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Ritz Method and Genocchi Polynomials for Solution of Fractional Partial Differential Equations

Mohammad Arab Firoozjaee

Department of Mathematics, University of Science and Technology of Mazandaran, P.O.Box 48518-78195, Behshahr, Iran

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Abstract: In this article, Ritz approximation based on Genocchi polynomials has been used to get numerical solutions large category of FPDEs (fractional partial differential equations). In this work, we use of Caputo fractional derivatives. In first, we transform FDEs into an optimization problem by Genocchi polynomial and achieve the nonlinear algebraic equation system. By solving this system, we get the polynomial expansion's coefficients. Then, we will show that the Ritz approximation based on Genoscchi polynomial is very efficient for a wide class of FPDEs. Some numerical instances are provided in the following to show the Genocchi polynomials efficiency.

Keywords: Caputo fractional derivative, fractional partial differential equations, Genocchi polynomials, Ritz method, the boundary and initial value fractional problem.

1 Introduction

Polynomials are one of the most important parts of mathematics and have many uses. In this work, we will use Genochi polynomials that Which has interesting properties. Extensively, these polynomials were employed in several various contexts in mathematics branches like complex analytic number theory and elementary number theory. Genochi polynomials are greatly established and utilized, such as the q–Genocchi polynomials advanced in [1, 2], their interpolating functions is also discussed in [3] and FDEs solutions based on Caputo–Fabrizio derived by operational matrix [4]. Genocchi polynomials are used as well in homotopy theory (spheres' constant homotopy groups), differential topology (differential structures on spheres), modular forms theory (Eisenstein series) and other fields. The classical Genocchi polynomials have shown with $G_n(x)$ and is generally explained through the exponential creating functions [2, 5].

$$\frac{2te^{xt}}{e^t+1} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \qquad (|t| < \pi), \tag{1}$$

that Genocchi polynomials of degree n are defined by

$$G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_k x^{n-k}. \tag{2}$$

The Genocchi polynomials of degree 1 to 4 are:

$G_1(x) = 1,$
$G_2(x) = 2x-1,$
$G_3(x) = 3x^2-2x,$
$G_4(x) = 4x^3-6x^2+1.$

* Corresponding author e-mail: m_firoozjaee@mazust.ac.ir

In this manuscript, Genocchi polynomials have been applied to obtain numerical solutions of FPDEs. Recently, the fractional differential equations have become one of the most important branches of mathematics. Therefore many researchers were interested to work in this field and are being progressively common owing to their practical claims in different fields of engineering and science [6,7]. The calculus of the fractional, in the integer order calculus, can describe the natural phenomena more realistic and efficiently better. The integer order PDEs' solutions explain the future modes of some physical procedures by understating their current conditions not dependent on their previous history. Nevertheless, for describing the hereditary and memory features of various materials and procedures, in which the previous states affect significantly way the future states, sufficient situation is not handled by integer–order models. In the stated cases, FPDEs offer superior instruments for describing the systems. For the first time Hopital in the Seventeenth century raised the theory of fractional calculus. After nearly four centuries numerous works were performed and several imperative books were presented in fractional calculs in which we can highlight Podlubny's books [8], Baleanu and Mustafa [9], Ortigueira [10] and Santanu Saha Ray [11]. By FDEs, most of the scientific phenomena and problems modeled. For example, in dynamical systems in control theory [12], mathematical chemical [13], in continuum and fluid mechanics [14], colored noises [15], Nuclear Reactor Dynamics [11].

Recently, many researchers have paid great attention to the theory of primary and boundary value problems for FPDE. According to the increasing number of current publications, the uniqueness and existence theory of the solutions to boundary value problems for fractional differential equations is obvious [16,17].

2 Ritz Method Based on Genocchi Polynomials for the Solution of FPDE

In this manuscript, the following is considered in the Fractional partial differential equations in the general form:

$$\begin{array}{l}\frac{\partial^{\alpha_1}u(x,t)}{\partial t^{\alpha_1}}+F\left(x,t,u(x,t),\frac{\partial^{\alpha_2}u(x,t)}{\partial t^{\alpha_2}},\ldots,\frac{\partial^{\alpha_m}u(x,t)}{\partial t^{\alpha_m}},\frac{\partial^{\gamma_1}u(x,t)}{\partial x^{\gamma_1}},\ldots,\frac{\partial^{\gamma_n}u(x,t)}{\partial x^{\gamma_n}}\right)=0,\\ i-1<\alpha_i\leq i, \qquad j-1<\gamma_j\leq j, \quad 0\leq t\leq 1, 0\leq x\leq 1,\end{array} \tag{3}$$

where $0<i\leq m$, $0<j\leq n$. We also assume boundary circumstances of the form $u(0,t)=f_0(t)$ and $u(1,t)=f_1(t)$ and a primary condition $u(x,0)=g(x)$, for $0<x<1$. In (3), $\frac{\partial^{\alpha_i}u(x,t)}{\partial t^{\alpha_i}}$ is the Caputo time–fractional derivative of order α_i and is determined as

$$\frac{\partial^{\alpha_i}u(x,t)}{\partial t^{\alpha_i}}=\frac{1}{\Gamma(i-\alpha_i)}\int_0^x\frac{\partial^i u(x,s)}{\partial s^i}(t-s)^{\alpha_i-i+1}ds. \tag{4}$$

$\frac{\partial^{\gamma_i}u(x,t)}{\partial x^{\gamma_i}}$ similar to definition(4). In this method, we convert the FPDE with boundary and primary conditions to optimizing problem and by Genocchi polynomial expand the solution with unidentified coefficients.

We consider $u_{ll'}(x,t)$ as approximation of $u(x,t)$.

$$u(x,t)\cong u_{ll'}(x,t)=\sum_{i=0}^{l}\sum_{j=0}^{l'}c_{ij}tx(x-1)G_i(x)G_j(t)+k(x,t), \tag{5}$$

where $G_i(x)$, $G_j(t)$ are Genocchi polynomials and c_{ij} are unknown coefficients. $k(x,t)$ is called "satisfier function" that, we define $k(x,t)$ satisfying the problem circumstances, known as "satisfying function". With the help of the interpolation function generally, $k(x,t)$ is selected as the following:

$$k(x,t)=g(x)+(1-x)(f_0(t)-f_0(0))+x(f_1(t)-f_1(0)).$$

Of course we can get it another way. It is necessary that $k(x,t)$ meets the problem condition.

Substituting(5) in (3)

$$\begin{array}{l}I[c_{00},\ldots,c_{0l'},\ldots,c_{l0},\ldots,c_{ll'}]=\int_0^1\int_0^1\left(\frac{\partial^{\alpha_1}u(x,t)}{\partial t^{\alpha_1}}+F\left(x,t,u(x,t),\frac{\partial^{\alpha_2}u(x,t)}{\partial t^{\alpha_2}},\ldots,\right.\right.\\ \left.\left.\frac{\partial^{\alpha_m}u(x,t)}{\partial t^{\alpha_m}},\frac{\partial^{\gamma_1}u(x,t)}{\partial x^{\gamma_1}},\ldots,\frac{\partial^{\gamma_n}u(x,t)}{\partial x^{\gamma_n}}\right)\right)^2dxdt.\end{array} \tag{6}$$

By the minimizing function I, we determine c_{ij}, then of (5) we get functions that approximate the minimum value of I in (6) and also this functions meets all boundary and initial circumstances. Then we use of the essential minimizing circumstances for the function (6), Hence, we will have following systems

$$\frac{\partial I}{\partial c_{ij}}=0,\ i=0,\ldots l\,, j=0,\ldots,l'. \tag{7}$$

By mathematica software, we solve this problem and get c_{ij}. This method is on the basis of the Ritz technique. For further information, an Iterested Researcher can see [18].

3 Illustrative Examples

In [18], we show that the method is convergence. In section show that if we use of Genocchi polynomials, convergence speed is very good. To show the efficiency of the suggested technique, in following we take into account some FPDEs.

Example 1:

Considering the fractional diffusion equation [19]

$$\frac{\partial u(x,t)}{\partial t} = h(x)\frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} + g(x,t), \quad 0 < x < 1, 0 < t \leq 1, \tag{8}$$

with primary condition: $u(x,0) = x^2(1-x)$, and boundary circumstances $u(0,t) = 0$ and $u(1,t) = 0$. For $h(x) = \Gamma(1.2)x^{1.8}$ and $g(x,t) = 3x^2(2x-1)e^{-t}$, the exact solution is

$$u(x,t) = x^2(1-x)e^{-t}.$$

In this problem, we used the presented method with $l = 2, l' = 2$ and solved (8).
we obtain unknown coefficients From (7)

$c_{00} = 0.40793, c_{01} = -0.000228957, c_{02} = 0.00019264,$
$c_{10} = 0.993683, c_{11} = -0.46253, c_{12} = 0.100529,$
$c_{20} = 0.000755434, c_{21} = -0.00372183, c_{22} = 0.00356356.$

To replace this coefficients in (5), we obtain

$$u(x,t) = (-0.602569t - 1.75107t^2 - 0.0691982t^3)x + 0.811497t + 1.66249t^2 + 0.0991357t^3)x^2 + (-1 - 0.208928t + 0.0885836t^2 - 0.0299375t^3)x^3 \ldots$$

This problem's absolute error is shown in Figure. 1. This figure shows that precise outcomes are provided by the current technique.

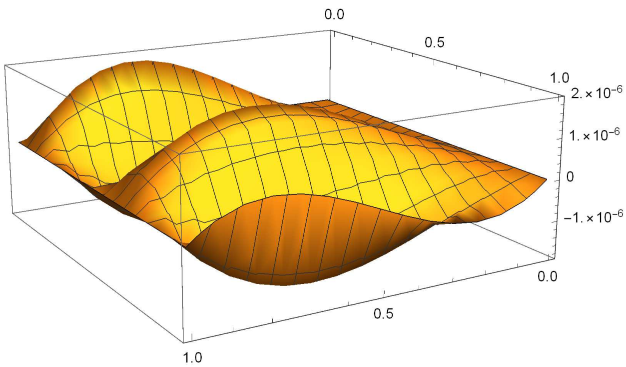

Fig.1. The Absolute Error within the numerical and exact solutions.

Example 2:

In this case, another FPDE is considered [20]

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{1}{2}x^2\frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < 1, 0 < t \leq 1, 0 < \alpha \leq 1, \tag{9}$$

subject to boundary conditions $u(0,t) = 0, u(1,t) = e^t$ and primary condition $u(x,0) = x^2$. For this problem with $\alpha = 1$, the exact solution is $u(x,t) = x^2e^t$.
We used the Ritz estimation that in section 2 explained. By solved system (9) with different valuesand of l, l', we attain

$$u(x,t) = \left(1.19696 \times 10^{-16}t - 9.71445 \times 10^{-17}t^2\right)x + \left(e^t - 2.1684 \times 10^{-16}t + 1.80411 \times 10^{-16}t^2\right)x^2 + \left(9.71445 \times 10^{-17}t - 8.32667 \times 10^{-17}t^2\right)x^3 + \ldots \cong x^2e^t.$$

Which obviously converges to the exact solution.
In Figure. 2, we provide the estimated solutions of $u(0.5,t)$ for $\alpha = 0.6, 0.8, 0.9, 1$ with considering $l = 1, l' = 1$ and compare $u(0.5,t)$ that is the exact solution. By seeing this figure, the effectiveness of the presented method demonstrate.

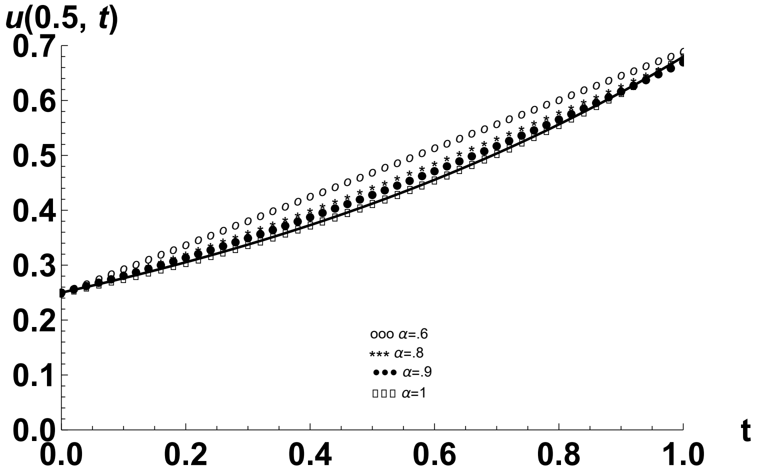

Fig2. Approximate (for $l = 1, l' = 1$) and exact solution(—) for $\alpha = 0.6, 0.8, 0.9, 1,$.

Example 3*:*

Considering the boundary and primary problem of FPDE [20]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + x\frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 u(x,t)}{\partial x^2} = 2t^\alpha + 2x^2 + 2, \quad 0 < \alpha \le 1, 0 < t \le 1, 0 < x < 1, \tag{10}$$

the primary condition $u(x,0) = x^2$ and the boundary circumstances:

$$u(0,t) = 2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}, \qquad u(1,t) = 1 + 2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha},$$

The exact solution of above problem is given in [21]

$$u(x,t) = x^2 + 2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}.$$

We used the Ritz estimate that in section 2 explained for $\alpha = 1$. By solved system (9) with different values and of l, l', we obtain

$c_{00} = 5.76757 \times 10^{-9}, c_{01} = -3.50534 \times 10^{-9}, c_{02} = 7.06229 \times 10^{-9}.$

To replace this coefficients in (5), we obtain

$$\begin{aligned} u(x,t) &= (-9.27291 \times 10^{-9}t + 2.81976 \times 10^{-8}t^2 - 2.11868 \times 10^{-8}t^3)x \\ &+ (1 + 9.27291 \times 10^{-8}t - 2.81975 \times 10^{-8}t^2 + 2.11868 \times 10^{-8}t^3)x^2 + t^2 \\ &+ \ldots \cong t^2 + x^2, \end{aligned}$$

Which obviously converges to the exact solution.
We show that if for α consider various value, the exact solution is obtained. For example, if consider $\alpha = 0.5$

$c_{00} = .80576 \times 10^{-10}, c_{01} = -2.011005 \times 10^{-10}, c_{02} = 3.83446 \times 10^{-10}.$

and then

$$\begin{aligned} u(x,t) &= 1.7724t + (2.81677 \times 10^{-10}t + 1.55254 \times 10^{-9}t^2 \\ &+ -1.15034 \times 10^{-9}t^3)x + (1 + 2.81677 \times 10^{-10}t - 1.55254 \times 10^{-9}t^2 \\ &+ 1.15034 \times 10^{-9}t^3)x^2 + \ldots \cong \frac{2\Gamma(0.5+1)}{\Gamma(2(0.5)+1)}t + x^2, \end{aligned}$$

and for $\alpha = 0.8$, we get

$c_{00} = 8.67468 \times 10^{-10}, c_{01} = -5.68841 \times 10^{-10}, c_{02} = 1.17515 \times 10^{-10}$.

and then

$$\begin{aligned} u(x,t) &= 1.30298t^{1.6} + (1 - 1.43631 \times 10^{-9}t + 4.66312 \times 10^{-9}t^2 \\ &- 3.52544 \times 10^{-9}t^3)x + (1 + 1.43630 \times 10^{-9}t - 4.66312 \times 10^{-9}t^2 \\ &- +3.52544 \times 10^{-9}t^3)x^2 + \dots \cong \frac{2\Gamma(0.8+1)}{\Gamma(2(0.8)+1)}t^{1.6} + x^2, \end{aligned}$$

that are the exact solutions.

Example 4:

In the latter example, the nonlinear time–fractional advection PDE is considered [22]

$$\begin{array}{c} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u(x,t)\frac{\partial u(x,t)}{\partial x} = x + xt^2, \\ t > 0,\ x \in [0,1],\ 0 < \alpha \le 1, \end{array} \tag{11}$$

subjected to the primary condition $u(x,0) = 0$ and the boundary circumstances:

$$u(0,t) = 0, \qquad u(1,t) = t.$$

For $\alpha = 1$, The exact solution is given $u(x,t) = xt$.
By using the technique explained in Section 2, we solved the problem and obtain

$$\begin{aligned} u(x,t) &= x(t + 1.64996 \times 10^{-9}t^2 + \dots) + x^2(5.51492 \times 10^{-9}t \\ &- 4.74718 \times 10^{-9}t^2 + \dots) + x^3(-3.63301 \times 10^{-9}t3.09739 \times 10^{-9}t^2 + \dots) \\ &\cong xt. \end{aligned}$$

In Figure. 3, to represent the exactitude and accuracy of presented method, we make a comparison between the numerical solutions for $x = 0.5$ and various values of α.

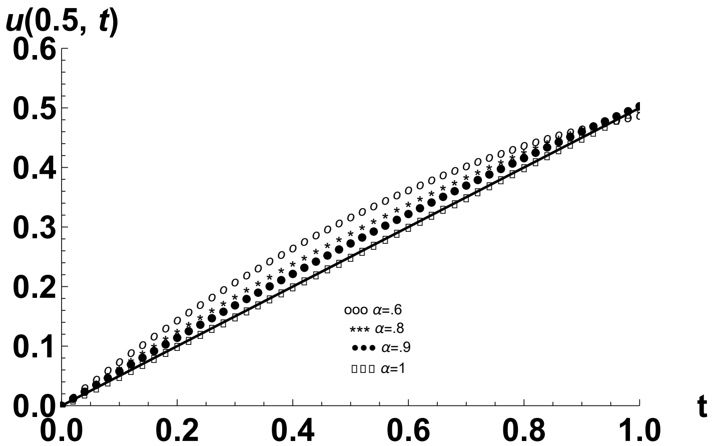

Fig3. approximate solution for $\alpha = 0.6, 0.8, 0.9, 1$, and Exact (—) solution for $\alpha = 1$ with $l = 1, l' = 1$..

4 Conclusion

Within this manuscript, we provide a numerical approach for a wide class of FPDEs. We demonstrate that the Ritz approximation based on Genoscchi polynomials are very effective for a wide class of FPDEs. Solving the equations' resultant system, the present estimated solution is defined, which can be effectually calculated utilizing on any personal computer. Demonstrative examples indicate that this technique based on Genocchi polynomials has high accuracy.

Conflict of Interest

The authors declare that they have no conflict of interest.

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