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The Eigenvalues and The Optimal Potential Functions of Sturm-Liouville Operators

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Abstract: In this article, we study the eigenvalues and potential functions of Sturm-Liouville operators with ordinary separated-type boundary condition. When the gap of the first two eigenvalues reaches minimum, we give the specific form of potential function. Meanwhile, for step potential function, we establish an one-to-one relationship between the eigenvalues and the nonnegative real roots of a class of algebraic equation, which provide an effective method for the approximate calculation of eigenvalues.

Keywords: Sturm-Liouville operators; eigenvalues; potential function.

1. Introduction

The eigenvalues of Sturm-Liouville operators such as $Schrödinger$ operators acting on $[0,\pi]$ are corresponding to the energy lever of the quantum state of particles. So, it is a meaningful work to calculate the eigenvalues of Sturm-Liouville operators, especially, the gap of the first two eigenvalues, which stands for the energy of excited state of particles. While, attributing to the restriction of mathematical methods, it is a challenging work to calculate the exact value of eigenvalues. Therefore, the estimate of eigenvalues, especially, the gap of the first two eigenvalues of Sturm-Liouville operators and the description about the form of corresponding potential functions have been receiving more and more attention from scholars recently.

For Sturm-Liouville problems:

$$\begin{cases} -y'' + v(x)y = \lambda y,\ x \in [0,\pi] \\ y(0) = y(\pi) = 0 \ or \ y'(0) = y'(\pi) = 0, \end{cases} \tag{1.1}$$

Singer, Wong, You gave a lower bound of the gap of the first two eigenvalues to S-L problem (1.1) in [1-2]. Meanwhile, they described the form of the corresponding potential function. In 1994, Richard, Lavine obtained the optimal lower bound of the gap of the first two eigenvalues to S-L problem (1.1). Simultaneously, they depicted the form of the corresponding potential function (see [3]). But they all relied on the assumption that the potential function is convex. In 2003, Miklás Horváth got rid of the assumption that the potential function must be convex (see [4]). However, owing to the restriction of methods of study, the foregoing results on Sturm-Liouville operators with ordinary separated type boundary condition have not been addressed so far.

Stimulated by [1-12], in this article, we study the S-L problem:

$$\begin{cases} -y'' + v(x)y = \lambda y,\ x \in [0,\pi] \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0 \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases} \tag{1.2}$$

where $v(x)$ is a single-well function (see definition 2.1), $\alpha, \beta \in (0,\pi)$. Owing to the complicacy of the boundary condition, it is very difficult to calculate the exact value of eigenvalues to S-L problem (1.2). In this article, we focus on the estimate of the optimal lower bound of the gap of the first two eigenvalues to S-L problem (1.2) and the description about the form of the optimal potential function(see definition 2.2).

Meanwhile, under step potential function, we establish one-to-one relationship between the eigenvalues of S-L problem (1.3) and the real roots of a class of algebraic equation, which makes the approximate calculation of eigenvalues to S-L problem (1.3) into reality.

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The rest of this article is organized as follows. In section 2, we give the explicit form of the optimal potential function of S-L problem (1.2). In section 3, Considering S-L problem:

$$\begin{cases} -y'' + q(x)y = \lambda y, \ x \in [0, \pi] \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0 \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases} \tag{1.3}$$

where $\alpha, \beta \in (0, \pi)$ and

$$q(x) = \begin{cases} m, \ x \in [0, \pi/2] \\ n, \ x \in (\pi/2, \pi], \end{cases} \tag{1.4}$$

where m and n are all real number. We establish an one-to-one relationship between the eigenvalues to the S-L problem (1.3) and the nonnegative real roots to a class of algebraic equation. An example and numerical simulations are presented in the final section.

2. The Optimal Potential Functions of Sturm-Liouville Operators

In this section, we will give the explicit form of the optimal potential function of S-L problem (2.1).

Definition 2.1 We call function v a single-well function on $[0, \pi]$, if v is decreasing on $[0, \pi/2]$ and increasing on $[\pi/2, \pi]$. The notation below will be used through this article:

$$SW_{[0,\ \pi]} = \{v(x) : v(x) \ is \ a \ single - well \ function \ on \ [0, \pi]\}.$$

Definition 2.2 Consider S-L problem

$$\begin{cases} -y'' + v(x)y = \lambda y, \ x \in [0, \pi] \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0 \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases} \tag{2.1}$$

where $v \in \ SW_{[0,\ \pi]}$, $\alpha, \beta \in (0, \pi)$, $\lambda_1(v), \lambda_2(v)$ are the first and the second eigenvalue to S-L problem (2.1), respectively. Denote $G(v) = \lambda_2(v) - \lambda_1(v)$. if there exists v_0 such that

$$G(v_0) = \inf_{v \in \ SW_{[\ 0,\ \pi]}} G(v),$$

we call function v_0 an optimal potential function of S-L problem (2.1).

Theorem 2.1(see[4]) For S-L problem

$$\begin{cases} -y'' + v(x,t) = \lambda y, \ x \in [0, \pi] \\ L_1(y) = 0 \\ L_2(y) = 0. \end{cases} \tag{2.2}$$

Suppose that $\lambda_n(t)$ is the n-th eigenvalue to S-L problem (2.2) and $u_n(x,t)$ is corresponding to the n-th normalizing eigenfunction, then

$$d\lambda_n(t)dt = \int_0^\pi \partial v(x,t)\partial t u_n^2(x,t)dx.$$

Lemma 2.1 Suppose that $u_1(x), u_2(x)$ are the first and the second eigenfunction of S-L problem (2.1), we can show that $u_2(x)u_1(x)$ is strictly monotonous decreasing on $[0, \pi]$.

proof : Since $u_1(x)$ is the first eigenfunction of S-L problem (2.1), there are not zero points of $u_1(x)$ on $[0, \pi]$, and we can suppose $u_1(x) > 0$ ($u_1(x) < 0$ is similar). And because $u_2(x)$ is the second eigenfunction of S-L problem (2.1), there exists only one zero point of $u_2(x)$ on $[0, \pi]$, denoted by x_0, then, $u_2(x) > 0, whilex \in [0, x_0)$ and $u_2(x) < 0, whilex \in (x_0, \pi]$. Next, we show $(u_2(x)u_1(x))' < 0, foreveryx \in [0, \pi]$.

Since

$$\Big(u_2(x)u_1(x)\Big)' = u_2'(x)u_1(x) - u_1'(x)u_2(x)[u_1(x)]^2,$$

note that

$$\int_0^x [u_2''(x)u_1(x) - u_1''(x)u_2(x)]dx = [u_2'(x)u_1(x) - u_1'(x)u_2(x)] - [u_2'(0)u_1(0) - u_1'(0)u_2(0)],$$

with the boundary condition $y(0)\cos\alpha + y'(0)\sin\alpha = 0$ and $\alpha \in (0,\pi)$, we can obtain $[u_2'(0)u_1(0) - u_1'(0)u_2(0)] = 0$. Considering $u_2''(x)u_1(x) - u_1''(x)u_2(x) = (\lambda_1 - \lambda_2)u_1(x)u_2(x)$. Then, when $0 \le x < x_0$, we have

$$\Big(u_2(x)u_1(x)\Big)' = u_2'(x)u_1(x) - u_1'(x)u_2(x)[u_1(x)]^2 = 1u_1^2(x)\int_0^x [(\lambda_1 - \lambda_2)u_1(x)u_2(x)]dx < 0.$$

On the other hand,

$$\int_x^\pi [u_2''(x)u_1(x) - u_1''(x)u_2(x)]dx = [u_2'(\pi)u_1(\pi) - u_1'(\pi)u_2(\pi)] - [u_2'(x)u_1(x) - u_1'(x)u_2(x)],$$

with the boundary condition $y(\pi)\cos\beta + y'(\pi)\sin\beta = 0$ and $\beta \in (0,\pi)$, we can derive $u_2'(\pi)u_1(\pi) - u_1'(\pi)u_2(\pi) = 0$. So,

$$\int_x^\pi [u_2''(x)u_1(x) - u_1''(x)u_2(x)]dx = -[u_2'(x)u_1(x) - u_1'(x)u_2(x)].$$

Then

$$(u_2(x)u_1(x))' = u_2'(x)u_1(x) - u_1'(x)u_2(x)[u_1(x)]^2 = -1u_1^2(x)\int_x^\pi [u_2''(x)u_1(x) - u_1''(x)u_2(x)]dx.$$

Note that $u_2''(x)u_1(x) - u_1''(x)u_2(x) = (\lambda_1 - \lambda_2)u_1(x)u_2(x)$. Then, when $x_0 < x \le \pi$, we have

$$(u_2(x)u_1(x))' = u_2'(x)u_1(x) - u_1'(x)u_2(x)[u_1(x)]^2 = -1u_1^2(x)\int_x^\pi [(\lambda_1 - \lambda_2)u_1(x)u_2(x)]dx < 0.$$

This completes the proof.

Lemma 2.2 Assume that $u_1(x), u_2(x)$ are the first and the second normalizing eigenfunction of S-L problem (2.1), respectively, then the equation $u_1^2(x) = u_2^2(x)$ has at least one solution and at most two solutions in $(0,\pi)$.

proof: (1) There is at least one solution to the equation $u_1^2(x) = u_2^2(x)$ in $(0,\pi)$.

Since $\int_0^\pi u_1^2(x)dx = \int_0^\pi u_2^2(x)dx = 1$, there is at least one solution to the equation $u_1^2(x) = u_2^2(x)$ in $(0,\pi)$.

(2)There is at most two solutions to the equation $u_1^2(x) = u_2^2(x)$ in $(0,\pi)$.

By the lemma 2.1, $u_2^2(x)u_1^2(x)$ is strictly monotonous decreasing in $(0, x_0)$, and strictly monotonous increasing in (x_0,π). Assume that $x_1, x_2, (0 < x_1 < x_2 < x_0)$ are two different solutions to the equation $u_1^2(x) = u_2^2(x)$, then

$$u_2^2(x_1)u_1^2(x_1) = u_2^2(x_2)u_1^2(x_2),$$

which is a contradiction. So, in $(0, x_0)$, there exists at most one solution to the equation $u_1^2(x) = u_2^2(x)$. Using similar arguments as above, we can show that there exists at most one solution of the equation $u_1^2(x) = u_2^2(x)$ in (x_0,π). So, there is at most two solutions to the equation $u_1^2(x) = u_2^2(x)$ in $(0,\pi)$. This completes the proof of Lemma 2.2.

Definition 2.3 (1) If for a given potential function $v(x) \in SW_{[0,\ \pi]}$, the equation $u_1^2(x) = u_2^2(x)$ has two solutions in $(0,\pi)$, where $u_1(x), u_2(x)$ are the first and the second normalizing eigenfunction of S-L problem (2.1), then, we call the potential function $v(x)$ the first class of potential function, and the set of all these potential functions is denoted by PF_I. (2) If for a given potential function $v(x) \in SW_{[0,\pi]}$, the equation $u_1^2(x) = u_2^2(x)$ has only one solution in $(0,\pi)$, where $u_1(x), u_2(x)$ are the first and the second normalizing eigenfunction of S-L problem (2.1), then we call the potential function $v(x)$ the second class of potential function, and the set of all these potential functions is denoted by PF_{II}.

By Lemma 2.1 and 2.2, we can easily obtain the following result.

Lemma 2.3 Suppose that $u_1(x), u_2(x)$ are the first and the second normalizing eigenfunction of S-L problem (2.1), then

(1) If $v \in PF_I$, then for $u_1(x)$ and $u_2(x)$ there is only one case to consider, that is, there exist x_+, x_- $(0 < x_- < x_+ \le \pi)$ such that

$$\begin{cases} u_2^2(x) - u_1^2(x) > 0, & x \in (0, x_-)\bigcup(x_+,\pi) \\ u_2^2(x) - u_1^2(x) < 0, & x \in (x_-, x_+). \end{cases}$$

(2) If $v \in PF_{II}$, then, then for $u_1(x)$ and $u_2(x)$ there are only following two cases to consider.

(i) Existing $x^* \in (0,\pi)$ such that

$$\begin{cases} u_2^2(x) - u_1^2(x) > 0, & x \in (0, x_*) \\ u_2^2(x) - u_1^2(x) < 0, & x \in (x_*,\pi). \end{cases}$$

(ii)Existing $x^* \in (0,\pi)$ such that

$$\begin{cases} u_2^2(x) - u_1^2(x) < 0, & x \in (0, x_*) \\ u_2^2(x) - u_1^2(x) > 0, & x \in (x_*,\pi). \end{cases}$$

Lemma 2.4 Define $A_M = \{V : 0 \leq V \leq M,\ V \in SW_{[0,\ \pi]}\}$, then $A_M \subset SW_{[0,\ \pi]}$ and A_M is a convex set.

proof: By the definition of A_M, $A_M \subset SW_{[0,\ \pi]}$ is obvious. Next, we show that A_M is a convex set. For all $v_1, v_2 \in A_M$ and for every $t \in [0,1]$, we have $0 \leq tv_1 + (1-t)v_2 \leq M$. And because $tv_1 \in SW_{[0,\ \pi]},\ (1-t)v_2 \in SW_{[0,\ \pi]}$, we have $tv_1 + (1-t)v_2 \in SW_{[0,\ \pi]}$. Thus, $tv_1 + (1-t)v_2 \in A_M$, namely, A_M is a convex set.

Lemma 2.5 Define $v(x,t) = tv_1(x) + (1-t)v_0(x), t \in [0,1],\ v_0(x), v_1(x) \in A_M$, where $v_0(x)$ is the optimal potential function, $\lambda_1(t), \lambda_2(t)$ are the first and the second eigenvalue to S-L problem

$$\begin{cases} -y'' + v(x,t) = \lambda y,\ x \in [0,\pi] \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0 \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0. \end{cases}$$

Then, we have

$$d(\lambda_2(t) - \lambda_1(t))dt = \int_0^\pi [v_1(x) - v_0(x)][u_2^2(x,t) - u_1^2(x,t)]dx$$

and

$$0 \leq d(\lambda_2(t) - \lambda_1(t))dt|_{t=0} = \int_0^\pi [v_1(x) - v_0(x)][u_2^2(x) - u_1^2(x)]dx.$$

Proof: By theorem 2.1, we have

$$d(\lambda_2(t) - \lambda_1(t))dt = \int_0^\pi \partial v(x,t)\partial t[u_2^2(x,t) - u_1^2(x,t)]dx = \int_0^\pi [v_1(x) - v_0(x)][u_2^2(x,t) - u_1^2(x,t)]dx.$$

When $t = 0$, we have $v(x,t) = v(x,0) = v_0(x)$. Since $v_0(x)$ is the optimal potential function, $\lambda_2(v(x,0)) - \lambda_1(v(x,0))$ reaches minimum, thus, along with the increase of t, $\lambda_2(v(x,t)) - \lambda_1(v(x,t))$ will be increase, so

$$0 \leq d(\lambda_2(t) - \lambda_1(t))dt|_{t=0} = \int_0^\pi [v_1(x) - v_0(x)][u_2^2(x) - u_1^2(x)]dx.$$

This finishes the proof.

Next, we give the first main result of this article.

Theorem 2.2 Suppose that $\lambda_1(v), \lambda_2(v), (v \in A_M)$ are the first and the second eigenvalue to S-L problem (2.1), then the optimal potential function of S-L problem (2.1) must be a step function on $[0,\pi]$.

proof: By [4], the optimal potential function of S-L problem (2.1) exists, when $v(x) \in A_M$, and we denote it by v_0. Next, we will discuss the explicit form of v_0 from the following two cases.

Case1. $v_0 \in PF_I$

By lemma 2.3, there exist x_+ and x_- ($0 < x_- < x_+ < \pi$) such that

$$\begin{cases} u_2^2(x) - u_1^2(x) > 0,\ \ x \in (0, x_-)\bigcup(x_+, \pi) \\ u_2^2(x) - u_1^2(x) < 0,\ \ x \in (x_-, x_+), \end{cases} \tag{2.3}$$

where, $u_1(x)$, $u_2(x)$ are the first and the second normalizing eigenfunction. Here, only to consider the following two cases.

(1) $x_- \leq \pi 2 < x_+$($x_- < \pi 2 \leq x_+$ is similar)

Let

$$v_1(x) = \begin{cases} v_0(x_-),\ \ x \in [0, \pi/2] \\ v_0(x_+),\ \ x \in (\pi/2, \pi]. \end{cases}$$

Obviously, $v_1(x) \in A_M$, and

$$\begin{cases} v_1(x) - v_0(x) \leq 0,\ \ x \in (0, x_-)\bigcup(x_+, \pi) \\ v_1(x) - v_0(x) \geq 0,\ \ x \in (x_-, x_+). \end{cases} \tag{2.4}$$

By lemma 2.5, the following result

$$\int_0^\pi (v_1(x) - v_0(x))(u_2^2(x) - u_1^2(x))dx \geq 0$$

holds. That is

$$\int_{(0,x_-)\bigcup(x_+,\pi)}(v_1(x)-v_0(x))(u_2^2(x)-u_1^2(x))dx+\int_{(x_-,x_+)}(v_1(x)-v_0(x))(u_2^2(x)-u_1^2(x))dx\geq 0. \tag{2.5}$$

By (2.3) and (2.4), (2.5) holds if and only if $v_0(x)=v_1(x)$. Thus, $v_0(x)$ must be a step function with the following form:

$$v_0(x)=\begin{cases} v_0(x_-), & x\in[0,\pi/2] \\ v_0(x_+), & x\in(\pi/2,\pi]. \end{cases}$$

(2) $\pi/2<x_-$ ($x_+<\pi/2$ is similar)
Let

$$v_1(x)=\begin{cases} v_0(\pi 2), & x\in[0,x_-] \\ v_0(x_+), & x\in(x_-,\pi]. \end{cases}$$

Analogously, by the same methods of (1), we can obtain $v_0=v_1$. Furthermore, we can also choose

$$v_1(x)=\begin{cases} 0, & x\in[0,x_-] \\ M, & x\in(x_-,\pi]. \end{cases}$$

By lemma 2.5, we have

$$\int_0^\pi(v_1(x)-v_0(x))(u_2^2(x)-u_1^2(x))dx\geq 0,$$

that is

$$\begin{aligned} &0\leq\int_0^\pi(v_1(x)-v_0(x))(u_2^2(x)-u_1^2(x))dx \\ &=-v_0(\pi 2)\int_0^{x_-}(u_2^2(x)-u_1^2(x))dx \\ &+(M-v_0(x_+))\int_{x_-}^\pi(u_2^2(x)-u_1^2(x))dx. \end{aligned} \tag{2.6}$$

Then, by the definition of x_-, we have

$$\int_0^{x_-}(u_2^2(x)-u_1^2(x))dx>0. \tag{2.7}$$

While,

$$\int_0^\pi(u_2^2(x)-u_1^2(x))dx=0,$$

so,

$$\int_{x_-}^\pi(u_2^2(x)-u_1^2(x))dx<0. \tag{2.8}$$

By (2.7) and (2.8), (2.6) holds if and only if $v_0(\pi 2)=0$ and $v_0(x_+)=M$. Namely, $v_0(x)$ must be a step function as following

$$v_0(x)=\begin{cases} 0, & x\in[0,x_-] \\ M, & x\in(x_-,\pi]. \end{cases}$$

Case 2. $v_0\in PF_{II}$. (By lemma 2.3, we only consider case (i), and case (ii) is similar)
(1) $0<x^*\leq\pi/2$
We choose

$$v_1(x)=\begin{cases} v_0(x^*), & x\in[0,\pi/2] \\ M, & x\in(\pi/2,\pi]. \end{cases} \tag{2.9}$$

Analogously, by lemma 2.5, we have

$$0\leq\int_0^\pi(v_1(x)-v_0(x))(u_2^2(x)-u_1^2(x))dx$$

$$=\int_0^{x^*}(v_1(x)-v_0(x))(u_2^2(x)-u_1^2(x))dx+\int_{x^*}^\pi(v_1(x)-v_0(x))(u_2^2(x)-u_1^2(x))dx. \tag{2.10}$$

By (2.9) and the definition of x^*, (2.10) holds if and only if $v_0(x) = v_1(x)$, namely, $v_0(x)$ must be a step function as following

$$v_0(x) = \begin{cases} n_1, & x \in [0, \pi/2] \\ n_2, & x \in (\pi/2, \pi]. \end{cases}$$

(2) $x^* > \pi/2$

Now let

$$v_1(x) = \begin{cases} 0, & x \in [0, x^*] \\ M, & x \in (x^*, \pi], \end{cases}$$

by the lemma 2.5, we can obtain that

$$0 \le \int_0^{\pi} (v_1(x) - v_0(x))(u_2^2(x) - u_1^2(x))dx$$

$$= \int_0^{x^*} (v_1(x) - v_0(x))(u_2^2(x) - u_1^2(x))dx + \int_{x^*}^{\pi} (v_1(x) - v_0(x))(u_2^2(x) - u_1^2(x))dx, \quad (2.11)$$

because of $0 \le v_0 \le M$ and the definition of x^*, (2.11) holds if and only if

$$v_0(x) = \begin{cases} 0, & x \in [0, x^*] \\ M, & x \in (x^*, \pi]. \end{cases}$$

So, comprehensive case 1 and case 2, and the optimal potential function of S-L problem (2.1) must be a step function on $[0, \pi]$. This completes the proof.

3. Eigenvalues of Sturm-Liouville Operators

In this section, we first establish an one-to-one relationship between the eigenvalues to the S-L problem (3.1) and the nonnegative real roots to a class of algebraic equations, which made the approximate calculation of eigenvalues to S-L problem (3.1) into reality.

we first consider S-L problem:

$$\begin{cases} -y'' + q(x)y = \lambda y, \ x \in [0, \pi] \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0 \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases} \quad (3.1)$$

where $\alpha, \ \beta \in (0, \pi)$ and

$$q(x) = \begin{cases} m, \ x \in [0, x_0] \\ n, \ x \in (x_0, \pi]. \end{cases}$$

For the sake of simplicity and convenience in the following expression, the notations below will be used in this section:

$$C_x(\lambda, y) = \cos(\sqrt{\lambda - x}y), \quad S_x(\lambda, y) = \sin(\sqrt{\lambda - x}y),$$

$$F_{n,\ \beta}(\lambda) = \sqrt{\lambda - n}\cos(\sqrt{\lambda - n}\pi)\sin\beta + \sin(\sqrt{\lambda - n}\pi)\cos\beta,$$

$$G_{n,\ \beta}(\lambda) = \sqrt{\lambda - n}\sin(\sqrt{\lambda - n}\pi)\sin\beta - \cos(\sqrt{\lambda - n}\pi)\cos\beta,$$

We then have the following result.

Theorem 3.1 Eigenvalues to S-L problem (3.1) are corresponding to the nonnegative real roots of the following algebraic equation

$$\sqrt{\lambda - m}C_m(\lambda,\ x_0) - \cot\alpha S_m(\lambda,\ x_0)(\lambda - m)S_m(\lambda,\ x_0) + \sqrt{\lambda - m}\cot\alpha C_m(\lambda,\ x_0)$$
$$= S_n(\lambda,\ x_0)G_{n,\ \beta}(\lambda) + C_n(\lambda, x_0)F_{n,\ \beta}(\lambda)\sqrt{\lambda - n}S_n(\lambda,\ x_0)F_{n,\ \beta}(\lambda) - \sqrt{\lambda - n}C_n(\lambda,\ x_0)G_{n,\ \beta}(\lambda) 3.2$$

Proof: Since $y(x) = c_1\cos\sqrt{\lambda - m}x + c_2\sin\sqrt{\lambda - m}x$ is the general solution to the boundary value problem:

$$\begin{cases} -y'' + my = \lambda y, \ x \in [0, x_0] \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0, \end{cases} \quad (3.3)$$

where m is a positive integer, $\alpha \in [0, \pi]$. By the boundary value condition $y(0)\cos\alpha + y'(0)\sin\alpha = 0$, it follows that

$$c_1\cos\alpha + c_2\sqrt{\lambda - m}\sin\alpha = 0.$$

Then

$$c_2 = -c_1 \cot\alpha\sqrt{\lambda - m}.$$

So, the general solution of boundary value problem(3.2) can be written as

$$y_1(x) = c_1 \cos\sqrt{\lambda - m}x - c_1 \cot\alpha\sqrt{\lambda - m}\sin\sqrt{\lambda - m}x.$$

For boundary value problem:

$$\begin{cases} -y'' + ny = \lambda y,\ x \in (x_0, \pi] \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases} \tag{3.4}$$

similar to above methods, we can write the general solution of boundary value problem (3.3) as

$$y_2(x) = d_1 \cos\sqrt{\lambda - n}x + d_1\sqrt{\lambda - n}\sin\sqrt{\lambda - n}\pi\sin\beta - \cos\sqrt{\lambda - n}\pi\cos\beta\sqrt{\lambda - n}\cos\sqrt{\lambda - n}\pi\sin\beta$$
$$+\sin\sqrt{\lambda - n}\pi\cos\beta\sin\sqrt{\lambda - n}x.$$

Then, the solution of boundary value problem (3.1) can be further simplified as

$$y(x) = \begin{cases} y_1(x),\ x \in [0, x_0] \\ y_2(x),\ x \in (x_0, \pi]. \end{cases} \tag{3.5}$$

The constants $c_1,\ d_1$ have to be chosen such that $y(x)$ is C^1-smooth at x_0. This can be done if and only if the quotients $y'y$, counted in x_0 from both sides, are the same, that is, when

$$\sqrt{\lambda - m}C_m(\lambda,\ x_0) - \cot\alpha S_m(\lambda,\ x_0)(\lambda - m)S_m(\lambda,\ x_0) + \sqrt{\lambda - m}\cot\alpha C_m(\lambda,\ x_0)$$
$$= S_n(\lambda,\ x_0)G_{n,\ \beta}(\lambda) + C_n(\lambda,\ x_0)F_{n,\ \beta}(\lambda)\sqrt{\lambda - n}S_n(\lambda,\ x_0)F_{n,\ \beta}(\lambda) - \sqrt{\lambda - n}C_n(\lambda,\ x_0)G_{n,\ \beta}(\lambda), 3.6$$

the eigenvalues of S-L problem (3.1) are the nonnegative real solutions of equation (3.6). Theorem 3.1 is proved.

4. Example and Numerical Simulation

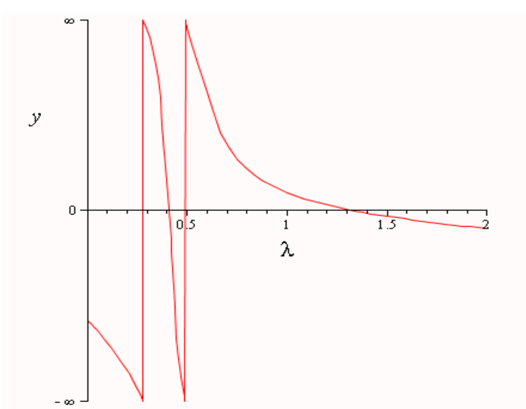

Figure 1 The first four eigenvalues in [0 ,2]

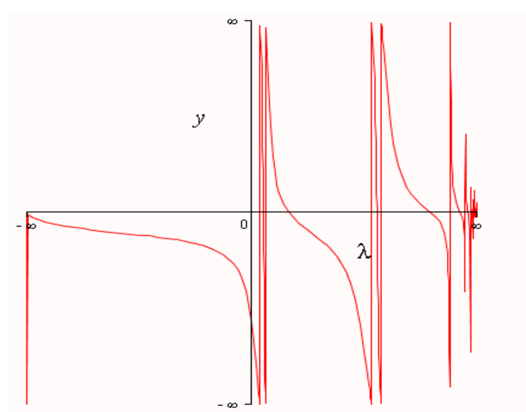

Figure 2 All eigenvalues in R

In this section, we give an example for Theorem 3.1 and calculate the approximates of its eigenvalues.
We consider S-L problem

$$\begin{cases} -y'' + q(x)y = \lambda y,\ x \in [0, \pi] \\ y(0)\cos\pi 3 + y'(0)\sin\pi 3 = 0 \\ y(\pi)\cos\pi 3 + y'(\pi)\sin\pi 3 = 0, \end{cases} \tag{4.1}$$

where

$$q(x) = \begin{cases} 1,\ x \in [0, \pi/2] \\ 0,\ x \in (\pi/2, \pi]. \end{cases}$$

By Theorem 3.1, the eigenvalues to S-L problem (4.1) are the nonnegative real roots of the following equation:

$$3\sqrt{\lambda-1}\cos(\sqrt{\lambda-1}2\pi)-\sqrt{3}\sin(\sqrt{\lambda-1}2\pi)3(\lambda-1)\sin(\sqrt{\lambda-1}2\pi)+\sqrt{3(\lambda-1)}\cos(\sqrt{\lambda-1}2\pi)$$
$$=\sqrt{3\lambda}\cos(\sqrt{\lambda}2\pi)+\sin(\sqrt{\lambda}2\pi)\sqrt{3}\lambda\sin(3\sqrt{\lambda}2\pi)-\sqrt{\lambda}\cos(\sqrt{\lambda}2\pi) \tag{4.2}$$

Using Matlab, we can calculate the approximates of nonnegative real roots to equation (4.2), thus, we obtain the approximates of eigenvalues to S-L problem (4.1). Here, a straightforward calculation shows the approximates of the first four eigenvalues: $\lambda_1 \approx 0.2791$, $\lambda_2 \approx 0.4085$, $\lambda_3 \approx 0.4898$, $\lambda_4 \approx 1.3225$ (see Figure 1). Meanwhile, we can validate that the number of the eigenvalues to S-L problem (4.1) is infinite and all eigenvalues are nonnegative (see Figure 2).

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