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On Relative m-Semi Logarithmically Convexity Functions.

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Abstract: We consider and study a new class of convex functions that are called relative $m-$semilogarithmic convex functions. Some Hermite-Hadamard inequalities for this a kind of generalized convex function are derived.

Keywords: relative $m-$semilogarithmic convex functions, Hermite-Hadamard inequalities

1 Introduction

It is well known that modern analysis directly or indirectly involve the applications of convexity (see [12]), due to its applications and significant importance.

Several generalizations have been introduced in recent years and extensions of the classical notion of convex function and in the theory of inequalities are produced important contributions in this regard (see [16,3,11,13, 14]).

In [11], Noor introduced a new class of convex set and convex function with respect to an arbitrary function; which are called relative convex set and relative convex function respectively, and in [10] established some Hadamard's type inequality for relative convex functions.

Let K be a nonempty closed set in a real Hilbert spaces H.

Definition 1([11]). *Let K_g be any set in H. The set K_g is said to be relative convex (g-convex) with respect to an arbitrary function $g : H \to H$ such that $(1-t)u+tg(v) \in K_g$, $\forall u,v \in H : u,g(v) \in K_g$, $t \in [0,1]$.*

Note that every convex set is relative convex, but the converse is not true.

Definition 2([11]). *A function $f : K_g \to H$ is said to be relative convex, if there exists an arbitrary function $g : H \to H$ such that*

$f((1-t)u+tg(v)) \leq (1-t)f(u)+tf(g(v)),$

$\forall u,v \in H : u,g(v) \in K_g, \quad t \in [0,1]$

Clearly every convex function is relative convex, but the converse is not true. The following inequality holds for any convex function f defined on $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \qquad (1)$$

It was firstly discovered by Hermite in 1881 in the journal Mathesis (see [4]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result [12]. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by Hadamard in 1893 [1]. In 1974,Mitrinovič found Hermites note in Mathesis [4]. In view of the fact that (1) was known as Hadamard's inequality, now the inequality is commonly referred as the Hermite- Hadamard inequality [12].

For the properties of relative convex functions and Hermite-Hadamard type inequalities (see [9,10,11]).

Theorem 1([10]). *Let $f : Kg = [a,g(b)] \to \mathbb{R}$ be a relative convex function. Then, we have*

$$f\left(\frac{a+g(b)}{2}\right) \leq \frac{1}{(g(b)-a)}\int_a^{g(b)} f(x)\,dx \leq \frac{f(a)+f(g(b))}{2}.$$

Noor in [8] introduced the class of relative h-convex functions and also discuss some special cases, in addition established some Hermite-Hadamard type inequalities related to relative h-convex functions.

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Definition 3([8]). *A function $f : K_g \to H$ is said to be relative h-convex function with respect to two functions $h : [0,1] \to (0,+\infty)$ and $g : H \to H$ such that K_g is a relative convex set, if*

$$f((1-t)u+tg(v)) \le h(1-t)f(u)+h(t)f(g(v)),$$

$$\forall u,v \in H : u,g(v) \in K_g, \quad t \in (0,1).$$

Theorem 2([8]). *Let $f : K_g \to \mathbb{R}$ be a relative h-convex function, such that $h(\frac{1}{2}) \neq 0$, then, we have*

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+g(b)}{2}\right) \le \frac{1}{(g(b)-a)} \int_a^{g(b)} f(x)\,dx$$

$$\le [f(a)+f(g(b))] \int_0^1 h(t)dt.$$

Definition 4. *A function $f : K_g \to [0,+\infty)$ is a semi logarithmically to be relative convex function with respect to $g : H \to H$ such that K_g is a relative convex set, if*

$$f((1-t)u+tg(v)) \le (f(u))^t (f(v))^{1-t},$$

$$\forall u,v \in H : u,g(v) \in K_g, \quad t \in (0,1).$$

Now, we combine definitions of Noor-convexity (relative convexity), m-convexity and semi *m*-logarithmically-convex for we obtain obtain the class of *m*-logarithmically- semi-convex functions, as the following.

Definition 5. *A function $f : K_g \to (0,+\infty)$ is: Relative m-logarithmically- semi-convex function with respect to $g : H \to H$ such that K_g is a relative convex set, if*

$$f(m(1-t)u+tg(v)) \le (f(u))^{m(1-t)}(f(v))^t, \tag{2}$$

$\forall u,v \in H : u,g(v) \in K_g, \quad t,m \in (0,1).$

If this inequality reverses, then we call f m-logarithmically semi-concave function.

Remark.

1. If we take $m = 1$ in (2), then we have the definition of **semi-logarithmically-convex function** .
2. If we take $g(x) = x$, in (2), then we have the definition of **(m)- logarithmically convex function** .

2 Preliminaries

Now we discuss some properties of relative *m*-logarithmically semi-convex functions.

Theorem 3. *Let $f : [a,g(b)) \to (0,+\infty)$, $m \in (0,1]$ then,*

(i) When $f \ge 1$, f is relative m-logarithmically- semi-convex function if and only if, $\ln f$ is an relative m-semi-convex function.

(ii) f is relative m-logarithmically- semi-convex function if and only if $\frac{1}{f}$ is a relative m-logarithmically- semi-concave function.

Proof. The inequality (2) may be written as

$$\ln(f(mtx+(1-t)g(y))) \le mt\ln(f(x))+(1-t)\ln(f(y)),$$

and

$$[f(mtx+(1-t)g(y))]^{-1} \ge [f(x)]^{-t} \cdot [f(y)]^{-m(1-t)},$$

for all $x,g(y) \in [a,g(b))$ and $t \in [0,1]$. The proof is completed.

Theorem 4. *Let $f,h : [a,g(b)) \to (0,+\infty)$ be relative m-logarithmically semi-convex function and let $m \in (0,1]$, then $f \cdot h$ is a relative m-logarithmically semi-convex function on $[a,g(b))$.*

Proof. Using the relative *m*-logarithmically semi convexity of f on $[a,g(b))$, we obtain

$$f(tmx+(1-t)g(y)) \le (f(x))^{mt} \cdot (f(y))^{1-t},$$

and

$$h(tmx+(1-t)g(y)) \le (h(x))^{mt} \cdot (h(y))^{1-t},$$

for all $x,g(y) \in [a,g(b))$ and $t \in [0,1]$. Multiplying both inequalities, we get

$$\begin{aligned} f(tmx+(1-t)g(y)) \;\cdot\; h(tmx+(1-t)g(y))^{mt} \\ \le (f(x))^{mt} \cdot (f(y))^{1-t} \cdot (h(x))^{mt} \cdot (h(y))^{1-t} \\ = (f(x) \cdot h(x))^{mt} (f(y) \cdot h(y))^{1-t} \end{aligned}$$

This completes the proof.

3 Main Results

In this section, we present and discuss Hermite-Hadamard type inequalities. First, let us recall the following definition we will using. For positive $a,b \in \mathbb{R}$:

1. Arithmetic mean: $A(a,b) = \frac{a+b}{2}$.
2. Geometric mean: $G(a,b) = \sqrt{ab}$.
3. Logarithmic mean: $L(a,b) = \frac{b-a}{\log(b)-\log(a)}$.

Theorem 5. *If $f : [a,g(b)) \to (0,+\infty)$ is Lebesgue integrable on $[a,g(b))$ and m-semi log-convex function relative to $g : (0,+\infty) \to (0,+\infty)$, $m \in (0,1]$, then*

$$\frac{1}{g(b)-a}\int_a^{g(b)} G(f(x), f(a+g(b)-x))dx \le G\left(f\left(\frac{a}{m}\right)^m, f(b)\right).$$

Proof. Since f is relative m-semi log-convex function, we have

$$f(ta+(1-t)g(b)) \leq f\left(\frac{a}{m}\right)^{mt} f(b)^{1-t},\ t \in (0,1)$$

and

$$f((1-t)a+tg(b)) \leq f\left(\frac{a}{m}\right)^{m(1-t)} f(b)^{t},\ t \in (0,1).$$

By multiplying these two latest inequalities, we obtain

$$f(ta+(1-t)g(b))f((1-t)a+tg(b)) \leq f\left(\frac{a}{m}\right)^{m} f(b).$$

Now, taking square root, we get

$$G(f(ta+(1-t)g(b)), f((1-t)a+tg(b)) \leq G\left(f\left(\frac{a}{m}\right)^{m}, f(b)\right).$$

Integrating over the interval $(0,1)$ with respect to x and replacing $x = ta+(1-t)g(b)$, we get the required inequality.

Theorem 6. *If $f : [a,g(b)) \to (0,+\infty)$ is Lebesgue integrable on $[a,g(b))$ and m-semi logarithmically convex function relative to $g : (0,+\infty) \to (0,+\infty)$ $m \in (0,1]$, then*

$$\frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx \leq L\left(f\left(\frac{a}{m}\right)^{m}, f(b)\right) \leq \frac{f\left(\frac{a}{m}\right)^{m}+f(b)}{2}.$$

Proof. Since f is relative m-semi log-convex function, we have

$$f(ta+(1-t)g(b)) \leq f\left(\frac{a}{m}\right)^{mt} f(b)^{1-t},\ t \in (0,1).$$

Integrating over the interval $(0,1)$, we get

$$\begin{aligned}\int_0^1 f(ta+(1-t)g(b))dt) &\leq \int_0^1 f\left(\frac{a}{m}\right)^{mt} f(b)^{1-t}dt \\ &= \int_0^1 f\left(\left(\frac{a}{m}\right)^{m}\right)^{t} f(b)^{1-t}dt \\ &= \frac{f(b)-f\left(\frac{a}{m}\right)^{m}}{\log(f(b))-\log(f\left(\frac{a}{m}\right)^{m})} \\ &= L\left(f(b), f\left(\frac{a}{m}\right)^{m}\right) \\ &\leq \frac{f(b)+f\left(\frac{a}{m}\right)^{m}}{2}.\end{aligned}$$

Substituting $x = ta+(1-t)g(b)$, we get the required inequality.

*Remark.*If $m = 1$ and the function g is such that $g(a) = a$ then we can get a coincidence with Theorem 4.4 in [3].

Theorem 7. *If the functions $f_1, f_2 : [a,g(b)) \to (0,+\infty)$ are Lebesgue integrable on $[a,g(b))$ and relative m-semi logarithmically convex functions and $g : (0,+\infty) \to (0,+\infty)$, $m \in (0,1]$, then*

$$\begin{aligned}&\frac{1}{g(b)-a}\int_a^{g(b)} f_1(x)f_2(x)dx \\ &\leq L\left(f_1(b)f_2(b), \left(f_1\left(\frac{a}{m}\right)f_2\left(\frac{a}{m}\right)\right)^{m}\right) \\ &\leq \frac{1}{4}\left\{\left(f_1\left(\frac{a}{m}\right)^{m}+f_1(b)\right)L\left(f_1(b), f_1\left(\frac{a}{m}\right)^{m}\right)\right. \\ &\quad \left.+\left(f_2\left(\frac{a}{m}\right)^{m}+f_2(b)\right)L\left(f_2(b), f_2\left(\frac{a}{m}\right)^{m}\right)\right\}\end{aligned}$$

Proof. Since f_1, f_2 are relative m-semi logarithmically convex functions, we have

$$f_1(ta+(1-t)g(b)) \leq f_1\left(\frac{a}{m}\right)^{mt} f_1(b)^{1-t},\ t \in (0,1)$$

and

$$f_2(ta+(1-t)g(b)) \leq f_2\left(\frac{a}{m}\right)^{mt} f_2(b)^{1-t},\ t \in (0,1)$$

We multiply both inequalities and integrating over the interval $(0,1)$, we get

$$\begin{aligned}&\int_0^1 f_1(ta+(1-t)g(b))f_2(ta+(1-t)g(b))dt \\ &\leq \int_0^1 \left(f_1\left(\frac{a}{m}\right)f_2\left(\frac{a}{m}\right)\right)^{mt}(f_1(b)f_2(b))^{1-t}dt \\ &= \int_0^1 \left(\left(f_1\left(\frac{a}{m}\right)f_2\left(\frac{a}{m}\right)\right)^{m}\right)^{t}(f_1(b)f_2(b))^{1-t}dt \\ &= \frac{f_1(b)f_2(b)-\left(f_1\left(\frac{a}{m}\right)f_2\left(\frac{a}{m}\right)\right)^{m}}{log(f_1(b)f_2(b))-log(\left(f_1\left(\frac{a}{m}\right)f_2\left(\frac{a}{m}\right)\right)^{m})} \\ &= L\left(f_1(b)f_2(b), \left(f_1\left(\frac{a}{m}\right)f_2\left(\frac{a}{m}\right)\right)^{m}\right).\end{aligned}$$

By Young's inequality, we have

$$\begin{aligned}&\int_0^1 \left(\left(f_1\left(\frac{a}{m}\right)f_2\left(\frac{a}{m}\right)\right)^{m}\right)^{t}(f_1(b)f_2(b))^{1-t}dt \\ &\leq \frac{1}{2}\int_0^1\left\{\left(f_1\left(\frac{a}{m}\right)^{mt}f_1(b)^{1-t}\right)^2+\left(f_2\left(\frac{a}{m}\right)^{mt}f_2(b)^{1-t}\right)^2\right\}dt \\ &= \frac{1}{4}\left[\frac{(f_1(b))^2-\left(f_1\left(\frac{a}{m}\right)^{m}\right)^2}{\log(f_1(b))-\log\left(f_1\left(\frac{a}{m}\right)^{m}\right)}+\frac{(f_2(b))^2-\left(f_2\left(\frac{a}{m}\right)^{m}\right)^2}{\log(f_2(b))-\log\left(f_2\left(\frac{a}{m}\right)^{m}\right)}\right] \\ &= \frac{1}{4}\left\{\left(f_1\left(\frac{a}{m}\right)^{m}+f_1(b)\right)L\left(f_1(b), f_1\left(\frac{a}{m}\right)^{m}\right)\right. \\ &\quad \left.+\left(f_2\left(\frac{a}{m}\right)^{m}+f_2(b)\right)L\left(f_2(b), f_2\left(\frac{a}{m}\right)^{m}\right)\right\}.\end{aligned}$$

Substituting $x = ta+(1-t)g(b)$, we get the required inequality.

Remark. If $m = 1$ and the function g is such that $g(a) = a$ we have a coincidence with Theorem 4.5 in [3].

Theorem 8. *If $f,h : [a,g(b)) \to (0,+\infty)$ such that $f.h \in L([a,g(b)])$ and f is m_1-semi log-convex function relative to $g : (0,+\infty) \to (0,+\infty)$ in $\left[0, \frac{g(b)}{m_1}\right]$ and h is m_2-semi* *log-convex function relative to* $g:(0,+\infty)\to(0,+\infty)$ *in* $\left[0,\dfrac{g(b)}{m_2}\right]$, *then*

$$\frac{1}{g(b)-a}\int_a^{g(b)} f(x)h(x)dx \leq L\left(f\left(\frac{a}{m_1}\right)^{m_1} h\left(\frac{a}{m_2}\right)^{m_2}, f(b).h(b)\right) \leq A\left(f\left(\frac{a}{m_1}\right)^{m_1} h\left(\frac{a}{m_2}\right)^{m_2}, f(b).h(b)\right).$$

Proof. Since f and h are relative m_i-semi log-convex functions, we have

$$f(ta+(1-t)g(b)) \leq f\left(\frac{a}{m_1}\right)^{m_1 t} f(b)^{1-t},\ t\in(0,1),$$

and

$$h(ta+(1-t)g(b)) \leq h\left(\frac{a}{m_2}\right)^{m_2 t} h(b)^{1-t},\ t\in(0,1).$$

By multiplying inequalities, we get

$$f(ta+(1-t)g(b))h(ta+(1-t)g(b) \leq f\left(\frac{a}{m_1}\right)^{m_1 t} h\left(\frac{a}{m_2}\right)^{m_2 t}(f(b).h(b))^{1-t}.$$

By integrating over interval $(0,1)$ and replacing $x=ta+(1-t)g(b)$, we get the required inequality.

Using the technique of [15], we can prove the following result.

Theorem 9. *If* $f,h:[a,g(b))\to(0,+\infty)$ *such that* $f.h^q\in L([a,g(b)])$ *and* f *and* h^q *are m-semi log-convex function relative to* $g:(0,+\infty)\to(0,+\infty)$ *in* $\left[0,\dfrac{g(b)}{m}\right]$, *for* $q\geq 1$, *then*

$$\frac{1}{g(b)-a}\int_a^{g(b)} f(x)h(x)dx \leq L^{1-\frac{1}{q}}\left(f\left(\frac{a}{m}\right)^m, f(b)\right)\cdot L^{\frac{1}{q}}\left(fh^q\left(\frac{a}{m}\right)^m, fh^q(b)\right) \leq A^{1-\frac{1}{q}}\left(f\left(\frac{a}{m}\right)^m, f(b)\right)\cdot A^{\frac{1}{q}}\left(fh^q\left(\frac{a}{m}\right)^m, fh^q(b)\right).$$

Proof. Since f and h are relative m-semi logarithmically-convex functions, we have

$$f(ta+(1-t)g(b)) \leq f\left(\frac{a}{m}\right)^{mt} f(b)^{1-t},\ t\in(0,1),$$

and

$$h(ta+(1-t)g(b)) \leq h\left(\frac{a}{m}\right)^{mt} h(b)^{1-t},\ t\in(0,1).$$

By Holder's inequality, we get

$$\begin{aligned}
&\frac{1}{g(b)-a}\int_a^{g(b)} f(x)h(x)dx \\
&= \int_0^1 f(ta+(1-t)g(b))h(ta+(1-t)g(b))dt \\
&\leq \left(\int_0^1 f(ta+(1-t)g(b))dt\right)^{1-\frac{1}{q}} \times \left(\int_0^1 f(ta+(1-t)g(b))h^q(ta+(1-t)g(b))dt\right)^{\frac{1}{q}} \\
&\leq \left(\int_0^1 f\left(\frac{a}{m}\right)^{mt}(f(b))^{1-t}dt\right)^{1-\frac{1}{q}} \times \left(\int_0^1 [fh^q\left(\frac{a}{m}\right)]^{mt}[fh^q(b)]^{1-t}dt\right)^{\frac{1}{q}} \\
&= L^{1-\frac{1}{q}}\left(f(b), f\left(\frac{a}{m}\right)^m\right) L^{\frac{1}{q}}\left(fh^q(b), fh^q\left(\frac{a}{m}\right)^m\right) \\
&\leq A^{1-\frac{1}{q}}\left(f(b), f\left(\frac{a}{m}\right)^m\right) A^{\frac{1}{q}}\left(fh^q(b), fh^q\left(\frac{a}{m}\right)^m\right).
\end{aligned}$$

This completes the proof.

Corollary 1. *Under the same hypotheses of theorem 9, if* $q=1$ *we get*

$$\frac{1}{g(b)-a}\int_a^{g(b)} f(x)h(x)dx \leq L\left(fh(b), fh\left(\frac{a}{m}\right)^m\right) \leq A\left(fh(b), fh\left(\frac{a}{m}\right)^m\right) = \frac{\left(fh\left(\frac{a}{m}\right)\right)^m + fh(b)}{2}.$$

Remark. if $q=1$ and $g(x)=x$ in the theorem 9 we get

$$\frac{1}{b-a}\int_a^{b} f(x)h(x)dx \leq L\left(fh(b), fh\left(\frac{a}{m}\right)^m\right) \leq A\left(fh(b), fh\left(\frac{a}{m}\right)^m\right) = \frac{\left(fh\left(\frac{a}{m}\right)\right)^m + fh(b)}{2}.$$

For the following results we need the following Lemma. Using the method in [2].

Lemma 1. *Let* $f:I\to R$ *be a differentiable function on* $int(I)$ *and* $g:R\to R$ *an arbitrary function. If* f' *is integrable on* $[a,g(b)]$ *con* $g(b)\geq a$, *then*

$$\frac{f(a)+f(g(b))}{2} - \frac{1}{(g(b)-a)}\int_a^{g(b)} f(x)dx = \frac{g(b)-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)g(b))dt.$$

Proof. Integrating by parts

$$\int_0^1 (1-2t)f'(ta+(1-t)g(b))dt$$

$$= \left.\frac{(1-2t)f(ta+(1-t)g(b))}{a-g(b)}\right|_0^1 + \frac{2}{a-g(b)}\int_0^1 f(ta+(1-t)g(b))dt$$

$$= \frac{-(f(a))}{a-g(b)} - \frac{f(g(b))}{a-g(b)} + \frac{2}{a-g(b)}\int_0^1 f(ta+(1-t)g(b))dt$$

$$= \frac{(f(a))}{g(b)-a} + \frac{f(g(b))}{g(b)-a} - \frac{2}{(a-g(b))^2}\int_a^{g(b)} f(x)dx$$

i.e.

$$\frac{g(b)-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)g(b))dt = \frac{f(a)+f(g(b))}{2} - \frac{1}{(g(b)-a)}\int_a^{g(b)} f(x)dx$$

Theorem 10. *Let $f: I \to R$ be a differentiable function on $int(I)$ and $g: R \to R$ an arbitrary function. Let $m \in (0,1)$. If $|f'|$ is $m-$semi logarithmic convex function respect to g and integrable over I we have*

$$\left|\frac{f(a)+f(g(b))}{2} - \frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx\right| \le \frac{g(b)-a}{4}\left|f'\right|(b)E_1(c)$$

where

$$c = \frac{|f'|\left(\frac{a}{m}\right)^m}{|f'|(b)}$$

and

$$E_1(c) = \begin{cases} \frac{1}{2} \ \text{if} \ c=1 \\ \frac{1}{\ln^2 c}(4\sqrt{c}-2c-2+(c-1)\ln c) \ \text{if} \ c \neq 1 \end{cases}$$

Proof. Using Lemma 1 and Definition 2 we have

$$\left|\frac{f(a)+f(g(b))}{2} - \frac{1}{(a-g(b))}\int_a^{g(b)} f(x)dx\right|$$

$$\le \frac{g(b)-a}{2}\int_0^1 |(1-2t)|\left|f'(ta+(1-t)g(b))\right|dt$$

$$\le \frac{g(b)-a}{2}\int_0^1 |(1-2t)|\left(\left|f'\right|\left(\frac{a}{m}\right)\right)^{mt}\left(\left|f'\right|(b)\right)^{1-t}dt$$

$$= \frac{g(b)-a}{2}\left(\int_0^{1/2}(1-2t)\left(\left|f'\right|\left(\frac{a}{m}\right)\right)^{mt}\left(\left|f'\right|(b)\right)^{1-t}dt + \int_{1/2}^1 (2t-1)\left(\left|f'\right|\left(\frac{a}{m}\right)\right)^{mt}\left(\left|f'\right|(b)\right)^{1-t}dt\right)$$

Doing $c = \frac{|f'|\left(\frac{a}{m}\right)^m}{|f'|(b)}$, we continue the proof by cases. Case $c \neq 1$: evaluating the first integral, we have

$$\int_0^{1/2}(1-2t)\left|f'(ta+(1-t)g(b))\right|dt \le \int_0^{1/2}(1-2t)\left(\left|f'\right|\left(\frac{a}{m}\right)\right)^{mt}\left(\left|f'\right|(b)\right)^{1-t}dt$$
$$= \left|f'\right|(b)\int_0^{1/2}(1-2t)\left(\frac{|f'|\left(\frac{a}{m}\right)^m}{|f'|(b)}\right)^t dt = \left|f'\right|(b)\int_0^{1/2}(1-2t)c^t dt = \left|f'\right|(b)\left(-\frac{1}{\ln c} + \frac{2\sqrt{c}}{\ln^2 c} - \frac{2}{\ln^2 c}\right)$$

in similar way, the second integral is

$$\int_{1/2}^1 (2t-1)\left|f'(ta+(1-t)g(b))\right|dt \le \left|f'\right|(b)\int_{1/2}^1 (2t-1)\left(\frac{|f'|\left(\frac{a}{m}\right)^m}{|f'|(b)}\right)^t dt = \left|f'\right|(b)\left(\frac{c}{\ln c} - \frac{2c}{\ln^2 c} + \frac{2\sqrt{c}}{\ln^2 c}\right)$$

adding the integrals

$$\int_0^{1/2}(1-2t)\left|f'(ta+(1-t)g(b))\right|dt + \int_{1/2}^1 (2t-1)\left|f'(ta+(1-t)g(b))\right|dt = \frac{|f'|(b)}{\ln^2 c}\left(4\sqrt{c}-2c-2+(c-1)\ln c\right)$$

Note that the expression $4\sqrt{c}-2c-2+(c-1)\ln c$ has the only zero in $c=1$ and is positive for $c \in (0,\infty)-\{1\}$

Case $c=1$:

Evaluating the first integral

$$\int_0^{1/2}(1-2t)\left|f'(ta+(1-t)g(b))\right|dt \le \int_0^{1/2}(1-2t)\left(\left|f'\right|\left(\frac{a}{m}\right)\right)^{mt}\left(\left|f'\right|(b)\right)^{1-t}dt$$
$$= \left|f'\right|(b)\int_0^{1/2}(1-2t)\left(\frac{|f'|\left(\frac{a}{m}\right)^m}{|f'|(b)}\right)^t dt = \left|f'\right|(b)\int_0^{1/2}(1-2t)dt = \frac{1}{4}\left|f'\right|(b)$$

The second integral

$$\int_{1/2}^{1}(2t-1)\left|f'(ta+(1-t)g(b))\right|dt$$
$$\leq \int_{1/2}^{1}(2t-1)\left(\left|f'\right|\left(\frac{a}{m}\right)\right)^{mt}\left(\left|f'\right|(b)\right)^{1-t}dt$$
$$=\left|f'\right|(b)\int_{1/2}^{1}(2t-1)\left(\frac{\left|f'\right|\left(\frac{a}{m}\right)^{m}}{\left|f'\right|(b)}\right)^{t}dt$$
$$=\left|f'\right|(b)\int_{1/2}^{1}(2t-1)dt=\frac{1}{4}\left|f'\right|(b)$$

adding the integrals

$$\int_{0}^{1/2}(1-2t)\left|f'(ta+(1-t)g(b))\right|dt$$
$$+\int_{1/2}^{1}(2t-1)\left|f'(ta+(1-t)g(b))\right|dt$$
$$=\frac{1}{2}\left|f'\right|(b)$$

Remark. If in the Theorem 10 we put $m=1$ and $g(x)=x$ we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\leq\frac{b-a}{4}\left|f'\right|(b)$$

if $c=1$, and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|$$
$$\leq\frac{\left|f'\right|(b)}{\ln^{2}c}\left(4\sqrt{c}-2c-2+(c-1)\ln c\right)$$

if $c\neq 1$, where

$$c=\frac{\left|f'\right|(a)}{\left|f'\right|(b)}.$$

Theorem 11. *Let $I\subset[0;\infty)$ be an open real interval and let $f:I\to(0;\infty)$ be a differentiable function on I such that $f'\in L([a;b])$ for $0\leq a<b<\infty$. If $|f'(x)|^{q}$ is m-logarithmically semi convex on $\left[\frac{a}{m},\infty\right]$ for $m\in(0;1]$, then for $q>1$*

$$[\quad\left|\frac{f(a)+f(g(b))}{2}-\frac{1}{g(b)-a}\int_{a}^{g(b)}f(x)dx\right|$$
$$\leq\frac{(g(b)-a)}{2}\left(\frac{1}{2}\right)^{1-1/q}|f\prime(b)|^{q}\left(E_{1}(c,q)\right)^{1/q}$$

where

$$c=\frac{\left|f'\right|\left(\frac{a}{m}\right)^{m}}{\left|f'\right|(b)},$$

and

$$E_{1}(c,q)=\begin{cases}\frac{1}{2},c=1\\ \frac{1}{q^{2}\ln^{2}c}\left((c^{q}-1)q\ln c+4c^{q/2}-2c^{q}-2\right),c\neq 1\end{cases}$$

Proof. When $q>1$, using Definition , Lemma (1) and Holder inequality we have

$$\left|\frac{f(a)+f(g(b))}{2}-\frac{1}{g(b)-a}\int_{a}^{g(b)}f(x)dx\right|$$
$$=\frac{(g(b)-a)}{2}\left|\int_{0}^{1}(1-2t)f'(ta+(1-t)g(b))dt\right|$$
$$\leq\frac{(g(b)-a)}{2}\int_{0}^{1}|(1-2t)|\left|f'(ta+(1-t)g(b))\right|dt$$
$$\leq\frac{(g(b)-a)}{2}\left(\int_{0}^{1}|(1-2t)|dt\right)^{1-1/q}\times$$
$$\left(\int_{0}^{1}|(1-2t)|\left|f'(ta+(1-t)g(b))\right|^{q}dt\right)^{1/q}$$
$$\leq\frac{(g(b)-a)}{2}\left(\frac{1}{2}\right)^{1-1/q}|f\prime(b)|^{q}\left(\int_{0}^{1}|(1-2t)|c^{qt}dt\right)^{1/q}.$$

If $c=1$ we have

$$\int_{0}^{1}|(1-2t)|c^{qt}dt=\int_{0}^{1}|(1-2t)|dt=\frac{1}{2}$$

If $c>1$, then

$$\int_{0}^{1}|(1-2t)|c^{qt}dt$$
$$=\int_{0}^{1/2}(1-2t)c^{qt}dt+\int_{1/2}^{1}(2t-1)c^{qt}dt$$
$$=\frac{1}{q\ln^{2}c}\left((c^{q}-1)q\ln c+4c^{q/2}-2c^{q}-2\right)$$

Remark. If $m=1$ and $g(x)=x$ the we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|$$
$$\leq\frac{(b-a)}{2}\left(\frac{1}{2}\right)^{1-1/q}|f'(b)|^{q}\left(E_{1}(c,q)\right)^{1/q}$$

where

$$c=\frac{\left|f'\right|(a)}{\left|f'\right|(b)},$$

and

$$E_{1}(c,q)=\begin{cases}\frac{1}{2},c=1\\ \frac{1}{q^{2}\ln^{2}c}\left((c^{q}-1)q\ln c+4c^{q/2}-2c^{q}-2\right),c\neq 1\end{cases}$$

Lemma 2. *Let $f:I\subset R\to R$ be a differentiable mapping on int(I) and $a,b\in I$ with $a<b$. If $f'\in L([a;b])$, then*

$$f\left(\frac{a+g(b)}{2}\right)-\frac{1}{g(b)-a}\int_{a}^{g(b)}f(x)dx$$
$$=(g(b)-a)\left[\int_{0}^{1/2}tf'(ta+(1-t)b)dt\right.$$
$$\left.+\int_{1/2}^{1}(1-t)f'(ta+(1-t)b)dt\right]$$

Theorem 12. *Let $f: I \to R$ be a differentiable function on $int(I)$ and $g: R \to R$ an arbitrary function. Let $m \in (0,1)$. If $|f'|$ is $m-$semi logarithmic convex function respect to g and integrable over I we have*

$$\left| f\left(\frac{a+g(b)}{2}\right) - \frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx \right| \leq (g(b)-a)\left|f'\right|(b)E_2(c)$$

where

$$c = \frac{\left(|f'|\left(\frac{a}{m}\right)\right)^m}{|f'|(b)} \quad \text{and } E_2(c) = \begin{cases} \frac{1}{4} & \text{if } \; c=1 \\ \frac{2c-4\sqrt{c}+2}{(\ln c)^2} & \text{if } c \neq 1 \end{cases}$$

Proof. Using lemma 2 we have

$$\left| f\left(\frac{a+g(b)}{2}\right) - \frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx \right|$$

$$= (g(b)-a)\left|\int_0^{1/2} t f'(ta+(1-t)g(b))dt + \int_{1/2}^1 (1-t)f'(ta+(1-t)g(b))dt\right|$$

$$\leq (g(b)-a)\left(\int_0^{1/2} t\left|f'(ta+(1-t)g(b))\right|dt + \int_{1/2}^1 (1-t)\left|f'(ta+(1-t)g(b))\right|dt\right)$$

Let

$$c = \frac{\left(|f'|\left(\frac{a}{m}\right)\right)^m}{|f'|(b)}$$

The first integral is

$$\int_0^{1/2} t\left|f'(ta+(1-t)g(b))\right|dt \leq \int_0^{1/2} t\left|f'(mt\frac{a}{m}+(1-t)g(b))\right|dt \leq \int_0^{1/2} t\left(|f'|\left(\frac{a}{m}\right)\right)^{mt}\left(|f'|(b)\right)^{1-t}dt = |f'|(b)\int_0^{1/2} t\left(\frac{\left(|f'|\left(\frac{a}{m}\right)\right)^m}{|f'|(b)}\right)^t dt$$

and

$$\int_{1/2}^1 (1-t)f'(ta+(1-t)g(b))dt \leq |f'|(b)\int_{1/2}^1 (1-t)\left(\frac{\left(|f'|\left(\frac{a}{m}\right)\right)^m}{|f'|(b)}\right)^t dt$$

If $c=1$ we have

$$\int_0^{1/2} t\left|f'(ta+(1-t)g(b))\right|dt \leq |f'|(b)\int_0^{1/2} t dt = \frac{|f'|(b)}{8}$$

and

$$\int_{1/2}^1 (1-t)f'(ta+(1-t)g(b))dt \leq |f'|(b)\int_{1/2}^1 (1-t)dt = \frac{|f'|(b)}{8}$$

Case $c \neq 1$ we have

$$\int_0^{1/2} t\left|f'(ta+(1-t)g(b))\right|dt \leq |f'|(b)\int_0^{1/2} tc^t dt = |f'|(b)\left(\frac{tc^t}{\ln c} - \frac{c^t}{(\ln c)^2}\bigg|_0^{1/2}\right) = |f'|(b)\left(\frac{\sqrt{c}(\ln c - 2)+2}{2(\ln c)^2}\right)$$

and

$$\int_{1/2}^1 (1-t)f'(ta+(1-t)g(b))dt \leq |f'|(b)\int_{1/2}^1 (1-t)c^t dt = |f'|(b)\left(\frac{(1-t)c^t}{\ln c} + \frac{c^t}{(\ln c)^2}\bigg|_{1/2}^1\right) = |f'|(b)\left(\frac{2c-\sqrt{c}(\ln c+2)}{2(\ln c)^2}\right)$$

Adding integrals

$$\int_0^{1/2} t\left|f'(ta+(1-t)g(b))\right|dt + \int_{1/2}^1 (1-t)f'(ta+(1-t)g(b))dt$$

$$= |f'|(b)\left(\frac{\sqrt{c}(\ln c-2)+2}{2(\ln c)^2} + \frac{2c-\sqrt{c}(\ln c+2)}{2(\ln c)^2}\right) = \frac{|f'|(b)}{(\ln c)^2}\left(2c-4\sqrt{c}+2\right)$$

So

$$\left| f\left(\frac{a+g(b)}{2}\right) - \frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx \right| \leq (g(b)-a)\left|f'\right|(b)E_2(c)$$

where

$$c = \frac{\left(|f'|\left(\frac{a}{m}\right)\right)^m}{|f'|(b)} \quad \text{and } E_2(c) = \begin{cases} \frac{1}{4} & \text{if } \; c=1 \\ \frac{2c-4\sqrt{c}+2}{(\ln c)^2} & \text{if } c \neq 1 \end{cases}$$

Theorem 13. *Let $f : I \to R$ be a differentiable function on $int(I)$ and $g : R \to R$ an arbitrary function. Let $m \in (0,1)$. If $|f'|^q$ is $m-$semi logarithmic convex function respect to g and integrable over I then for $q > 1$ we have*

$$\left| f\left(\frac{a+g(b)}{2}\right) - \frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx \right| \leq \frac{(g(b)-a)}{4}\left(\frac{1}{2}\right)^{1-3/q} |f'|(b)E_2(c,q)$$

where

$$c = \frac{\left(|f'|\left(\frac{a}{m}\right)\right)^m}{|f'|(b)}$$

and

$$E_2(c,q) = \begin{cases} 2\left(\frac{1}{8}\right)^{1/q}, c = 1 \\ \left(\frac{\sqrt{c^q}(q\ln c-2)+2}{2(q\ln c)^2}\right)^{1/q} \\ \quad + \left(\frac{2c^q-\sqrt{c^q}(q\ln c+2)}{2(q\ln c)^2}\right)^{1/q}, c \neq 1 \end{cases}$$

Proof. Using lemma 2 and holder inequality we have

$$\begin{aligned}
&\left| f\left(\frac{a+g(b)}{2}\right) - \frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx \right| \\
&= (g(b)-a)\left|\int_0^{1/2} t f'(ta+(1-t)g(b))dt + \int_{1/2}^1 (1-t) f'(ta+(1-t)g(b))dt\right| \\
&\leq (g(b)-a)\left(\int_0^{1/2} t\left|f'(ta+(1-t)g(b))\right|dt + \int_{1/2}^1 (1-t)\left|f'(ta+(1-t)g(b))\right|dt\right) \\
&\leq (g(b)-a)\times \\
&\left[\left(\int_0^{1/2} tdt\right)^{1-1/q}\left(\int_0^{1/2} t\left|f'(ta+(1-t)g(b))\right|^q dt\right)^{1/q} + \left(\int_{1/2}^1 (1-t)dt\right)^{1-1/q} \times \right. \\
&\left.\left(\int_{1/2}^1 (1-t)\left|f'(ta+(1-t)g(b))\right|^q dt\right)^{1/q}\right] \\
&\leq \frac{(g(b)-a)}{4}\left(\frac{1}{2}\right)^{1-3/q} \times \\
&\left[\left(\int_0^{1/2} t\left|f'(ta+(1-t)g(b))\right|^q dt\right)^{1/q} + \left(\int_{1/2}^1 (1-t)\left|f'(ta+(1-t)g(b))\right|^q dt\right)^{1/q}\right]
\end{aligned}$$

Case $c \neq 1$: Solving the first integral

$$\begin{aligned}
&\int_0^{1/2} t\left|f'(ta+(1-t)g(b))\right|^q dt \\
&\leq |f'|^q(b)\int_0^{1/2} t\left(\frac{\left(|f'|^q\left(\frac{a}{m}\right)\right)^m}{|f'|^q(b)}\right)^t dt \\
&= |f'|^q(b)\int_0^{1/2} tc^{qt}dt \\
&= |f'|^q(b)\left(\frac{\sqrt{c^q}(q\ln c-2)+2}{2(q\ln c)^2}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{1/2}^1 (1-t)\left|f'(ta+(1-t)g(b))\right|^q dt \\
&\leq |f'|^q(b)\int_{1/2}^1 (1-t)\left(\frac{\left(|f'|^q\left(\frac{a}{m}\right)\right)^m}{|f'|^q(b)}\right)^t dt \\
&= |f'|^q(b)\int_{1/2}^1 (1-t)c^{qt}dt \\
&= |f'|^q(b)\left(\frac{2c^q-\sqrt{c^q}(q\ln c+2)}{2(q\ln c)^2}\right)
\end{aligned}$$

then

$$\begin{aligned}
&\left| f\left(\frac{a+g(b)}{2}\right) - \frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx \right| \\
&\leq \frac{(g(b)-a)}{4}\left(\frac{1}{2}\right)^{1-3/q}|f'|(b)\times \\
&\left(\left(\frac{\sqrt{c^q}(q\ln c-2)+2}{2(q\ln c)^2}\right)^{1/q}\times \left(\frac{2c^q-\sqrt{c^q}(q\ln c+2)}{2(q\ln c)^2}\right)^{1/q}\right)
\end{aligned}$$

Case $c = 1$:

$$\int_0^{1/2} t\left|f'(ta+(1-t)g(b))\right|^q dt \leq |f'|^q(b)\int_0^{1/2} tdt = \frac{|f'|^q(b)}{8}$$

and

$$\int_{1/2}^1 (1-t)\left|f'(ta+(1-t)g(b))\right|^q dt \leq |f'|^q(b)\int_{1/2}^1 (1-t)dt = \frac{|f'|^q(b)}{8}$$

then

$$\left| f\left(\frac{a+g(b)}{2}\right) - \frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx \right| \leq \frac{(g(b)-a)}{4}\left(\frac{1}{2}\right)^{1-3/q}|f'|(b)2\left(\frac{1}{8}\right)^{1/q}$$

Now, we can establish

$$\left| f\left(\frac{a+g(b)}{2}\right) - \frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx \right| \leq \frac{(g(b)-a)}{4}\left(\frac{1}{2}\right)^{1-3/q} |f'|(b)E_2(c,q)$$

where

$$c = \frac{\left(|f'|\left(\frac{a}{m}\right)\right)^m}{|f'|(b)}$$

and

$$E_2(c,q) = \begin{cases} 2\left(\frac{1}{8}\right)^{1/q}, c = 1 \\ \left(\frac{\sqrt{c^q}(q\ln c - 2) + 2}{2(q\ln c)^2}\right)^{1/q} \\ \qquad + \left(\frac{2c^q - \sqrt{c^q}(q\ln c + 2)}{2(q\ln c)^2}\right)^{1/q}, c \neq 1 \end{cases}$$

4 Conclusions

We expect that the ideas and techniques used in this paper may inspire interested readers to explore some new applications of these newly introduced functions in various fields of pure and applied sciences.

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