpha} y_{n+j}\right)(x) = f_n(x); \tag{2.3}$$

and its corresponding homogeneous equation will be:

$$\left(\mathscr{D}_{a+}^{N\alpha} y_n\right)(x) + \sum_{j=1}^{N} a_{N-j} \left(\mathscr{D}_{a+}^{(N-j)\alpha} y_{n+j}\right)(x) = 0. \tag{2.4}$$

A Sequential Fractional Differential Equation with recurrence relationship of order $N\alpha$ is given by the following expression

$$F\left[n, x, \mathrm{E}^N y_n(x), \left(\mathscr{D}^{\alpha} \mathrm{E}^{N-1} y_n\right)(x), \left(\mathscr{D}^{2\alpha} \mathrm{E}^{N-2} y_n\right)(x), \ldots, \left(\mathscr{D}^{N\alpha} y_n\right)(x)\right] = f_n(x), \tag{2.5}$$

with $\alpha > 0$, $F(x, y_1, y_2, \ldots, y_N)$ and $f(x)$ are known functions, and E^k is the Shift Operator (See, for example, [14]).

The equation (2.1) represents a recurrence relationship between the sequential derivative up to order N and the consecutive terms of the sequence of functions $(y_n(x))_{n=0}^{\infty}$. Thus, solve the equation is to find $y_n(x)$.

In the development of this paper we will limit ourselves to work with sequential derivative (1.18), considering the Riemann-Liouville fractional derivative.

It will be denoted with $\Delta^{N\alpha}(a,b)$ the set of functions that admit sequential derivatives $\mathscr{D}_{a+}^{k\alpha}$, $1 \leq k \leq N$, in (a,b).

Definition 4. *The solution of the LFDERR will be given by the sequence of functions $(y_n(x))_{n=0}^{\infty}$ which verified (2.1).*

Now, we define Initial Values Problem (IVP).

Definition 5.

$$\begin{cases} [\mathbf{R}_{N\alpha}(y_n(t))_{n=0}^{\infty}](x) = f_n(x) \quad (n \in \mathbb{N}_0,\ x > a) \\ y_0(x) = c_0(x) \\ y_1(x) = c_1(x) \\ \vdots \\ y_{N-1}(x) = c_{N-1}(x). \end{cases} \tag{2.6}$$

where $c_1(x)$, ... ,$c_N(x)$ are known function.

Theorem 3. *The IVP (2.6), with $c_0(x)$,$c_1(x)$, ...,$c_{N-1}(x) \in \Delta^{\infty\alpha}(a,b)$, and $(f_n(x))_{n=0}^{\infty} \in [\Delta^{\infty\alpha}(a,b)]^{\mathbb{N}}$, admits unique solution. Also, the solution to (2.6) will be given by the sequence $(y_n(x))_{n=0}^{\infty}$, such that*

$$\begin{cases} y_0(x) = c_0(x) \\ y_1(x) = c_1(x) \\ \vdots \\ y_{N-1}(x) = c_{N-1}(x) \\ y_n(x) = \frac{1}{a_0}\{-\mathscr{D}_{a+}^{N\alpha} y_{n-N}(x) - a_{N-1}\mathscr{D}_{a+}^{(N-1)\alpha} y_{n-(N-1)}(x) - \ldots - a_1 \mathscr{D}_{a+}^{\alpha} y_{n-1}(z) + f_n(x)\} \quad \text{if } n \geq N. \end{cases} \tag{2.7}$$

Proof. To obtain the solution, a forward iteration is performed from the known functions. From (2.3) we obtain:

$$y_{n+N}(x) = \frac{1}{a_0}\{-\mathscr{D}^{N\alpha}_{a+}y_n(x) - a_{N-1}\mathscr{D}^{(N-1)\alpha}_{a+}y_{n+1}(x) - \dots - a_1\mathscr{D}^{\alpha}_{a+}y_{n+N-1}(x) + f_n(x)\}. \tag{2.8}$$

The functions $y_p(x)$, wiht $p \geq N$, can be obtained by recurrence using the initial conditions:

$$\begin{aligned} y_0(x) &= c_0(x) \\ y_1(x) &= c_1(x) \\ &\vdots \\ y_{N-1}(x) &= c_{N-1}(x) \end{aligned} \tag{2.9}$$

in (2.8). The uniqueness of the solution results from the construction. In addition, from (2.8), we obtain:

$$\begin{aligned} y_N(x) &= \frac{1}{a_0}\{-\mathscr{D}^{N\alpha}_{a+}c_0(x) - a_{N-1}\mathscr{D}^{(N-1)\alpha}_{a+}c_1(x) - \dots - a_1\mathscr{D}^{\alpha}_{a+}c_{N-1}(x) + f_0(x)\} \\ y_{N+1}(x) &= \frac{1}{a_0}\{-\mathscr{D}^{N\alpha}_{a+}c_1(x) - a_{N-1}\mathscr{D}^{(N-1)\alpha}_{a+}c_2(x) - \dots - a_1\mathscr{D}^{\alpha}_{a+}y_N(x) + f_1(x)\} \\ &\vdots \\ y_{N+m}(x) &= \frac{1}{a_0}\{-\mathscr{D}^{N\alpha}_{a+}y_m(x) - a_{N-1}\mathscr{D}^{(N-1)\alpha}_{a+}y_{m+1}(x) - \dots - a_1\mathscr{D}^{\alpha}_{a+}y_{m+(N-1)}(x) + f_m(x)\}, \end{aligned} \tag{2.10}$$

where $m \in \mathbb{N}$. Therefore, if we call $n = N + m$, for $n \geq N$ results:

$$y_n(x) = \frac{1}{a_0}\{-\mathscr{D}^{N\alpha}_{a+}y_{n-N}(x) - a_{N-1}\mathscr{D}^{(N-1)\alpha}_{a+}y_{n-N+1}(x) - \dots - a_1\mathscr{D}^{\alpha}_{a+}y_{n-1}(x) + f_{n-N}(x)\} \tag{2.11}$$

Definition 6. *We will call fundamental set of solutions of the equation (2.3) to a set of N linearly independent functions, in some interval $V \subset (a,b)$, which are solutions to this equation.*

Theorem 4. *Let $\left\{(y^1_n(x))^{\infty}_{n=0}, (y^2_n(x))^{\infty}_{n=0}, \dots, (y^N_n(x))^{\infty}_{n=0}\right\} \subset [\Delta^{\infty\alpha}(a,b)]^{\mathbb{N}}$ be, a fundamental set of solutions to equation (2.4), then for each $x \in (a,b)$, the Casoratian*[1] $|\mathbf{W}_0(y^1_n(x), y^2_n(x), \dots, y^N_n(x))| \neq 0$.

Proof. If there exist some $x_0 \in (a,b)$ such that

$$|\mathbf{W}_0(y^1_n(x_0), y^2_n(x_0), \dots, y^N_n(x_0))| = 0; \tag{2.12}$$

then the following system, with variables c_1, c_2,..., c_N:

$$\begin{cases} c_1y^1_0(x_0) + c_2y^2_0(x_0) + \cdots + c_Ny^N_0(x_0) = 0 \\ c_1y^1_1(x_0) + c_2y^2_1(x_0) + \cdots + c_Ny^N_1(x_0) = 0 \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad \ddots \qquad \vdots \qquad\quad \vdots \\ c_1y^1_{N-1}(x_0) + c_2y^2_{N-1}(x_0) + \cdots + c_Ny^N_{N-1}(x_0) = 0. \end{cases} \tag{2.13}$$

It has infinite solutions: In particular, there exist c^0_1, c^0_2,..., c^0_N real constants, not all zero, that solve the system (2.13), and with these values we can define a sequence of functions: $(z_n(x))^{\infty}_{n=0}$, where $x \in (a,b)$, such that

$$z_n(x) = \sum_{k=1}^{N} c^0_k y^k_n(x). \tag{2.14}$$

[1] Where $|\mathbf{W}_0(y^1_n(x), y^2_n(x), \dots, y^N_n(x))| = |\mathbf{W}_{n_0}(y^1_n(x), y^2_n(x), \dots, y^N_n(x))|$ with $n_0 = 0$ (see, for exaple [14]).

By definition $z_n(x)$ is a solution of (2.3):

$$[\mathbf{R}_{N\alpha}(z_n(t))_{n=0}^{\infty}](x) = \left(\mathscr{D}_{a+}^{N\alpha}\left(\sum_{k=1}^{N} c_k^0 y_n^k(t)\right)\right)(x) + \sum_{j=1}^{N} a_{N-j}\left(\mathscr{D}_{a+}^{(N-j)\alpha}\left(\sum_{k=1}^{N} c_k^0 y_{n+j}^k(t)\right)\right)(x) \tag{2.15}$$

$$= \sum_{k=1}^{N} c_k^0\left(\mathscr{D}_{a+}^{N\alpha} y_n^k(t)\right)(x) + \sum_{k=1}^{N} c_k^0 \sum_{j=1}^{N} a_{N-j}\left(\mathscr{D}_{a+}^{(N-j)\alpha} y_{n+j}^k(t)\right)(x)$$

$$= \sum_{k=1}^{N} c_k^0\left[\left(\mathscr{D}_{a+}^{N\alpha} y_n^k(t)\right)(x) + \sum_{j=1}^{N} a_{N-j}\left(\mathscr{D}_{a+}^{(N-j)\alpha} y_{n+j}^k(t)\right)(x)\right]$$

$$= \sum_{k=1}^{N} c_k^0\left[\mathbf{R}_{N\alpha}\left(y_n^k(t)\right)_{n=0}^{\infty}\right](x) = 0, \tag{2.16}$$

since $\left(y_n^1(x)\right)_{n=0}^{\infty}, \ldots, \left(y_n^N(x)\right)_{n=0}^{\infty}$ verify (2.4).

Rewriting the system (2.13), and taking into account (2.14) such that:

$$z_0(x_0) = z_1(x_0) = \ldots = z_{N-1}(x_0) = 0, \tag{2.17}$$

so from (2.16) and (2.17), we have to verify an initial values problem like the following:

$$\begin{cases} [\mathbf{R}_{N\alpha}(z_n(t))_{n=0}^{\infty}](x) = 0 \quad (n \in \mathbb{N}_0) \\ z_0(x) = d_0(x) \\ z_1(x) = d_1(x) \\ \vdots \\ z_{N-1}(x) = d_{N-1}(x). \end{cases} \tag{2.18}$$

where $d_0(x)$, $d_1(x)$, ..., $d_{N-1}(x) \in \Delta^{\infty\alpha}(a,b)$, such that:

$$d_0(x_0) = d_1(x_0) = \ldots = d_{N-1}(x_0) = 0. \tag{2.19}$$

On the other hand, the sequence zero, i.e. $(w_n(x))_{n=0}^{\infty}$ such that $w_n(x) = 0$ for all $x \in (a,b)$, verify trivially (2.18); but from Theorem 3, the problem (2.18) admits a unique solution, i.e.:

$$(z_n(x))_{n=0}^{\infty} = (w_n(x))_{n=0}^{\infty}. \tag{2.20}$$

Then, for which $n \in \mathbb{N}_0$:

$$\sum_{k=1}^{N} c_k^0 y_n^k(x) = z_n(x) = w_n(x) = 0. \tag{2.21}$$

Finally, we found a null combination, not trivial, of $y_n^1(x)$, $y_n^2(x)$, ..., $y_n^N(x)$. Hence, $(y_n^1(x))_{n=0}^{\infty}, (y_n^2(x))_{n=0}^{\infty}, \ldots, (y_n^N(x))_{n=0}^{\infty}$ are lineally dependent.

Theorem 5. *Let* $\mathbf{G} = \left\{(y_n^1(x))_{n=0}^{\infty}, (y_n^2(x))_{n=0}^{\infty}, \ldots, (y_n^N(x))_{n=0}^{\infty}\right\} \subset [\Delta^{\infty\alpha}(a,b)]^{\mathbb{N}}$ *be. If there exists* $x_0 \in (a,b)$ *such that the Casoratian*

$$|\mathbf{W}_0(y_n^1(x_0), y_n^2(x_0), \ldots, y_n^N(x_0))| \neq 0, \tag{2.22}$$

then the set $\mathbf{G}$ *is linearly independent.*

Proof. Let $x_0 \in (a,b)$ be such that:

$$|\mathbf{W}_0(y_n^1(x_0), y_n^2(x_0), \ldots, y_n^N(x_0))| \neq 0. \tag{2.23}$$

Let $x \in (a,b)$. We propose the following linear combination:

$$c_1 y_n^1(x) + c_2 y_n^2(x) + \ldots + c_N y_n^N(x) = 0, \tag{2.24}$$

where $n \in \mathbb{N}_0$. Also, we can obtain the following system of equations, where c_1, c_2,..., c_N are unknown constants:

$$\begin{cases} c_1 y_0^1(x_0) + c_2 y_0^2(x_0) + \cdots + c_N y_0^N(x_0) = 0 \\ c_1 y_1^1(x_0) + c_2 y_1^2(x_0) + \cdots + c_N y_1^N(x_0) = 0 \\ \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \\ c_1 y_{N-1}^1(x_0) + c_2 y_{N-1}^2(x_0) + \cdots + c_N y_{N-1}^N(x_0) = 0 \end{cases} \tag{2.25}$$

Since the determinant of the matrix of this homogeneous system is nonzero, i.e. $|\mathbf{W}_0(y_n^1(x_0), y_n^2(x_0), ..., y_n^N(x_0))| \neq 0$, it admits a unique solution, the trivial: $c_1 = c_2 = ... = c_N = 0$. Thus, the unique linear combination possible to (2.24) is the trivial; then $\left(y_n^1(x)\right)_{n=0}^{\infty}, ..., \left(y_n^N(x)\right)_{n=0}^{\infty}$ are linear independent.

Now we consider the family of functions

$$y_n(x) = \gamma^n e_\alpha^{\lambda\gamma(x-a)}, \quad (n \in \mathbb{N},\ \gamma \neq 0). \tag{2.26}$$

If we replace $\gamma^n e_\alpha^{\lambda\gamma(x-a)}$ in the left hand of (2.3):

$$\begin{aligned} \left[\mathbf{R}_{N\alpha}\left(\gamma^n e_\alpha^{\lambda(t-a)}\right)_{n=0}^{\infty}\right](x) &= \gamma^n \left\{ \left[\mathscr{D}_{a+}^{N\alpha}\left(e_\alpha^{\lambda(t-a)}\right)\right](x) + \sum_{j=1}^{N} a_{N-j}\gamma^j \left[\mathscr{D}_{a+}^{(N-j)\alpha} e_\alpha^{\lambda(t-a)}\right](x) \right\} \\ &= \gamma^n \left\{ \lambda^N e_\alpha^{\lambda(x-a)} + \sum_{j=1}^{N} a_{N-j}\gamma^j \lambda^{N-j} e_\alpha^{\lambda(x-a)} \right\} \\ &= \gamma^n e_\alpha^{\lambda(x-a)} \left\{ \lambda^N + \sum_{j=1}^{N} \left(a_{N-j}\gamma^j\right) \lambda^{N-j} \right\}. \end{aligned} \tag{2.27}$$

Definition 7. *We called characteristic γ-polynomial associated with (2.3) is the expression that is between brace (2.27), and we will write:*

$$P_{N,\gamma}(\lambda) = \lambda^N + \sum_{j=1}^{N} \left(a_{N-j}\gamma^j\right) \lambda^{N-j}. \tag{2.28}$$

Remark. Since $a_0 \neq 0$ and $\gamma \neq 0$, from the definition, the roots of (2.28) are nonzero.

Theorem 6. *If λ is a root of the characteristic polynomial (2.28), then the succession $\left(\gamma^n e_\alpha^{\lambda\gamma(x-a)}\right)_{n=0}^{\infty}$ is solution to the equation (2.4).*

Proof. Taking into account (2.27), the proof is completed.

Lemma 3. *Let $\left\{(y_n^k(x))_{n=0}^{\infty}\right\}_{k=1}^{N} \subset [\Delta^{\infty\alpha}(a,b)]^{\mathbb{N}}$ be a fundamental set of solutions to equation (2.4). Then, any solution $(y_n(x))_{n=0}^{\infty}$ to the equation (2.4), in (a,b), can be written, as follows:*

$$y_n(x) = \sum_{k=1}^{N} c_k y_n^k(x) \tag{2.29}$$

where c_1, c_2, ... ,c_N are arbitrary constants.

Proof. Let $x \in (a,b)$. From Theorem 4, we have

$$|\mathbf{W}_0(y_n^1(x), y_n^2(x), ..., y_n^N(x))| \neq 0. \tag{2.30}$$

On the other hand, let $(y_n(x))_{n=0}^{\infty}$ be any solution to (2.4) and we suppose:

$$\begin{cases} y_0(x) = d_0(x) \\ y_1(x) = d_1(x) \\ \vdots \\ y_{N-1}(x) = d_{N-1}(x). \end{cases} \tag{2.31}$$

From (2.31) we have that the system

$$\begin{cases} \sum_{j=i}^{N} c_j y_0^j(x) & = d_0(x) \\ \sum_{j=i}^{N} c_j y_1^j(x) & = d_1(x) \\ \quad \vdots & \quad \vdots \\ \sum_{j=i}^{N} c_j y_{N-1}^j(x) & = d_{N-1}(x), \end{cases} \tag{2.32}$$

admits a unique solution, where $x \in (a,b)$. Then, there is a unique $(c_1^0, c_2^0, ..., c_N^0)$ that verifies (2.32). Let's define

$$w_n(x) = \sum_{j=1}^{N} c_j^0 y_n^j(x). \tag{2.33}$$

Therefore, by (2.32), the sequence $(w_n(x))_{n=0}^{\infty}$ verifies the following initial conditions

$$\begin{cases} w_0(x) & = d_0(x) \\ w_1(x) & = d_1(x) \\ & \vdots \\ w_{N-1}(x) & = d_{N-1}(x). \end{cases} \tag{2.34}$$

where $x \in (a,b)$. Finally, taking into account Theorem 3, the result is, as follow:

$$y_n(x) = w_n(x) = \sum_{k=1}^{N} c_k y_n^k(x). \tag{2.35}$$

Definition 8. *We will denote $\boldsymbol{E}_N^0(a,b)$ the set of solutions to equation (2.4), $x \in (a,b)$, with the operations vector addition, "$+$", and scalar multiplication "$\cdot$", defined as follows:*

$$(y_n^1(x))_{n=0}^{\infty} + (y_n^2(x))_{n=0}^{\infty} = ((y_n^1 + y_n^2)(x))_{n=0}^{\infty} \tag{2.36}$$

$$d(y_n^1(x))_{n=0}^{\infty} = ((dy_n^1)(x))_{n=0}^{\infty}, \tag{2.37}$$

whenever $(y_n^1(x))_{n=0}^{\infty}$, $(y_n^2(x))_{n=0}^{\infty} \in \boldsymbol{E}_N^0(a,b)$, and d is a scalar.

Theorem 7. *Any linear combination of solutions to the homogeneous equation (2.2), is also solution of the equation (2.2).*

Proof. If $\left\{(y_n(x))_{n=0}^{\infty}\right\}_{k=1}^{M}$ are a set of M solutions to (2.2), and c_1, c_2, ..., c_M are arbitrary constants, then

$$\begin{aligned} \left[\mathbf{R}_{N\alpha}\left(\sum_{k=1}^{M} c_k y_n^k(t)\right)_{n=0}^{\infty}\right](x) &= \left[\mathscr{D}_{a+}^{N\alpha} \sum_{k=1}^{M} c_k y_n^k(t)\right](x) + \sum_{j=0}^{N} a_{N-j}(x)\left[\mathscr{D}_{a+}^{(N-j)\alpha} \sum_{k=1}^{M} c_k y_{n+j}^k(t)\right](x) \\ &= \sum_{k=1}^{M} c_k \left[\mathscr{D}_{a+}^{N\alpha} y_n^k(t)\right](x) + \sum_{j=0}^{N} a_{N-j}(x) \sum_{k=1}^{M} c_k \left[\mathscr{D}_{a+}^{(N-j)\alpha} y_{n+j}^k(t)\right](x) \\ &= \sum_{k=1}^{M} c_k \left\{\left[\mathscr{D}_{a+}^{N\alpha} y_n^k(t)\right](x) + \sum_{j=0}^{N} a_{N-j}(x)\left[\mathscr{D}_{a+}^{(N-j)\alpha} y_{n+j}^k(t)\right](x)\right\} \\ &= \sum_{k=1}^{M} c_k \underbrace{\left[\mathbf{R}_{N\alpha}\left(y_n^k(t)\right)_{n=0}^{\infty}\right](x)}_{=0} = 0. \end{aligned} \tag{2.38}$$

Corollary 3. *For $\left\{(y_n^k(x))_{n=0}^{\infty}\right\}_{k=1}^{M} \subset [\Delta^{\infty\alpha}(a,b)]^{\mathbb{N}}$, we have*

$$\left[\mathbf{R}_{N\alpha}\left(\sum_{k=1}^{M} c_k y_n^k(t)\right)_{n=0}^{\infty}\right](x) = \sum_{k=1}^{M} c_k \left[\mathbf{R}_{N\alpha}\left(y_n^k(t)\right)_{n=0}^{\infty}\right](x), \tag{2.39}$$

where $\left[\mathbf{R}_{N\alpha}\left(y_n^k(t)\right)_{n=0}^{\infty}\right](x)$ is given in Definition 3.

Theorem 8. *The set* $\boldsymbol{H} = \boldsymbol{E}_N^0(a,b) \cap [\Delta^{\infty\alpha}(a,b)]^{\mathbb{N}}$ *is a vector space of N dimensions.*

Proof. It is clear that $(0)_{n=0}^{\infty} \in \mathbf{H}$, so $\mathbf{H} \neq \emptyset$. Let $\left(y_n^1(x)\right)_{n=0}^{\infty}$, $\left(y_n^2(x)\right)_{n=0}^{\infty} \in \mathbf{H}$, and $\eta, \mu \in \mathbb{C}$. From Corollary 3, we know that:

$$\mathbf{R}_{N\alpha}\left[\eta\,(y_n^1(x))_{n=0}^{\infty} + \mu\,(y_n^2(x))_{n=0}^{\infty}\right] = \eta\mathbf{R}_{N\alpha}\left[(y_n^1(x))_{n=0}^{\infty}\right] + \mu\mathbf{R}_{N\alpha}\left[(y_n^2(x))_{n=0}^{\infty}\right] = 0. \tag{2.40}$$

Finally, taking into account Lemma 3, the thesis is concluded.

Theorem 9. *Let* $(y_n^p(x))_{n=0}^{\infty}$ *be a particular solution (2.3) and let* $(y_n^h(x))_{n=0}^{\infty}$ *be a solution to (2.4). Then, any solution* $(y_n(x))_{n=0}^{\infty}$ *to (2.3) can be written as follows:*

$$(y_n(x))_{n=0}^{\infty} = ((y_n^h + y_n^p)(x))_{n=0}^{\infty}. \tag{2.41}$$

Proof. The thesis results from applying Corollary 3.

Corollary 4. *Let* $(y_n(x))_{n=0}^{\infty}$ *be a solution to the equation (2.1). Then, for all* $n_0 \in \mathbb{N}$, $(y_{n_0+n}(x))_{n=0}^{\infty}$ *is also a solution of (2.1).*

Proof. The thesis is shown verifying (2.1).

In the following Lemma we will establish a relationship between the roots of the polynomial $P_N(\lambda)$, defined in (1.25), with the roots of the polynomial $P_{N,\gamma}(\lambda)$ defined in (2.28).

Lemma 4. *If* λ_1 *is a root of the polynomial* $P_N(\lambda)$*, then* $\gamma\lambda_1$ *is a root of the polynomial* $P_{N,\gamma}(\lambda)$*. Moreover, if* λ_1 *is a root of multiplicity* ℓ *of* $P_N(\lambda)$*, then* $\gamma\lambda_1$ *is a root of multiplicity* ℓ *of* $P_{N,\gamma}(\lambda)$*.*

Proof. Taking into account (1.25) and (2.28) we can write:

$$P_{N,\gamma}(\gamma\lambda) = (\gamma\lambda)^N + \sum_{j=1}^{N}\left(a_{N-j}\gamma^j\right)(\gamma\lambda)^{N-j} = (\gamma\lambda)^N + \sum_{j=1}^{N}\left(a_{N-j}\gamma^N\right)\lambda^{N-j} = \gamma^N P_N(\lambda). \tag{2.42}$$

Then $P_{N,\gamma}(\gamma\lambda_1) = \gamma^N P_N(\lambda_1) = 0$.

From (2.42), we have that:

$$\frac{d^k P_{N,\gamma}}{d\lambda^k}(\gamma\lambda) = \gamma^N \frac{d^k P_N}{d\lambda^k}(\lambda), \quad 0 \leq k \leq \ell. \tag{2.43}$$

On the other hand, since λ_1 is a root of multiplicity ℓ of $P_N(\lambda)$, we know that

$$P_N(\lambda_1) = \frac{dP_N}{d\lambda}(\lambda_1) = \ldots = \frac{d^{\ell-1}P_N}{d\lambda^{\ell-1}}(\lambda_1) = 0 \text{ and } \frac{d^{\ell}P_N}{d\lambda^{\ell}}(\lambda_1) \neq 0. \tag{2.44}$$

Finally, applying (2.43) and (2.44), we obtain:

$$P_{N,\gamma}(\gamma\lambda_1) = \frac{dP_{N,\gamma}}{d\lambda}(\gamma\lambda_1) = \ldots = \frac{d^{\ell-1}P_{N,\gamma}}{d\lambda^{\ell-1}}(\gamma\lambda_1) = 0 \text{ and } \frac{d^{\ell}P_{N,\gamma}}{d\lambda^{\ell}}(\gamma\lambda_1) \neq 0, \tag{2.45}$$

i.e., $\gamma\lambda_1$ is a root of multiplicity ℓ of $P_{N,\gamma}(\lambda)$.

2.1 Solution of the homogeneous LFDERR via $e_{\alpha}^{\lambda x}$

Lemma 3 asserts that the problem to obtaining a general solution to the Homogeneous LFDERR can be reduced to finding N linearly independent solutions. In what follows we will show how, in different cases, we can obtain $N-1$ solutions from the first and such that the N solutions form a fundamental set of solutions.

It is possible to find a fundamental set of solutions of the equation (2.4) using the well known function, α-Exponential, (1.8). The solution will be given by the potential function γ^n multiplied by the function $e_{\alpha}^{\lambda x}$.

Proposition 7. *If λ_1 is a root of the characteristic polynomial (1.25), with multiplicity ℓ_1; then*

$$\left\{ \left(y_n^{1,j}(x) \right)_{n=0}^{\infty} \right\}_{j=0}^{\ell_1 - 1} \subset \mathbf{E}_N^0(a,b) \cap [\Delta^{\infty\alpha}(a,b)]^{\mathbb{N}}, \tag{2.46}$$

with

$$y_n^{1,j}(x) = \gamma^n (x-a)^{j\alpha} e_{\alpha,j}^{\lambda_1 \gamma(x-a)}, \tag{2.47}$$

where $e_{\alpha,j}^{\lambda_1(x-a)}$ is given by (1.11), and $\gamma \neq 0$.

Proof. Since λ_1 is a root of multiplicity ℓ_1 of $P_N(\lambda)$; from Property 4, it results that $\gamma\lambda_1$ is a root of multiplicity ℓ_1 of $P_{N,\gamma}(\lambda)$, i.e. (2.45) is valid.

Let $0 \leq j \leq \ell_1 - 1$. Therefore, taking into account Lemma 2 and proceeding as in (2.27), we obtain:

$$\begin{aligned}
&\left\{ \left[\mathbf{R}_{N\alpha} \left(\gamma^n (t-a)^{j\alpha} e_{\alpha,j}^{\lambda(t-a)} \right)_{n=0}^{\infty} \right](x) \right\}_{\lambda=\lambda_1\gamma} = \\
&= \left\{ \left[\mathbf{R}_{N\alpha} \left(\gamma^n \frac{\partial^j}{\partial \lambda^j} e_{\alpha}^{\lambda(t-a)} \right)_{n=0}^{\infty} \right](x) \right\}_{\lambda=\lambda_1\gamma} = \\
&= \left\{ \left[\mathscr{D}_{a+}^{N\alpha} \left(\gamma^n \frac{\partial^j}{\partial \lambda^j} e_{\alpha}^{\lambda(t-a)} \right)(x) \right] + \sum_{\sigma=1}^{N} a_{N-\sigma} \left[\mathscr{D}_{a+}^{(N-\sigma)\alpha} \left(\gamma^{n+\sigma} \frac{\partial^j}{\partial \lambda^j} e_{\alpha}^{\lambda(t-a)} \right)(x) \right] \right\}_{\lambda=\lambda_1\gamma} \\
&= \left\{ \gamma^n \frac{\partial^j}{\partial \lambda^j} \left(\left[\mathscr{D}_{a+}^{N\alpha} \left(e_{\alpha}^{\lambda(t-a)} \right)(x) \right] + \sum_{\sigma=1}^{N} a_{N-\sigma} \gamma^{\sigma} \left[\mathscr{D}_{a+}^{(N-\sigma)\alpha} e_{\alpha}^{\lambda(t-a)} \right](x) \right) \right\}_{\lambda=\lambda_1\gamma} = \\
&= \left\{ \gamma^n \frac{\partial^j}{\partial \lambda^j} \left[P_{N,\gamma}(\lambda) e_{\alpha}^{\lambda(x-a)} \right] \right\}_{\lambda=\lambda_1\gamma}.
\end{aligned} \tag{2.48}$$

Finally, applying the Leibniz rule in (2.48), we obtain:

$$\left\{ \left[\mathbf{R}_{N\alpha} \left(\gamma^n (t-a)^{j\alpha} e_{\alpha,j}^{\lambda(t-a)} \right)_{n=0}^{\infty} \right](x) \right\}_{\lambda=\lambda_1\gamma} = \gamma^n \sum_{k=0}^{j} \binom{j}{k} \left\{ \frac{\partial^{j-k}}{\partial \lambda^{j-k}} \left(e_{\alpha}^{\lambda(x-a)} \right) \right\}_{\lambda=\lambda_1\gamma} \underbrace{\left\{ \frac{\partial^k P_{N,\gamma}}{\partial \lambda^k}(\lambda) \right\}_{\lambda=\lambda_1\gamma}}_{=0} = 0, \tag{2.49}$$

since (2.45) is valid; i.e. $\left(\gamma^n (x-a)^{j\alpha} e_{\alpha,j}^{\lambda_1\gamma(x-a)} \right)_{n=0}^{\infty} \in \mathbf{E}_N^0(a,b)$. Moreover, proceeding as in(2.1):

$$\left(\mathscr{D}_{a+}^{N\alpha} \gamma^n (t-a)^{N\alpha} e_{\alpha,j}^{\lambda(t-a)} \right)(x) = \gamma^n \frac{\partial^j}{\partial \lambda^j} \left(\mathscr{D}_{a+}^{N\alpha} e_{\alpha}^{\lambda(t-a)} \right)(x) = \gamma^n \lambda^N \frac{\partial^j}{\partial \lambda^j} e_{\alpha}^{\lambda(x-a)} = \lambda^N \gamma^n (t-a)^{j\alpha} e_{\alpha,j}^{\lambda(x-a)}, \tag{2.50}$$

so,

$$\left(\gamma^n (x-a)^{j\alpha} e_{\alpha,j}^{\lambda_1\gamma(x-a)} \right)_{n=0}^{\infty} \in [\Delta^{\infty\alpha}(a,b)]^{\mathbb{N}}. \tag{2.51}$$

Corollary 5. *Let $\{\lambda_j\}_{j=1}^{M}$ be M different roots of multiplicity $\{\ell_j\}_{j=1}^{M}$, respectively, of the characteristic polynomial (1.25). Then, the sequence of functions*

$$\bigcup_{k=1}^{M} \left\{ \left(y_n^{k,j}(x) \right)_{n=0}^{\infty} \right\}_{j=0}^{\ell_k - 1} \tag{2.52}$$

where

$$y_n^{k,j}(x) = \gamma^n (x-a)^{j\alpha} e_{\alpha,j}^{\lambda_k \gamma(x-a)}, \tag{2.53}$$

$\gamma \neq 0$, they form a fundamental set of solutions to the equation (2.4).

Proof. We will prove the $M = 2$ case. Let $\{\lambda_1, \lambda_2\}$ be two distinct roots of $P_N(\lambda)$, where λ_1 has multiplicity ℓ_1, and λ_2 has multiplicity ℓ_2, with $\ell_1 + \ell_2 = N$. Then, from Lemma 4, we know that $\gamma\lambda_1$ and $\gamma\lambda_2$ are roots of multiplicity ℓ_1 y ℓ_2, respectively, of $P_{N,\gamma}(\lambda)$. By Proposition 7, we know that

$$\left\{\left(\gamma^n (x-a)^{j\alpha} e_{\alpha,j}^{\lambda_1\gamma(x-a)}\right)_{n=0}^{\infty}\right\}_{j=0}^{\ell_1-1} \bigcup \left\{\left(\gamma^n (x-a)^{j\alpha} e_{\alpha,j}^{\lambda_2\gamma(x-a)}\right)_{n=0}^{\infty}\right\}_{j=0}^{\ell_2-1} \tag{2.54}$$

represents a set of solutions to the equation (2.4).

On the other hand, from Corollary 1, and Lemma 4, we know that the functions

$$\left\{(x-a)^{j\alpha} e_{\alpha,j}^{\lambda_1\gamma(x-a)}\right\}_{j=0}^{\ell_1-1} \bigcup \left\{(x-a)^{j\alpha} e_{\alpha,j}^{\lambda_2\gamma(x-a)}\right\}_{j=0}^{\ell_2-1} \tag{2.55}$$

are linearly independent. Hence the functions

$$\left\{\gamma^n (x-a)^{j\alpha} e_{\alpha,j}^{\lambda_1\gamma(x-a)}\right\}_{j=0}^{\ell_1-1} \bigcup \left\{\gamma^n (x-a)^{j\alpha} e_{\alpha,j}^{\lambda_2\gamma(x-a)}\right\}_{j=0}^{\ell_2-1} \tag{2.56}$$

also form a linearly independent set, with $n \in \mathbb{N}_0$. Then, we can construct the $\ell_1 + \ell_2$ sequences of (2.54), where their general terms are the functions in (2.56). Then, it has been verified that (2.54) is a fundamental set of solutions of (2.4).

Corollary 6. *If λ and $\overline{\lambda}$ ($\lambda = b + ic$, $c \neq 0$) are two complex solutions of multiplicity ℓ, of the characteristic polynomial (1.25), then the sequences of functions*

$$\left\{\left(\gamma^n \mathfrak{Re}\left[(x-a)^{j\alpha} e_{\alpha,j}^{\lambda\gamma(x-a)}\right]\right)_{n=0}^{\infty}\right\}_{j=0}^{\ell_1-1} \bigcup \left\{\left(\gamma^n \mathfrak{Im}\left[(x-a)^{j\alpha} e_{\alpha,j}^{\lambda\gamma(x-a)}\right]\right)_{n=0}^{\infty}\right\}_{j=0}^{\ell_1-1} \tag{2.57}$$

$\gamma \neq 0$, form a subset of 2ℓ sequences linearly independent belonging to $\mathbf{E}_N^0(a,b) \cap [\Delta^{\infty\alpha}(a,b)]^{\mathbb{N}}$.

Proof. By Corollary 5, we know that

$$\left\{\left(\gamma^n (x-a)^{j\alpha} e_{\alpha,j}^{\lambda\gamma(x-a)}\right)_{n=0}^{\infty}\right\}_{j=0}^{\ell_1-1} \bigcup \left\{\left(\gamma^n (x-a)^{j\alpha} e_{\alpha,j}^{\overline{\lambda}\gamma(x-a)}\right)_{n=0}^{\infty}\right\}_{j=0}^{\ell_1-1} \tag{2.58}$$

is a set of solution of (2.4).

On the other hand, for each $n \in \mathbb{N}_0$, we can write:

$$\begin{aligned} \mathfrak{Re}\left[\gamma^n (x-a)^{j\alpha} e_{\alpha,j}^{\lambda\gamma(x-a)}\right] &= \frac{1}{2}\gamma^n (x-a)^{j\alpha} e_{\alpha,j}^{\lambda\gamma(x-a)} + \frac{1}{2}\overline{\left(\gamma^n (x-a)^{j\alpha} e_{\alpha,j}^{\lambda\gamma(x-a)}\right)} \\ &= \frac{1}{2}\gamma^n (x-a)^{j\alpha} e_{\alpha,j}^{\lambda\gamma(x-a)} + \frac{1}{2}\gamma^n (x-a)^{j\alpha} e_{\alpha,j}^{\overline{\lambda}\gamma(x-a)}; \end{aligned} \tag{2.59}$$

then, for each $0 \leq j \leq \ell_1 - 1$, from Theorem 8, the sequence

$$\left(\gamma^n \mathfrak{Re}\left[(x-a)^{j\alpha} e_{\alpha,j}^{\lambda\gamma(x-a)}\right]\right)_{n=0}^{\infty} \in \mathbf{E}_N^0(a,b) \cap [\Delta^{\infty\alpha}(a,b)]^{\mathbb{N}}. \tag{2.60}$$

Analogously, we can prove that

$$\left(\gamma^n \mathfrak{Im}\left[(x-a)^{j\alpha} e_{\alpha,j}^{\lambda\gamma(x-a)}\right]\right)_{n=0}^{\infty} \in \mathbf{E}_N^0(a,b) \cap [\Delta^{\infty\alpha}(a,b)]^{\mathbb{N}} \tag{2.61}$$

Finally, from Proposition 5, we know that:

$$\left\{\left(\gamma^n \mathfrak{Re}\left[(x-a)^{j\alpha} e_{\alpha,j}^{\lambda\gamma(x-a)}\right]\right)_{n=0}^{\infty}\right\}_{j=0}^{\ell_1-1} \bigcup \left\{\left(\gamma^n \mathfrak{Im}\left[(x-a)^{j\alpha} e_{\alpha,j}^{\lambda\gamma(x-a)}\right]\right)_{n=0}^{\infty}\right\}_{j=0}^{\ell_1-1}, \tag{2.62}$$

is a linearly independent set.

Corollary 7. *Let $\{r_j;\overline{r_j}\}_{j=1}^{p}$ be ($r_j = b_j + ic_j$) the set of conjugate complex roots of $P_N(\lambda)$ with multiplicity $\{\sigma_j\}_{j=1}^{p}$, respectively, such that $2\sum_{j=1}^{p}\sigma_j = N$, then the sequences of functions*

$$\bigcup_{k=1}^{p}\left\{\left(\gamma^n \mathfrak{Re}\left[\gamma^n (x-a)^{j\alpha} e_{\alpha,j}^{r_k\gamma(x-a)}\right]\right)_{n=0}^{\infty}\right\}_{j=0}^{\sigma_k-1} \quad \text{and} \quad \bigcup_{k=1}^{p}\left\{\left(\gamma^n \mathfrak{Im}\left[\gamma^n (x-a)^{j\alpha} e_{\alpha,j}^{r_k\gamma(x-a)}\right]\right)_{n=0}^{\infty}\right\}_{j=0}^{\sigma_k-1} \tag{2.63}$$

form a fundamental set of solution to equation (2.4).

Proof. The proof is similar to that of Corollary 6.

Theorem 10. *Let $\{\lambda_j\}_{j=1}^{M}$ be roots of $P_N(\lambda)$, with multiplicity $\{\ell_j\}_{j=1}^{M}$ respectively, and let $\{r_j;\overline{r_j}\}_{j=1}^{p}$ be ($r_j = b_j + ic_j$) the set of pairs conjugate complex roots of $P_N(\lambda)$ with multiplicity $\{\sigma_j\}_{j=1}^{p}$, respectively, such that $\sum_{j=1}^{M}\ell_j + 2\sum_{j=1}^{p}\sigma_j = N$, then the sequence of functions*

$$\bigcup_{k=1}^{M}\left\{\left(\gamma^n (x-a)^{j\alpha} e_{\alpha,j}^{\lambda_k\gamma(x-a)}\right)_{n=0}^{\infty}\right\}_{j=0}^{\ell_j-1}; \tag{2.64}$$

$$\bigcup_{k=1}^{p}\left\{\left(\gamma^n \mathfrak{Re}\left[\gamma^n e_{\alpha,j}^{r_k\gamma(x-a)}\right]\right)_{n=0}^{\infty}\right\}_{j=0}^{\sigma_k-1} \tag{2.65}$$

and

$$\bigcup_{k=1}^{p}\left\{\left(\gamma^n \mathfrak{Im}\left[\gamma^n e_{\alpha,j}^{r_k\gamma(x-a)}\right]\right)_{n=0}^{\infty}\right\}_{j=0}^{\sigma_k-1} \tag{2.66}$$

$\gamma \neq 0$, they form a fundamental set of solutions to (2.4).

Proof. The proof of Theorem follows immediately from the Corollaries 5 and 7.

Example 1. We will consider the following LFDERR:

$$\left(\mathscr{D}_{a+}^{2\alpha} y_n\right)(x) - y_{n+2}(x) = 0. \tag{2.67}$$

We have that $P_2(\lambda) = (\lambda-1)(\lambda+1)$; from Corollary 5, the equation (2.67) has the following fundamental set of solutions:

$$\left\{\left(\gamma^n e_\alpha^{\gamma(x-a)}\right)_{n=0}^{\infty}; \left(\gamma^n e_\alpha^{-\gamma(x-a)}\right)_{n=0}^{\infty}\right\}. \tag{2.68}$$

To verify that statement is sufficient to take $y_n(x) = \gamma^n e_\alpha^{\gamma\lambda(x-a)}$; and from (1.15), we obtain:

$$\left[\mathscr{D}_{a+}^{2\alpha}\left(\gamma^n e_\alpha^{\gamma\lambda(t-a)}\right)\right](x) - \gamma^{n+2} e_\alpha^{\gamma\lambda(x-a)} = \gamma^{n+2}\lambda^2 e_\alpha^{\gamma\lambda(x-a)} - \gamma^{n+2} e_\alpha^{\gamma\lambda(x-a)} = 0, \tag{2.69}$$

if $\lambda = 1$ or $\lambda = -1$. From Corollary 1 (when $\ell_1 = \ell_2 = 1$ and $M = 2$), we know that the functions $e_\alpha^{\lambda(x-a)}$ and $e_\alpha^{-\lambda(x-a)}$ are linearly independent. Hence, we obtain the sequences of functions $\left(\gamma^n e_\alpha^{\gamma(x-a)}\right)_{n=0}^{\infty}$ and $\left(\gamma^n e_\alpha^{-\gamma(x-a)}\right)_{n=0}^{\infty}$ that are linearly independent. In addition, we can see in (2.69) that both sequences verify the equation (2.67).

Example 2. Now, we will consider the following LFDERR:

$$\left(\mathscr{D}_{a+}^{2\alpha} y_n\right)(x) - 2\left(\mathscr{D}_{a+}^{\alpha} y_{n+1}\right)(x) + y_{n+2}(x) = 0. \tag{2.70}$$

From Corollary 5, this equation has the following fundamental set of solutions

$$\left\{\left(\gamma^n e_\alpha^{\gamma(x-a)}\right)_{n=0}^{\infty}; \left(\gamma^n (x-a)^{\alpha} e_{\alpha,1}^{-\gamma(x-a)}\right)_{n=0}^{\infty}\right\}, \tag{2.71}$$

since $P_2(\lambda) = (\lambda-1)^2$, i.e. λ is a root of multiplicity 2. Proceeding analogously to the previous example, it can be verified that the sequences in (2.71) solve (2.70).

Example 3. Given the following EDFLSRR:

$$\left(\mathscr{D}_{a+}^{2\alpha}y_n\right)(x)+\nu^2y_{n+2}(x)=0,\ \ (\nu>0), \tag{2.72}$$

we have that the characteristic polynomial associated with this equation $P_{2,\gamma}(\lambda)=(\lambda-i\nu\gamma)(\lambda+i\nu\gamma)$; while $P_2(\lambda)=(\lambda-i\nu)(\lambda+i\nu)$. Then, according to Corollary 6, we can obtain the fundamental set of solutions

$$\left\{\left(\gamma^n\mathfrak{Re}\left[e_{\alpha,0}^{\nu\gamma(x-a)}\right]\right)_{n=0}^{\infty}\ ;\ \left(\gamma^n\mathfrak{Im}\left[e_{\alpha,0}^{\nu\gamma(x-a)}\right]\right)_{n=0}^{\infty}\right\}. \tag{2.73}$$

2.2 Solution to the nonhomogeneous LFDERR

For this section, we will consider the affirmation of Theorem 9.

Lemma 5. *Let $f_n(x)\in\Delta^{\alpha\infty}(a,b)$ be, for each $n\in\mathbb{N}_0$. A solution to the LFDERR of order α:*

$$a_0y_{n+1}(x)-\left(\mathscr{D}_{a+}^{\alpha}y_n\right)(x)=f_n(x)\ \ \ (n\geq1) \tag{2.74}$$

is given by the sequence $(y_n(x))_{n=1}^{\infty}$, with

$$y_n(x)=\sum_{j=0}^{n-1}{a_0}^{j-n}\left(\mathscr{D}_{a+}^{(n-j-1)\alpha}f_j\right)(x). \tag{2.75}$$

Proof. It results by verification: merely replacing (2.75) in (2.74):

$$\begin{aligned}&\left(\mathscr{D}_{a+}^{\alpha}y_n\right)(x)-a_0y_{n+1}(x)=\\&=\left[\mathscr{D}_{a+}^{\alpha}\left(\sum_{j=0}^{n-1}{a_0}^{j-n}\left(\mathscr{D}_{a+}^{(n-j-1)\alpha}f_j\right)(t)\right)\right](x)-a_0\sum_{j=0}^{n}{a_0}^{j-n-1}\left(\mathscr{D}_{a+}^{(n-j)\alpha}f_j\right)(x)=\\&=\sum_{j=0}^{n-1}{a_0}^{j-n}\left(\mathscr{D}_{a+}^{(n-j)\alpha}f_j\right)(x)-\left\{\sum_{j=0}^{n-1}{a_0}^{j-n}\left(\mathscr{D}_{a+}^{(n-j)\alpha}f_j\right)(x)+f_n(x.)\right\}=-f_n(x).\end{aligned} \tag{2.76}$$

Theorem 11. *Let $f_n(x)\in\Delta^{\alpha\infty}(a,b)$, $n\in\mathbb{N}_0$. A solution to the LFDERR of order α,*

$$a_0y_{n+1}(x)-\left(\mathscr{D}_{a+}^{\alpha}y_n\right)(x)=f_n(x)\ \ \ (n\in\mathbb{N}_0) \tag{2.77}$$

is given by $(y_n(x))_{n=0}^{\infty}$, with

$$y_n(x)=\begin{cases}-z_n(x)+\sum_{j=0}^{n-1}{a_0}^{j-n}\left(\mathscr{D}_{a+}^{(n-j-1)\alpha}f_j\right)(x) & \text{if } n\geq1,\\ -z_0(x) & \text{if } n=0,\end{cases} \tag{2.78}$$

where $(z_n(x))_{n=0}^{\infty}$ is a solution to the Homogeneous LFDERR of first order

$$\left(\mathscr{D}_{a+}^{\alpha}z_n\right)(x)-a_0z_{n+1}(x)=0. \tag{2.79}$$

Proof. Let $n=0$ be:

$$y_1(x)=-z_1(x)+a_0^{-1}f_0(x) \tag{2.80}$$

Since $(z_n(x))_{n=0}^{\infty}$ is solution of (2.79), the result is

$$z_1(x)=a_0^{-1}\left(\mathscr{D}_{a+}^{\alpha}z_0\right)(x). \tag{2.81}$$

Replacing (2.81) in (2.80), we have:

$$y_1(x) = -a_0^{-1}\left(\mathscr{D}_{a+}^{\alpha} z_0\right)(x) + a_0^{-1} f_0(x) = a_0^{-1}\left(\mathscr{D}_{a+}^{\alpha} y_0\right)(x) + a_0^{-1} f_0(x), \tag{2.82}$$

i.e.

$$a_0 y_1(x) - \left(\mathscr{D}_{a+}^{\alpha} y_0\right)(x) = f_0(x). \tag{2.83}$$

Now, let $n \geq 1$ be:

$$y_{n+1}(x) = -z_{n+1}(x) + \sum_{j=0}^{n} a_0^{j-n-1}\left(\mathscr{D}_{a+}^{(n-j)\alpha} f_j\right)(x). \tag{2.84}$$

Since $(z_n(x))_{n=0}^{\infty}$ is a solution of (2.79), we obtain the following:

$$z_{n+1}(x) = a^{-1}\mathscr{D}_{a+}^{\alpha} z_n(x). \tag{2.85}$$

Replacing (2.85) in (2.84), we have:

$$\begin{aligned} y_{n+1}(x) &= -a_0^{-1}\left(\mathscr{D}_{a+}^{\alpha} z_n\right)(x) + \sum_{j=0}^{n} a_0^{j-n-1}\left(\mathscr{D}_{a+}^{(n-j)\alpha} f_j\right)(x) = \\ &= -a_0^{-1}\left(\mathscr{D}_{a+}^{\alpha} z_n\right)(x) + \left[\sum_{j=0}^{n-1} a_0^{j-n-1}\left(\mathscr{D}_{a+}^{(n-j)\alpha} f_j\right)(x) + a_0^{-1} f_n(x)\right] = \\ &= a_0^{-1}\left[-\left(\mathscr{D}_{a+}^{\alpha} z_n\right)(x) + \sum_{j=0}^{n-1} a_0^{j-n}\left(\mathscr{D}_{a+}^{(n-j)\alpha} f_j\right)(x)\right] + a_0^{-1} f_n(x) = \\ &= a_0^{-1}\mathscr{D}_{a+}^{\alpha}\left[-z_n(t) + \sum_{j=0}^{n-1} a_0^{j-n}\left(\mathscr{D}_{a+}^{(n-j-1)\alpha} f_j\right)(t)\right](x) + a_0^{-1} f_n(x). \end{aligned} \tag{2.86}$$

Therefore, applying (2.78) in (2.86):

$$y_{n+1}(x) = a_0^{-1}\left(\mathscr{D}_{a+}^{\alpha} y_n\right)(x) + a_0^{-1} f_n(x), \tag{2.87}$$

i.e.,

$$a_0 y_{n+1}(x) - \left(\mathscr{D}_{a+}^{\alpha} y_n\right)(x) = f_n(x) \tag{2.88}$$

is valid.

Corollary 8. *Let $f_n(x) \in \Delta^{\alpha\infty}(a,b)$, $n \in \mathbb{N}_0$, and let $\gamma \neq 0$. A solution of (2.77) is given by $(y_n(x))_{n=0}^{\infty}$, with*

$$y_n(x) = \begin{cases} -\gamma^n e_{\alpha}^{a_0\gamma(x-a)} + \sum_{j=0}^{n-1} a_0{}^{j-n}\left(\mathscr{D}_{a+}^{(n-j-1)\alpha} f_j\right)(x) & \text{if } n \geq 1, \\ -e_{\alpha}^{a_0\gamma(x-a)} & \text{if } n = 0. \end{cases} \tag{2.89}$$

Proof. From Proposition 7, we know that, $(z_n(x))_{n=0}^{\infty}$, where

$$z_n(x) = y_n^{1,0}(x) = \gamma^n (x-a)^{(0)\alpha} e_{\alpha,0}^{-a_0\gamma(x-a)} = \gamma^n e^{a_0\gamma(x-a)}, \tag{2.90}$$

is solution of

$$\left(\mathscr{D}_{a+}^{\alpha} z_n\right)(x) - a_0 z_{n+1}(x) = 0. \tag{2.91}$$

Then, from Theorem 11, the proof is completed.

Proposition 8. *Let* $f_n(x) = \gamma^n f_0(x)$*, where* $f_0 \in L_1(a,b) \cap C(a,b)$*, and* $\gamma \neq 0$*. Then, the equation*

$$\left(\mathscr{D}^{\alpha}_{a+} y_n\right)(x) - \lambda y_{n+1}(x) = f_n(x) \; (n \in \mathbb{N}_0, \; x > a) \tag{2.92}$$

admits as a solution $(y_n(x))_{n=0}^{\infty}$*, where*

$$y_n(x) = c\gamma^n e_{\alpha}^{\gamma\lambda x} + y_n^p(x) \tag{2.93}$$

with

$$y_n^p(x) = \gamma^n e_{\alpha}^{\lambda\gamma x} *^a f(x) \tag{2.94}$$

where $\left(y_n^p(x)\right)_{n=0}^{\infty}$ *is a particular solution of (2.92), c is an arbitrary constant, and* $*^a$ *represents the convolution:*

$$(g *^a f)(x) = \int_a^x g(x-t) f(t) dt. \tag{2.95}$$

Proof. From Proposition 7, since λ is a root of the characteristic polynomial associated with the equation

$$\left(\mathscr{D}^{\alpha}_{a+} y_n\right)(x) - \lambda y_{n+1}(x) = 0, \tag{2.96}$$

we know that $\left(\gamma^n e_{\alpha}^{\gamma\lambda(x-a)}\right)_{n=0}^{\infty}$ is a solution of (2.96). Then, from Theorem 9, it is enough to verify that $(y_n^p(x))_{n=0}^{\infty}$ solves (2.92). To prove that $(y_n(x))_{n=0}^{\infty}$ is solution of (2.92), applying Lemma 1 and taking into account (1.16), we obtain:

$$\begin{aligned}
\left(\mathscr{D}^{\alpha}_{a+} y_n^p\right)(x) = \mathscr{D}^{\alpha}_{a+}\left[\gamma^n e_{\alpha}^{\lambda\gamma t} *^a f(t)\right](x) \; &= \; \mathscr{D}^{\alpha}_{a+}\left[\int_a^t \gamma^n e_{\alpha}^{\lambda(\tau-a)} f(\tau) d\tau\right](x) = \\
= \gamma^n \int_a^x \left\{\mathscr{D}^{\alpha}_{a+} e_{\alpha}^{\lambda\gamma(t-a)}\right\}(\tau) f(x-\tau+a) d\tau &+ \gamma^n f(x) \underbrace{\lim_{x\to a+}\left\{I^{1-\alpha}_{a+} e_{\alpha}^{\lambda\gamma(t-a)}\right\}(x)}_{=1} = \\
= \lambda\gamma^{n+1} \int_a^x e_{\alpha}^{\lambda\gamma(\tau-a)} f(x-\tau+a) d\tau + \gamma^n f(x) \; &= \; \lambda\left[\gamma^{n+1} e_{\alpha}^{\lambda\gamma\tau} *^a f(\tau)\right](x) + \gamma^n f(x) = \\
&= \lambda y_{n+1}^p(x) + \gamma^n f(x).
\end{aligned} \tag{2.97}$$

Theorem 12. *Let* $f_0 \in L_1(a,b) \cap C(a,b)$*,* $f_n(x) = \gamma^n f_0(x)$ *and* $\gamma \neq 0$*. Then, a particular solution of (2.3):*

$$[\mathbf{R}_{N\alpha}(y_n)_{n=0}^{\infty}](x) = f_n(x) \;\; (n \in \mathbb{N}_0, \; x > a), \tag{2.98}$$

is given by $(y_n^p(x))_{n=0}^{\infty}$ *with*

$$y_n^p(x) = \gamma^n (G_{\alpha,\gamma} *^a f_0)(x) \tag{2.99}$$

with

$$G_{\alpha,\gamma}(x) = \gamma^n \prod_{j=1}^{M}{}^{*^a} \left(\prod_{k=1}^{\ell_k}{}^{*^a} e_{\alpha}^{\lambda_j\gamma(x-a)}\right) \tag{2.100}$$

where $\{\lambda_j\}_{j=1}^{M}$ *are the M distinct roots of multiplicity* $\{\ell_k\}_{k=1}^{M}$ *of the characteristic polynomial (1.25), respectively, i.e.:*

$$P_N(\lambda) = (\lambda-\lambda_1)^{\ell_1}(\lambda-\lambda_2)^{\ell_2}\ldots(\lambda-\lambda_M)^{\ell_M}, \tag{2.101}$$

and $\ell_1 + \ell_2 + \ldots + \ell_M = N$.

Proof. Assuming the existence of $y(x)$ such that, for each $n \in \mathbb{N}_0$, we can write

$$y_n(x) = \gamma^n y(x); \tag{2.102}$$

and replacing (2.102) in (2.98), and taking into account that $f_n(x) = \gamma^n f_0(x)$, it results

$$\left(\mathscr{D}^{N\alpha}_{a+} \gamma^n y\right)(x) + \sum_{j=1}^{N} a_j \left(\mathscr{D}^{(N-j)\alpha}_{a+} \gamma^{n+j} y\right)(x) = \gamma^n f_0(x). \tag{2.103}$$

Thus,

$$
\begin{aligned}
f_0(x) = \gamma^N \left\{ \left(\mathscr{D}_{a+}^{N\alpha}\gamma^{-N}y\right)(x) + \sum_{j=1}^{N} a_j \left(\mathscr{D}_{a+}^{(N-j)\alpha}\gamma^{j-N}y\right)(x) \right\} = \\
= \gamma^N \left\{ \left[\left(\gamma^{-1}\mathscr{D}_{a+}^{\alpha}\right)^N y\right](x) + \sum_{j=1}^{N-1} a_j \left[\left(\gamma^{-1}\mathscr{D}_{a+}^{\alpha}\right)^{(N-j)} y\right](x) \right\} = \\
= \gamma^N \left\{ \left(\gamma^{-1}\mathscr{D}_{a+}^{\alpha}\right)^N + \sum_{j=1}^{N-1} a_j \left(\gamma^{-1}\mathscr{D}_{a+}^{\alpha}\right)^{(N-j)} \right\} y(x) = \\
= \gamma^N \left\{ P_N\left(\gamma^{-1}\mathscr{D}_{a+}^{\alpha}\right)\right\} y(x) \;=\; \gamma^{1+\ell_2+\ldots+\ell_M} P_N\left(\gamma^{-1}\mathscr{D}_{a+}^{\alpha}\right) y(x) = \\
= \gamma^{\ell_1+\ell_2+\ldots+\ell_M} \left(\gamma^{-1}\mathscr{D}_{a+}^{\alpha} - \lambda_1\right)^{\ell_1} \left(\gamma^{-1}\mathscr{D}_{a+}^{\alpha} - \lambda_2\right)^{\ell_2} \ldots \left(\gamma^{-1}\mathscr{D}_{a+}^{\alpha} - \lambda_M\right)^{\ell_M} y(x) = \\
= \left(\mathscr{D}_{a+}^{\alpha} - \gamma\lambda_1\right)^{\ell_1} \left(\mathscr{D}_{a+}^{\alpha} - \gamma\lambda_2\right)^{\ell_2} \ldots \left(\mathscr{D}_{a+}^{\alpha} - \gamma\lambda_M\right)^{\ell_M} y(x) = \\
= \left(\prod_{j=1}^{M} \left(\mathscr{D}_{a+}^{\alpha} - \lambda_j\gamma\right)^{\ell_j} y \right)(x). \qquad (2.104)
\end{aligned}
$$

Then, from Theorem 2, we conclude:

$$
y(x) = \left(\prod_{j=1}^{M}{}^{*^a} \left(\prod_{k=1}^{\ell_k}{}^{*^a} e_{\alpha}^{\lambda_j\gamma(t-a)} \right) *^a f_0 \right)(x). \qquad (2.105)
$$

From (2.102) and (2.105), the thesis is obtained.

The following result establishes a relationship between the LFDERR and the LFDE.

Theorem 13. *Let $y(x)$ be a solution to the Homogeneous LFDERR (1.23). Then, a solution of (2.2) is given by $(y_n(x))_{n=0}^{\infty}$ with $y_n(x) = y(x)$, $n \in \mathbb{N}_0$. Moreover,*

$$
\left[\mathbf{R}_{N\alpha}(y_n(t))_{n=0}^{\infty}\right](x) = \left[\mathbf{L}_{N\alpha}(y)\right](x). \qquad (2.106)
$$

Proof. The proof is evident.

3 Reduction of the LFDERR to a recurrence relationship

Let us consider the following sequence of functions:

$$
(y_n(x))_{n=0}^{\infty} = \left(z_n e_{\alpha}^{x-a}\right)_{n=0}^{\infty}, \qquad (3.1)
$$

where $(z_n)_{n=0}^{\infty}$ is a numerical sequence. From (2.4) we obtain:

$$
\begin{aligned}
\left[\mathbf{R}_{N\alpha}\left(z_n e_{\alpha}^{x-a}\right)_{n=0}^{\infty}\right](x) = \left(\mathscr{D}_{a+}^{N\alpha} z_n e_{\alpha}^{t-a}\right)(x) + \sum_{j=1}^{N} a_{N-j} \left(\mathscr{D}_{a+}^{(N-j)\alpha} z_{n+j} e_{\alpha}^{t-a}\right)(x) = \\
= z_n \left(\mathscr{D}_{a+}^{N\alpha} e_{\alpha}^{t-a}\right)(x) + \sum_{j=1}^{N} a_{N-j} z_{n+j} \left(\mathscr{D}_{a+}^{(N-j)\alpha} e_{\alpha}^{t-a}\right)(x) = z_n e_{\alpha}^{x-a} + \sum_{j=1}^{N} a_{N-j} z_{n+j} e_{\alpha}^{x-a} = \\
= \left(z_n + \sum_{j=1}^{N} a_{N-j} z_{n+j} \right) e_{\alpha}^{x-a}. \qquad (3.2)
\end{aligned}
$$

Then, from (3.2), the succession of functions (3.1) solve (2.4) and must be $(z_n)_{n=0}^{\infty}$ a solution of the recurrence equation:

$$
z_n + \sum_{j=1}^{N} a_{N-j} z_{n+j} = 0. \qquad (3.3)
$$

If we call $b_N = a_0^{-1}$ and $b_{N-j} = a_{N-j}a_0^{-1}$, for each $1 \le j \le N-1$, from (3.3), we obtain the following Linear Difference Equation[2]:

$$z_{n+N} + b_1 z_{n+N-1} + \ldots + b_{N-1}z_{n+1} + b_N z_n = 0 \tag{3.4}$$

Accordingly, we show the following Lemma.

Lemma 6. *If $(z_n)_{n=0}^{\infty}$ is a solution to (3.3), then $(z_n e_\alpha^{x-a})_{n=0}^{\infty}$ is a solution to (2.4).*

On the other hand, we know that if $\{\lambda_j\}_{j=1}^M$, are M different roots of multiplicity $\{\ell_j\}_{j=1}^M$, respectively, of the characteristic polynomial

$$\mathbf{P}_N(\lambda) = \lambda^n + \sum_{j=1}^{N} a_{N-j}\lambda^{n+j}, \tag{3.5}$$

associated with (3.3), then,

$$\bigcup_{k=1}^{M} \left\{ \left(\binom{n}{j} \lambda_k^{n-j} \right)_{n=0}^{\infty} \right\}_{j=0}^{\ell_k - 1} \tag{3.6}$$

is a fundamental set of solutions of (3.3), and since e_α^{x-a} is independent of n, fromm Lemma 6, the following theorem is proved.

Theorem 14. *If $\{\lambda_j\}_{j=1}^M$, are M different roots of multiplicity $\{\ell_j\}_{j=1}^M$, respectively, of (3.5). Then,*

$$\bigcup_{k=1}^{M} \left\{ \left(\binom{n}{j} \lambda_k^{n-j} e_\alpha^{x-a} \right)_{n=0}^{\infty} \right\}_{j=0}^{\ell_k - 1} \tag{3.7}$$

is a fundamental set of solutions (2.4).

Example 4. We will compare the solutions of an LFDE and a linear difference equation of first order, in the following sense

$$\left(\mathscr{D}_{a+}^{\alpha} y_n\right)(x) + a_0 y_{n+1}(x) = \gamma^n f_0(x), \tag{3.8}$$

with $x \in (a,b)$, and a $x_0 \in (a,b)$:

$$y_n(x_0) + a_0 y_{n+1}(x_0) = \gamma^n f_0(x_0). \tag{3.9}$$

The above mentioned example can be written, as follows:

$$(-a_0)y_{n+1}(x) - \left(\mathscr{D}_{a+}^{\alpha} y_n\right)(x) = -\gamma^n f_0(x) \tag{3.10}$$

and

$$y_{n+1}(x_0) - (-a_0^{-1})y_n(x_0) = a^{-1}\gamma^n f_0(x_0). \tag{3.11}$$

In (3.10) and (3.11), we will call $\gamma = -a_0^{-1}$ y $f_0(x) = e_\alpha^{x-a}$:

$$\gamma^{-1} y_{n+1}(x) - \left(\mathscr{D}_{a+}^{\alpha} y_n\right)(x) = -\gamma^n e_\alpha^{x-a} \tag{3.12}$$

and

$$y_{n+1}(x_0) - \gamma y_n(x_0) = -\gamma^{n+1} e_\alpha^{x_0-a}. \tag{3.13}$$

The equation (3.12) is of the type stated in (2.77), while (3.13) is an Linear Difference Equation of first order (see, for example [14]). From Corollary 8 we know that the solution to (3.12) is given by $(y_n(x))_{n=0}^{\infty}$ where

$$y_n(x) = \begin{cases} -\gamma^n e_\alpha^{x-a} - \sum_{j=0}^{n-1} \gamma^{n-j} \left(\gamma^j e_\alpha^{x-a}\right) & \text{if } n \ge 1, \\ -e_\alpha^{a-x} & \text{if } n = 0. \end{cases} \tag{3.14}$$

Then, if we fix $x = x_0$ and $y_0(x_0) = -e_\alpha^{x_0-a}$ in (3.14), we obtain

$$y_n(x_0) = \begin{cases} \gamma^n y_0(x_0) + \sum_{j=0}^{n-1} \gamma^{n-j-1} \left(-\gamma^{j+1} e_\alpha^{x_0-a}\right) & \text{if } n \ge 1, \\ y_0(x_0) & \text{if } n = 0. \end{cases} \tag{3.15}$$

We know (see, for instance [14]) that $(y_n(x_0))_{n=0}^{\infty}$ represents a solution to (3.13), under the initial condition $y_0(x_0) = -e_\alpha^{x_0-a}$.

[2] Also called recurrence equation (see, for example [14])

Example 5. We will analyze the following initial values problem:

$$\begin{cases} \left(\mathscr{D}_{0+}^{2\alpha} y_n\right)(x) + \left(\mathscr{D}_{0+}^{\alpha} y_{n+1}\right)(x) - y_{n+2}(x) = 0 \\ y_0(x) = 0 \\ y_1(x) = \frac{e_\alpha^x}{e_\alpha^1}, \end{cases} \tag{3.16}$$

where $n \in \mathbb{N}_0,\ x \in (0,+\infty)$. The equation

$$\left(\mathscr{D}_{0+}^{2\alpha} y_n\right)(x) + \left(\mathscr{D}_{0+}^{\alpha} y_{n+1}\right)(x) - y_{n+2}(x) = 0 \tag{3.17}$$

is a LFDERRH of order 2α, and its associated characteristic polynomial is $P_2(\lambda) = \lambda^2 + \lambda - 1$, whose roots are $\lambda_1 = -\frac{1-\sqrt{5}}{2}$ and $\lambda_1 = -\frac{1+\sqrt{5}}{2}$. By Corollary 5 it is known that, if $\gamma \neq 0$, the sequences

$$\left(\gamma^n e_\alpha^{\lambda_1 \gamma x}\right)_{n=0}^{\infty} \quad \text{and} \quad \left(\gamma^n e_\alpha^{\lambda_2 \gamma x}\right)_{n=0}^{\infty} \tag{3.18}$$

are two linearly independent solutions of the equation (3.17). In particular,

$$\left((-\lambda_2)^n e_\alpha^{-\lambda_1 \lambda_2 x}\right)_{n=0}^{\infty} \quad \text{and} \quad \left((-\lambda_1)^n e_\alpha^{-\lambda_2 \lambda_1 x}\right)_{n=0}^{\infty} \tag{3.19}$$

also form (see Theorem 14) a fundamental set of solutions of (3.17). Since $\lambda_1\lambda_2 = -1$, the solutions in (3.19) can be written as

$$\left((-\lambda_2)^n e_\alpha^x\right)_{n=0}^{\infty} \quad \text{and} \quad \left((-\lambda_1)^n e_\alpha^x\right)_{n=0}^{\infty}. \tag{3.20}$$

In addition, according to Lemma 3, the solution $(y_n(x))_{n=0}^{\infty}$ of the equation (3.17) may be written as:

$$y_n(x) = A(-\lambda_2)^n e_\alpha^x + B(-\lambda_1)^n e_\alpha^x = \left[A(-\lambda_2)^n + B(-\lambda_1)^n\right] e_\alpha^x, \tag{3.21}$$

where $x \in (0,+\infty)$, A and B arbitrary constants. By the initial conditions, we have

$$\begin{cases} y_0(x) = (A+B)\, e_\alpha^x = 0 \\ y_1(x) = \left[A(-\lambda_2) + B(-\lambda_1)\right] e_\alpha^x = \frac{e_\alpha^x}{e_\alpha^1}, \end{cases} \tag{3.22}$$

i.e. $A = -B = \left(\sqrt{5}\, e_\alpha^1\right)^{-1}$. Therefore, it follows that the solution to the IVP (3.16) is given by $(y_n(x))_{n=0}^{\infty}$ with

$$y_n(x) = \left[\frac{1}{\sqrt{5}\, e_\alpha^1}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}\, e_\alpha^1}\left(\frac{1-\sqrt{5}}{2}\right)^n\right] e_\alpha^x, \tag{3.23}$$

where $x \in (0,+\infty)$.

Finally, it can be seen in (3.23) that the conditions $y_0(1) = 0$ and $y_1(1) = 1$ are verified, then we have

$$F_n = y_n(1) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \tag{3.24}$$

that is the general expression of the well-known Fibonacci sequence :

$$\begin{cases} F_{n+2} = F_n + F_{n+1} \quad,\ n \geq 2 \\ F_0 = 0 \\ F_1 = 1. \end{cases} \tag{3.25}$$

4 Conclusion

It was possible to define and prove a new type of equation. In different cases, it was shown that it was possible to solve these equations by means of the α-Exponential Function. Moreover, it was possible to establish relationships between LFDE and RE through this solution, rethink about the already known problems, and study them using LFDE or RE. Furthermore, we obtained a “theoretical interpolation” between the theories already known, from the point of view of the LFDERR.

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