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Generalization of Chi-square Distribution

Gauhar Rahman[1], *Shahid Mubeen*[2,*] *and Abdur Rehman*[2]

[1] International Islamic University Islamabad, Pakistan
[2] University of Sargodha, Sargodha, Pakistan

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Abstract: In this paper, we define a generalized chi-square distribution by using a new parameter $k>0$. we give some properties of the said distribution including the moment generating function and characteristic function in terms of k. Also, we establish a relationship in central moments involving the parameter $k>0$. If $k=1$, we have all the results of classical χ^2 distribution.

Keywords: k-gamma functions, chi-square distribution, moments

1 Introduction and basic definitions

The chi-square distribution was first introduced in 1875 by F.R. Helmert, a German physicist. Later in 1900, Karl Pearson proved that as n approaches infinity, a discrete multinomial distribution may be transformed and made to approach a chi-square distribution. This approximation has broad applications such as a test of goodness of fit, as a test of independence and as a test of homogeneity.

The chi-square distribution contains only one parameter, called the number of degrees of freedom, where the term degree of freedom represent the number of independent random variables that express the chi-square. If the random variables entering a chi-square are subjected to linear restrictions, then the number of degrees of freedom is reduced by the number of restrictions involved. we generalize the chi-square distribution in the form of a new parameter k where $k>0$.

Here, we give some definitions which provide a base for our main results. The definitions $(1.1-1.2)$ are given in [1] while $(1.3-1.4)$ are introduced in [2]. Also, we have taken some statistics related definitions $(1.5-1.11)$ from [3-6].

1.1 Pochhmmer's Symbol.

The factorial function is denoted and defined by

$$(a)_n = \begin{cases} a(a+1)(a+2)\cdots(a+n-1); & \text{for} \quad n\geq 1, a\neq 0 \\ 1 \quad \text{if} \quad n=0. \end{cases} \tag{1.1}$$

The function $(a)_n$ defined in relation (1.1) is also known as Pochhmmer's symbol.

1.2 Gamma Function.

Let $z\in\mathbb{C}$, the Euler gamma function is defined by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^{z-1}}{(z)_n} \tag{1.2}$$

* Corresponding author e-mail: smjhanda@gmail.com

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and the integral form of gamma function is given by

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt, \quad \mathbb{R}(z)>0. \tag{1.3}$$

From the relation (1.3), using integration by parts, we can easily show that

$$\Gamma(z+1)=z\Gamma(z) \tag{1.4}$$

The relation between Pochhammer's symbol and gamma function is given by

$$(z)_n=\frac{\Gamma(z+n)}{\Gamma(z)}. \tag{1.5}$$

1.3 Pochhammer k-Symbol.

For $k>0$, the Pochhammer k-symbol is denoted and defined by

$$(a)_{n,k}=\begin{cases} a(a+k)(a+2k)\cdots(a+(n-1)k) & \text{for} \quad n\geq 1, a\neq 0 \\ 1 \quad \text{if } n=0. \end{cases} \tag{1.6}$$

1.4 k-Gamma Function.

For $k>0$ and $z\in\mathbb{C}$, the k-gamma function is defined as

$$\Gamma_k(z)=\lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{z}{k}-1}}{(z)_{n,k}} \tag{1.7}$$

and the integral representation of k-gamma function is

$$\Gamma_k(z)=\int_0^\infty t^{z-1}e^{\frac{-t^k}{k}}dt. \tag{1.8}$$

Also, the researchers [7-12] have worked on the generalized k-gamma and k-beta functions and discussed the different properties of these functions. Here we give some of the properties of k-gamma function as:

$$\Gamma_k(x+k)=x\Gamma_k(x) \tag{1.9}$$

$$(x)_{n,k}=\frac{\Gamma_k(x+nk)}{\Gamma_k(x)} \tag{1.10}$$

$$\Gamma_k(k)=1,\, k>0. \tag{1.11}$$

For more details about the theory of k-special functions like, k-gamma function, k-beta function, k- hypergeometric functions, solutions of k-hypergeometric differential equations, contegious functions relations, inequalities with applications and integral representations with applications involving k-gamma and k-beta functions, k-gamma and k-beta probability distributions and so forth (See [13-19]).

1.5 Moments:

A moment designates the power to which the deviations are raised before averaging them. In statistics, we have three kinds of moments as:
(i) Moments about any value $x=A$ is the rth power of the deviation of variable from A and is called the rth moment of the distribution about A. **(ii)** Moments about $x=0$ is the rth power of the deviation of variable from 0 and is called the rth moment of the distribution about 0. **(iii)** Moments about mean i.e., $x=\mu$ is the rth power of the deviation of variable from mean and is called the rth moment of the distribution about mean. These moments are also called central moments or mean moments and are used to describe the set of data.

Note: The moments about any number $x=A$ and about $x=0$ are denoted by μ'_r while about mean position, it is denoted by μ_r and $\mu_0=\mu'_0=1$.

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1.6 Moments of the continuous distribution.

If a random variable X assumes all the values from a to b, then for a continuous distribution, the rth moment about the arbitrary number A and mean μ respectively, are given by

$$\mu_r' = \int_a^b (x-A)^r f(x)dx \tag{1.12}$$

and

$$\mu_r = \int_a^b (x-\mu)^r f(x)dx. \tag{1.13}$$

1.7 Probability Distribution and Expected values:

In a random experiment with n outcomes, suppose a variable X assumes the values $x_1, x_2, x_3, ..., x_n$ with corresponding probabilities $p_1, p_2, p_3, ..., p_n$, then this collection is called probability distribution and $\Sigma p_i = 1$ (in case of discrete distributions). Also, if $f(x)$ is a continuous probability distribution function defined on an interval $[a,b]$, then $\int_a^b f(x)dx = 1$. The expected value of the variate is defined as the first moment of the probability distribution about $x = 0$ i.e.,

$$\mu_1' = E(X) = \int_a^b x f(x)dx \tag{1.14}$$

and the rth moment about mean of the probability distribution is defined as $E(X-\mu)^r$, where μ is the mean of the distribution.

1.8 Moment generating function:

Let $f(x)$ be the probability density function of a variate X in the distribution, then the expected value of e^{tX} is called moment generating function of the distribution about $x = 0$ and is denoted by $M_0(t)$, where t is a positive real number independent of x. Thus, for a continuous random variable X assuming all the values from a to b, the $m.g.f.$ is given by

$$M_0(t) = E(e^{tX}) = \int_a^b e^{tx} f(x)dx. \tag{1.15}$$

Similarly, the moment generating function about assumed mean A and mean μ respectively, are given by

$$M_A(t) = E(e^{t(X-A)}) = \int_a^b e^{t(x-A)} f(x)dx \tag{1.16}$$

and

$$M_\mu(t) = E(e^{t(X-\mu)}) = \int_a^b e^{t(x-\mu)} f(x)dx. \tag{1.17}$$

A link between the above moment generating functions is established as

$$M_\mu(t) = e^{-t\mu} M_0(t), \qquad M_A(t) = e^{-tA} M_0(t). \tag{1.18}$$

Note: The moment generating function of the sum of two independent variates is equal to the product of their respective moment generating functions.

1.9 Normal Distribution

A normal distribution is defined by the p.d.f. as

$$f(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{(z-\mu)^2}{2\sigma^2}}, \text{for} -\infty < z < \infty, \text{and}\, \sigma > 0, \tag{1.19}$$

where μ is the mean and σ is the standard deviation, the two parameters of the normal distribution, and it is usually denoted by $N(\mu, \sigma^2)$.

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1.10 Standardized Normal Distribution

The normal probability distribution which has zero mean and unit variance is called the standardized normal distribution or unit normal distribution and is denoted by $N(0,1)$. The distribution function of the standard normal distribution, is given by

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{\frac{-t^2}{2}} dt. \tag{1.20}$$

1.11 Chi-Square Distribution

Let $Z_1, Z_2, \cdots, Z_n$ be normally and independently distributed random variables with zero mean and unit variances. Then the random variable expressed by the quantity as

$$\chi^2 = \sum_{i=1}^{n} Z_i^2,$$

is defined as chi-square random variable with n degrees of freedom. That is, a χ^2 random variable is defined as the sum of squares of independent standard normal random variable and its density function is defined as

$$f(\chi^2) = \frac{1}{2\Gamma(\frac{n}{2})} (\frac{\chi^2}{2})^{\frac{n}{2}-1} e^{-\frac{(\chi^2)}{2}}; \qquad ,0 < \chi^2 < \infty. \tag{1.21}$$

The random variable χ^2 having the above density function is said to possess the chi-square distribution with n degrees of freedom, denoted by $\chi^2(n)$, where the parameter n is a positive integer.

2 Main Results: Generalized Form of Chi-Square Distribution.

Here, we introduce the generalized form of chi-square distribution with a new parameter $k > 0$. Also, we prove some properties of the distribution involving the said parameter k. If $k = 1$, we get the classical results.

2.1 Definition

Let $Z_{1,k}, Z_{2,k}, \cdots, Z_{n,k}$ (where $k > 0$) be normally and independently distributed random variables with zero mean and unit variances, then the random variables expressed by the quantity as

$$\chi_k^2 = \sum_{i=1}^{n} Z_{i,k} \tag{2.1}$$

and is called the generalized chi-square random variable with n degrees of freedom. That is, a χ_k^2 random variable is defined as the sum of squares of independent standard normal random variables. Its density function is defined as;

$$f(\chi_k^2) = \frac{1}{2\Gamma_k(\frac{n}{2})} (\frac{\chi_k^2}{2})^{\frac{n}{2}-1} e^{-\frac{(\chi_k^2)^k}{2k}}; \qquad 0 < \chi_k^2 < \infty, k > 0. \tag{2.2}$$

Random variables χ_k^2 having the above density function are said to possess the chi-square distributions with n degrees of freedom, denoted by $\chi_k^2(n)$, where the parameter n is a positive integer. As χ_k^2 distribution full fill the definition of chi-square distribution, so the curve of χ_k^2 will remain unchanged for each positive value of k given earlier.

Remarks: If $k = 1$, then $Z_1, Z_2, \cdots, Z_n$ are in the classical form of normally and independently distributed random variables with zero mean and unit variances.

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2.2 Proposition:

The generalized chi-square distribution function defined in (2.2), with a new parameter $k > 0$, has the following properties:
(i)The mean of generalized chi-square distribution is equal to the number of degrees of freedom.

(ii)The variance of generalized chi-square distribution is equal to $2nk$.

Proof: Using the definition of expected values, we have

$$\mu_k = E(\chi_k^2) = \frac{2}{2\Gamma_k(\frac{n}{2})}\int_0^\infty (\frac{\chi_k^2}{2})^{\frac{n}{2}} e^{-\frac{(\chi_k^2)^k}{2k}} d\chi_k^2. \quad (2.3)$$

Substituting $y = \frac{\chi_k^2}{2}$ implies $dy = \frac{d\chi_k^2}{2}$, the equation (2.3) gives

$$\mu_k = \frac{2}{\Gamma_k(\frac{n}{2})}\int_0^\infty y^{\frac{n}{2}} e^{-\frac{y^k}{k}} dy = \frac{2}{\Gamma_k(\frac{n}{2})}\Gamma_k(\frac{n}{2}+k)$$

$$= \frac{2}{\Gamma_k(\frac{n}{2})}\frac{n}{2}\Gamma_k(\frac{n}{2}) = n,$$

which shows the required result.

(ii): To find the variance, we proceed as

$$E\left((\chi_k^2)^2\right) = \frac{4}{2\Gamma_k(\frac{n}{2})}\int_0^\infty (\frac{\chi_k^2}{2})^{\frac{n}{2}+1} e^{-\frac{(\chi_k^2)^k}{2k}} d\chi_k^2 \quad (2.4)$$

Putting $y = \frac{\chi_k^2}{2}$ implies $dy = \frac{d\chi_k^2}{2}$ and equation (2.4) takes the form

$$E[(\chi_k^2)^2] = \frac{4}{\Gamma_k(\frac{n}{2})}\int_0^\infty y^{\frac{n}{2}+1} e^{-\frac{y^k}{k}} dy = \frac{4}{\Gamma_k(\frac{n}{2})}\Gamma_k(\frac{n}{2}+2k)$$

$$= \frac{4}{\Gamma_k(\frac{n}{2})}\frac{n}{2}(\frac{n}{2}+k)\Gamma_k(\frac{n}{2}) = n(n+2k).$$

Now, variance is given by

$$Var(\chi_k^2) = E\left((\chi_k^2)^2\right) - \left(E(\chi_k^2)\right)^2 = n(n+2k) - n^2 = 2nk.$$

This implies that the variance of χ_k^2 is equal to the twice of degrees of freedom multiplied by the new parameter k.

2.3 Theorem.

For $k > 0$, the algebraic moments of order r, of generalized chi-square distribution are given by

$$\mu'_{r,k} = 2^r(\frac{n}{2})(\frac{n}{2}+k)\cdots(\frac{n}{2}+(r-1)k) \quad (2.5)$$

Proof: Using the definition of expected values, along with the generalized form of chi-square distribution, we get

$$\mu'_{r,k} = E((\chi_k^2)^r) = \frac{2^r}{2\Gamma_k(\frac{n}{2})}\int_0^\infty (\frac{\chi_k^2}{2})^{\frac{n}{2}+r-1} e^{-\frac{(\chi_k^2)^k}{2k}} d\chi_k^2. \quad (2.6)$$

Let $y = \frac{\chi_k^2}{2} \Rightarrow dy = \frac{d\chi_k^2}{2}$. Thus, the right hand side of equation (2.6) becomes,

$$\mu'_{r,k} = \frac{2^r}{\Gamma_k(\frac{n}{2})}\int_0^\infty y^{\frac{n}{2}+r-1} e^{-\frac{y^k}{k}} dy = \frac{2^r}{\Gamma_k(\frac{n}{2})}\Gamma_k(\frac{n}{2}+rk)$$

By the properties of k-gamma function given in the relation (1.9), we get the desired theorem.

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2.4 Theorem.

The characteristic function, for the generalized chi-square distribution with n degrees of freedom, is given by

$$\phi_k(t) = (1-2kt)^{-\frac{n}{2k}}; \qquad t < \frac{1}{2k}, k > 0. \tag{2.7}$$

Proof: By the definition of expected values, we see that

$$\phi_k(t) = E(e^{t\chi_k^2}) = \frac{1}{2\Gamma_k(\frac{n}{2})}\int_0^\infty (\frac{\chi_k^2}{2})^{\frac{n}{2}-1} e^{-\frac{1}{2k}(1-2kt)(\chi_k^2)^k} d\chi_k^2.$$

Setting $y = \frac{\chi_k^2(1-2kt)^{\frac{1}{k}}}{2}$ gives $\frac{dy}{(1-2kt)^{\frac{1}{k}}} = \frac{d\chi_k^2}{2}$ and above equation becomes

$$\begin{aligned}\phi_k(t) &= \frac{1}{2\Gamma_k(\frac{n}{2})(1-2kt)^{\frac{n}{2k}-\frac{1}{k}}}\int_0^\infty (y)^{\frac{n}{2}-1} e^{-\frac{y^k}{k}} \frac{dy}{(1-2kt)^{\frac{1}{k}}}\\ &= \frac{1}{\Gamma_k(\frac{n}{2})(1-2kt)^{\frac{n}{2k}}}\int_0^\infty (y)^{\frac{n}{2}-1} e^{-\frac{y^k}{k}} dy\\ &= (1-2kt)^{-\frac{n}{2k}}; \qquad t < \frac{1}{2k}, k > 0.\end{aligned}$$

Remarks: If $X_{(k)}$ and $Y_{(k)}$ are independent χ_k^2 random variables with n_1 and n_2 degrees of freedom respectively, then the sum $X_{(k)} + Y_{(k)}$ is a χ_k^2-random variable with $n_1 + n_2$ degrees of freedom. OR the sum of two χ_k^2 variates is equal to a χ_k^2 variate.

Proof: The moment generating function of the sum of two independent variates is equal to the product of their respective moment generating functions. Thus, we have

$$\begin{aligned}M(X_{(k)} + Y_{(k)}) &= M(X_{(k)}).M(Y_{(k)})\\ &= (1-2kt)^{\frac{-n_1}{2k}}.(1-2kt)^{\frac{-n_2}{2k}}\\ &= (1-2kt)^{-\frac{(n_1+n_2)}{2k}},\end{aligned}$$

which shows the desired result.

2.5 Theorem.

The χ_k^2-distribution tends to normal distribution as the number of degrees approaches infinity.

Proof: The moment generating function of $\chi_{k(n)}^2$ is $M_o(t) = (1-2kt)^{-\frac{n}{2k}}$. Let us consider the χ_k^2 standard variables $\frac{\chi_k^2 - n}{\sqrt{2nk}}$. Then its m.g.f. about mean is

$$\begin{aligned}M_\mu(t) &= e^{-\frac{nt}{\sqrt{2nk}}} M_o(\frac{t}{\sigma})\\ &= e^{-\frac{nt}{\sqrt{2nk}}}(1-\frac{2kt}{\sqrt{2nk}})^{-\frac{n}{2k}}.\end{aligned}$$

Taking natural log, we have

$$\begin{aligned}\ln M_\mu(t) &= -\frac{nt}{\sqrt{2nk}} - \frac{n}{2k}\ln(1-\frac{2kt}{\sqrt{2nk}})\\ &= -\frac{nt}{\sqrt{2nk}} + \frac{n}{2k}\Big[\frac{2kt}{\sqrt{2nk}} + \frac{1}{2}\frac{4k^2t^2}{2nk} + \text{higher powers of } \frac{1}{n}\Big]\\ &= \frac{t^2}{2} + \text{higher powers of } \frac{1}{n}.\end{aligned}$$

Thus as $n \to \infty$, then $\ln M_\mu(t) \to \frac{t^2}{2}$ so that $M_\mu(t) \to e^{\frac{t^2}{2}}$ which is the m.g.f. of standard normal variable. Hence the random variable $\frac{\chi_k^2 - n}{\sqrt{2nk}}$ tends to standard normal distribution and consequently the χ_k^2-distribution tends to normality as n approaches infinity.

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2.6 Theorem.

The curve of χ_k^2 distribution is positively skewed.

Proof: From the moment generating function of χ_k^2 distribution, we see that

$$\mu'_{1,k} = n, \qquad \mu'_{2,k} = n(n+2k),$$

$$\mu'_{3,k} = n(n+2k)(n+4k), \qquad \mu'_{4,k} = n(n+2k)(n+4k)(n+6k).$$

Now, the moment about mean (central moments) are given by

$$\mu_{1,k} = 0$$

$$\mu_{2,k} = \mu'_{2,k} - (\mu'_{1,k})^2 = n(n+2k) - n^2 = 2nk$$

$$\mu_{3,k} = \mu'_{3,k} - 3\mu'_{2,k}\mu'_{1,k} + 2(\mu'_{1,k})^3 = 8nk^2$$

$$\mu_{4,k} = \mu'_{4,k} - 4\mu'_{3,k}\mu'_{1,k} + 6\mu'_{2,k}(\mu'_{1,k})^2 - 3(\mu'_{1,k})^4 = 12nk^2(n+4k)$$

and

$$\beta_1 = \frac{(\mu_{3,k})^2}{(\mu_{2,k})^3} = \frac{k}{n}, \quad \text{and} \quad \beta_2 = \frac{\mu_{4,k}}{(\mu_{2,k})^2} = \frac{12nk^2(n+4k)}{4n^2k^2} = 3 + \frac{12k}{n}.$$

Remarks: As $\beta_1 \neq 0$ implies that the distribution is not symmetrical. Also, $\beta_2 => 3$ implies that it is extremely skewed. Actually, the curve of χ_k^2 distribution is positively skewed and the skewness decreases as n increases. For $n = 1$ and $k = 1$, we have

$$f(\chi_1^2) = \frac{1}{\sqrt{2\pi\chi_1^2}} e^{-\frac{\chi_1^2}{2}},$$

the curve is extremely J-shaped.

2.7 Theorem.

For $k > 0$, the χ_k^2 distribution with n degrees of freedom , the central moments obey the relation

$$\mu_{r+1,k} = 2r(\mu_{r,k} + n\mu_{r-1,k}). \tag{2.8}$$

Proof: The rth moment of χ_k^2 distribution about mean are given by

$$\mu_{r,k} = \frac{1}{2^{\frac{n}{2}}\Gamma_k(\frac{n}{2})} \int_0^\infty (\chi^2 - \mu_k)^r e^{-\frac{(\chi^2)^k}{2k}} (\chi^2)^{\frac{n}{2}-1} d(\chi^2).$$

and

$$\mu_{r-1,k} = \frac{1}{2^{\frac{n}{2}}\Gamma_k(\frac{n}{2})} \int_0^\infty (\chi^2 - \mu_k)^{r-1} e^{-\frac{(\chi^2)^k}{2k}} (\chi^2)^{\frac{n}{2}-1} d(\chi^2).$$

Using $\mu_k = n$, from the above equations, with some algebraic calculations along with integration by parts, we can easily conclude the required result.

Conclusion. In this paper, the authors conclude that

(i)The mean of the generalized chi-square remain the same for each positive value of k and the variance of the generalized chi-square will be multiplied with parameter k. If k tends to 1 then the variance will tends to the variance of chi-square.

(ii)The m.g.f of the generalized chi-square is $(1-2kt)^{\frac{-n}{2k}}$, if $k = 1$ then it will be equal to the m.g.f of simple chi-square.

(iii)The moments of the generalized chi-square is μ_1 is equal to n for each value of $k > 0$ and the moments $\mu_2, \mu_3, \cdots$ will change with the parameter k, if k tends to 1 then it will simply be the moments of chi-square.

(iv)The generalized chi-square tends to a standard normal distribution for each positive value of k.

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Conflict of Interests.
The author(s) declare(s) that there is no conflict of interests regarding the publication of this article.

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