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On System of Nonlinear Fractional Differential Equations Involving Hadamard Fractional Derivative with Nonlocal Integral Boundary Conditions

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Abstract: This article discuss the existence and uniqueness of solutions for a system of nonlinear fractional differential equations involving Hadamardfractional derivative with nonlocal mixed boundary conditions with multiple orders. Example is given to demonstrate application of our results.

Keywords: Existence and uniqueness, contraction mapping principle, Hadamard fractional operator, boundary value problem.

1 Introduction, motivation and preliminaries

Approximately 322-years old ago fractional calculus was paid attention to most of the available fractional differential equations based on Riemann-Liouville and Caputo operators. One of the important characteristics of fractional operators is their nonlocal nature. Counting for the hereditary properties of many phenomena and processes involved.

The theory of fractional order differential equations involving different kinds of boundary conditions has been a field of interest in pure and applied sciences. In addition to the classical two-point boundary conditions, great attention is paid to nonlocal multipoint and integral boundary conditions. Nonlocal conditions are used to describe certain features of physical, chemical or other processes occurring in the internal positions of the given region, while integral boundary conditions provide a plausible and practical approach to modeling the problems of blood flow. For more details and explanation, see, for instance [1].

The efficient of the fractional differentiation approach has been proven in various sciences branches such as physics, chemistry, epidemiology, finance and biology sciences [1,2,3,4,5,6], these are few of them just to mention. Hadamard derivative differs from the preceding ones in the sense that the kernel of the integral contains a logarithmic function of arbitrary exponent. Details and properties of the Hadamard fractional derivative and integral can be found in [7,8,9,10, 11,12].

However, differential equations with Hadamard derivatives is still studied less than that of Riemann-Liouville and Caputo fractional differential equations, see [13,14,15,16,17,18,19,20,21]. The purpose of this article is to investigate the existence of solutions for the following system of nonlinear fractional derivative subject to the mixed Hadamard fractional derivative and Hadamard fractional integral conditions with multiple orders.

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$$\begin{aligned} {}_HD^{\alpha}_{a^+}u(t) &= f\left(t,u\left(t\right),v(t)\right)\ ,\ 1<\alpha\leq 2\ ,\ a\leq t\leq T,\\ {}_HD^{\beta}_{a^+}v(t) &= g\left(t,u\left(t\right),v(t)\right)\ ,\ 1<\beta\leq 2\ ,\ a\leq t\leq T,\\ u(a) &= 0,\ \ k_1{}_HI^{p_1}_{a^+}u(T)+M_1{}_HD^{p_2}_{a^+}u(T)=\delta_1\int_a^{\xi}u\left(s\right)ds+\varepsilon_1,\\ v(a) &= 0,\ \ k_2{}_HI^{q_1}_{a^+}v(T)+M_2{}_HD^{q_2}_{a^+}v(T)=\delta_2\int_a^{\eta}v\left(s\right)ds+\varepsilon_2, \end{aligned} \tag{1}$$

where ${}_HD^{\theta}_{a^+}$ is the Hadamard fractional derivative of order $\theta_1=\{\alpha,\beta,p_2,q_2\}$ and ${}_HI^{\theta_2}_{a^+}$ is the Hadamard fractional integral of order $\theta_2=\{p_1,q_1\}$ with $\alpha-1<p_1,p_2<\alpha$, $\beta-1<q_1,q_2<\beta, a<\xi,\eta\leq T$, $\delta_1,\delta_2,\ \varepsilon_1,\varepsilon_2,k_1,k_2,M_1,M_2\in\mathbb{R}$ $\ f,g\in C\left([a,T]\times\mathbb{R}^2,\mathbb{R}\right)$.
The reminder of the article is as follow: in the next Section, we present a basic concepts of Hadamard fractional calculus. In Section 3, main results are given to investigate the existence and uniqueness of solutions for the problem 1. This paper ends in Section 4 with a concluding remarks.

2 Preliminaries

In this section we introduce definition of the Hadamard fractional integral and derivative and present an auxiliary lemma to define the solution of problem 1.
Definition 2.1 The Hadamard fractional integrals of order α for a continuous function $\varphi:[a,\infty]\to\mathbb{R}$ are defined by

$${}_HI^{\alpha}_{a^+}\varphi(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\left(ln\frac{x}{t}\right)^{\alpha-1}\varphi(t)\frac{dt}{t}\ .$$

Definition 2.2 The Hadamard fractional derivatives of order α for a continuous function $\varphi:[a,\infty]\to\mathbb{R}$ are defined by ${}_HD^{\alpha}_{a^+}\varphi(x)=\delta^n\left({}_HI^{n-\alpha}_{a^+}\varphi\right)(x)=\left(x\frac{d}{dx}\right)^n\frac{1}{\Gamma(n-\alpha)}\int_a^x\left(ln\frac{x}{t}\right)^{n-\alpha-1}\varphi(t)\frac{dt}{t}$,

$${}_HD^{\alpha}_{a^+}\left(ln\frac{t}{a}\right)^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(ln\frac{t}{a}\right)^{\beta-\alpha-1}.$$

then

$${}_HI^{\alpha}_{a^+}\left({}_HD^{\alpha}_{a^+}x\right)(t)=x(t)-\sum_{j=1}^{n}c_j\left(ln\frac{t}{a}\right)^{\alpha-1}$$

.
Notations

$$\Delta_1=k_1\frac{\Gamma(p_1)}{\Gamma(p_1+\alpha)}\left(ln\frac{T}{a}\right)^{\alpha+p_1-1}+M_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-p_2)}\left(ln\frac{T}{a}\right)^{\alpha-p_2-1}-\delta_1\int_a^{\xi}\left(\ln\frac{s}{a}\right)^{\alpha-1}ds,$$

$$\lambda_1(t)=\frac{\varepsilon_1}{\Delta_1}\left(ln\frac{t}{a}\right)^{\alpha-1},\ \lambda_2(t)=\frac{K_1}{\Delta_1}\left(ln\frac{t}{a}\right)^{\alpha-1},\ \lambda_3(t)=\frac{M_1}{\Delta_1}\left(ln\frac{t}{a}\right)^{\alpha-1},\ \lambda_4(t)=\frac{\delta_1}{\Delta_1}\left(ln\frac{t}{a}\right)^{\alpha-1},$$

$$\Delta_2=k_2\frac{\Gamma(q_1)}{\Gamma(q_1+\beta)}\left(ln\frac{T}{a}\right)^{\beta+q_1-1}+\frac{M_2\Gamma(\beta)}{\Gamma(\beta-q_2)}\left(ln\frac{T}{a}\right)^{\beta-q_2-1}-\delta_2\int_a^{\eta}\left(\ln\frac{s}{a}\right)^{\beta-1}ds,$$

$$\phi_1(t)=\frac{\varepsilon_2}{\Delta_2}\left(ln\frac{t}{a}\right)^{\beta-1},\ \phi_2(t)=\frac{K_2}{\Delta_2}\left(ln\frac{t}{a}\right)^{\beta-1},\ \ \phi_3(t)=\frac{M_2}{\Delta_2}\left(ln\frac{t}{a}\right)^{\beta-1},\ \phi_4(t)=\frac{\delta_2}{\Delta_2}\left(ln\frac{t}{a}\right)^{\beta-1}.$$

Lemma 3.2 The following system given by

$$\begin{cases} {}_HD^{\alpha}_{a^+}u(t)=w_1(t)\ ,\ 1<\alpha\leq 2\ ,\ 0<a\leq t\leq T,\\ {}_HD^{\beta}_{a^+}v(t)=w_2(t)\ ,\ 1<\beta\leq 2\ ,\ a\leq t\leq T,\\ u(a)=0,\ \ k_1{}_HI^{p_1}_{a^+}u(T)+M_1{}_HD^{p_2}_{a^+}u(T)=\delta_1\int_a^{\xi}u(s)\ ds\ +\varepsilon_1,\\ v(a)=0,\ \ k_2{}_HI^{q_1}_{a^+}v(T)+M_2{}_HD^{q_2}_{a^+}v(T)=\delta_2\int_a^{\eta}v(s)\ ds+\varepsilon_2, \end{cases}$$

is equivalent to the following integral equations

$$u(t) = \lambda_1(t) + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} w_1(s)\frac{ds}{s} - \frac{\lambda_2(t)}{\Gamma(\alpha+P_1)} \int_a^T \left(ln\frac{T}{a}\right)^{\alpha+p_1-1} w_1(s)\frac{ds}{s}$$

$$-\frac{\lambda_3(t)}{\Gamma(\alpha-P_2)} \int_a^T \left(ln\frac{T}{a}\right)^{\alpha-p_2-1} w_1(s)\frac{ds}{s} - \frac{\lambda_4(t)}{\Gamma(\alpha)} \int_a^\xi \left(\int_a^s \left(\ln\frac{s}{\tau}\right)^{\alpha-1} w_1(\tau)\frac{d\tau}{\tau}\right) ds,$$

$$v(t) = \phi_1(t) + \frac{1}{\Gamma(\beta)} \int_a^t \left(\ln\frac{t}{s}\right)^{\beta-1} w_2(s)\frac{ds}{s} - \frac{\phi_2(t)}{\Gamma(\beta+q_1)} \int_a^T \left(ln\frac{T}{a}\right)^{\beta+q_1-1} w_2(s)\frac{ds}{s}$$

$$-\frac{\phi_3(t)}{\Gamma(\beta-q_2)} \int_a^T \left(ln\frac{T}{a}\right)^{\beta-q_2-1} w_2(s)\frac{ds}{s} + \frac{\phi_4(t)}{\Gamma(\beta)} \int_a^\eta \left(\int_a^s \left(\ln\frac{s}{\tau}\right)^{\beta-1} w_2(\tau)\frac{d\tau}{\tau}\right) ds.$$

Proof . Solving the linear equations

$$\begin{aligned} {}_HD^{\alpha}_{a^+}u(t) &= w_1(t), \\ {}_HD^{\beta}_{a^+}v(t) &= w_2(t), \end{aligned}$$

we get

$$u(t) = {}_HI^{\alpha}_{a^+}w_1(t) + c_1\left(ln\frac{t}{a}\right)^{\alpha-1} + c_2\left(ln\frac{t}{a}\right)^{\alpha-2}, \tag{2}$$

$$v(t) = {}_HI^{\beta}_{a^+}w_2(t) + d_1\left(ln\frac{t}{a}\right)^{\beta-1} + d_2\left(ln\frac{t}{a}\right)^{\beta-2}. \tag{3}$$

The boundary conditions $u(a) = 0, v(a) = 0$ implies $c_2 = d_2 = 0$.

$$u(t) = {}_HI^{\alpha}_{a^+}w_1(t) + c_1\left(ln\frac{t}{a}\right)^{\alpha-1}, \tag{4}$$

$$v(t) = {}_HI^{\beta}_{a^+}w_2(t) + d_1\left(ln\frac{t}{a}\right)^{\beta-1}, \tag{5}$$

observe that

$$\begin{aligned} {}_HI^{p_1}_{a^+}u(t) &= {}_HI^{\alpha+p_1}_{a^+}w_1(t) + c_1\frac{\Gamma(p_1)}{\Gamma(p_1+\alpha)}\left(ln\frac{t}{a}\right)^{\alpha+p_1-1}, \\ {}_HD^{p_2}_{a^+}u(t) &= {}_HI^{\alpha-p_2}_{a^+}w_1(t) + c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-p_2)}\left(ln\frac{t}{a}\right)^{\alpha-p_2-1}, \\ {}_HI^{q_1}_{a^+}v(t) &= {}_HI^{\beta+q_1}_{a^+}w_2(t) + d_1\frac{\Gamma(q_1)}{\Gamma(\beta+q_1)}\left(ln\frac{t}{a}\right)^{\beta+q_1-1}, \\ {}_HD^{q_2}_{a^+}v(t) &= {}_HI^{\beta-q_2}_{a^+}w_2(t) + d_1\frac{\Gamma(\beta)}{\Gamma(\beta-q_2)}\left(ln\frac{t}{a}\right)^{\beta-q_2-1}, \end{aligned}$$

then

$$\begin{aligned} K_1{}_HI^{p_1}_{a^+}u(t) + M_1{}_HD^{p_2}_{a^+}u(t) &= K_1{}_HI^{\alpha+p_1}_{a^+}w_1(t) + c_1K_1\frac{\Gamma(p_1)}{\Gamma(p_1+\alpha)}\left(ln\frac{t}{a}\right)^{\alpha+p_1-1} \\ &\quad + M_1{}_HI^{\alpha-p_2}_{a^+}w_1(t) + c_1M_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-p_2)}\left(ln\frac{t}{a}\right)^{\alpha-p_2-1} \\ &= \delta_1\int_a^\xi \left({}_HI^{\alpha}_{a^+}w_1(s) + c_1\left(ln\frac{t}{a}\right)^{\alpha-1}\right) ds + \varepsilon_1. \end{aligned} \tag{6}$$

(6) implies that

$$c_1 = \frac{1}{\Delta_1}\left(\delta_1 \int_a^{\xi} {}_H I_{a^+}^{\alpha} w_1(s)ds + \varepsilon_1 - K_{1H} I_{a^+}^{\alpha+p_1} w_1(T) - M_{1H} I_{a^+}^{\alpha-p_2} w_1(T)\right), \Delta_1 \neq 0 \tag{7}$$

and

$$\begin{aligned} K_{2H} I_{a^+}^{q_1} v(t) + M_{2H} D_{a^+}^{q_2} v(t) d_1 &= K_{2H} I_{a^+}^{\beta+q_1} w_2(t) + K_2 d_1 \frac{\Gamma(q_1)}{\Gamma(\beta+q_1)}\left(ln\frac{t}{a}\right)^{\beta+q_1-1} M_{2H} I_{a^+}^{\beta-q_2} w_2(t) \\ &+ M_2 \frac{d_1 \Gamma(\beta)}{\Gamma(\beta-q_2)}\left(ln\frac{t}{a}\right)^{\beta-q_2-1} \\ &= \delta_2 \int_a^{\eta} \left({}_H I_{a^+}^{\beta} w_2(s) + d_1\left(ln\frac{s}{a}\right)^{\beta-1}\right) ds + \varepsilon_2. \end{aligned} \tag{8}$$

(8) implies that

$$d_1 = \frac{1}{\Delta_2}\left(\delta_2 \int_a^{\eta} {}_H I_{a^+}^{\beta} w_2(s)ds + \varepsilon_2 - K_{1H} I_{a^+}^{\beta+q_1} w_2(T) - M_{1H} I_{a^+}^{\beta-q_2} w_2(T)\right), \Delta_2 \neq 0.$$

Substitute (7),(8) in (4),(5) respectively, we get

$$\begin{aligned} u(t) &= {}_H I_{a^+}^{\alpha} w_1(t) + \frac{\delta_1}{\Delta_1}\left(\ln\frac{t}{a}\right)^{\alpha-1} \int_a^{\xi} I_{a^+}^{\alpha} w_1(s)ds + \frac{\varepsilon_1}{\Delta_1}\left(\ln\frac{t}{a}\right)^{\alpha-1} \\ &- \frac{K_1}{\Delta_1}\left(\ln\frac{t}{a}\right)^{-1} {}_H I_{a^+}^{\alpha+p_1} w_1(T) - \frac{M_1}{\Delta_1}\left(\ln\frac{t}{a}\right)^{-1} {}_H I_{a^+}^{\alpha-p_2} w_1(T), \\ v(t) &= {}_H I_{a^+}^{\beta} w_2(t) + \frac{\delta_2}{\Delta_2}\left(\ln\frac{t}{a}\right)^{\beta-1} \int_a^{\eta} I_{a^+}^{\beta} w_2(s)ds + \frac{\varepsilon_2}{\Delta_2}\left(\ln\frac{t}{a}\right)^{\beta-1} \\ &- \frac{K_2}{\Delta_2}\left(\ln\frac{t}{a}\right)^{-1} {}_H I_{a^+}^{\beta+q_1} w_2(T) - \frac{M_2}{\Delta_2}\left(\ln\frac{t}{a}\right)^{\beta-1} {}_H I_{a^+}^{\beta-q_2} w_2(T), \end{aligned}$$

the converse holds by direct computation which completes the proof.

3 Main results

Let the space $C([a,T],\mathbb{R})$ denote the Banach space of all continuous function from $[a,T]$ to $\mathbb{R}$.

Introducing $U = \{u(t) \mid u(t) \in C[a,T]\}$ end ousted with the norm defined by $|u| = \sup_{a\leq t\leq T} |u(t)|$. Obviously $(U, |.|)$ is a Banach space. Also let $V = \{v(t) \mid v(t) \in C[a,T]\}$ with the norm $|v| = \sup_{a\leq t\leq T} |v(t)|$, clearly the product space $(U \times V, |(u,v)|)$ is a Banach space with norm
$|(u,v)| = |u| + |v|$,

Define an operator $T = \binom{T_1(u,v)}{T_2(u,v)}$, where

$$(T_1(u,v))(t)=\lambda_1(t)+\frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1}f(s,u(s),v(s))\frac{ds}{s}$$

$$-\frac{\lambda_2(t)}{\Gamma(\alpha+p_1)}\int_a^T\left(\ln\frac{T}{s}\right)^{\alpha+p_1-1}f(s,u(s),v(s))\frac{ds}{s}$$

$$-\frac{\lambda_3(t)}{\Gamma(\alpha-p_2)}\int_a^T\left(\ln\frac{T}{s}\right)^{\alpha-p_2-1}f(s,u(s),v(s))\frac{ds}{s}$$

$$+\frac{\lambda_4(t)}{\Gamma(\alpha)}\int_a^\xi\int_a^s\left(ln\frac{s}{\tau}\right)^{\alpha-1}f(\tau,u(\tau),v(\tau))\frac{d\tau}{\tau}ds,$$

$$(T_2(u,v))(t)=\phi_1(t)+\frac{1}{\Gamma(\beta)}\int_a^t\left(\ln\frac{t}{s}\right)^{\beta-1}g(s,u(s),v(s))\frac{ds}{s}$$

$$-\frac{\phi_2(t)}{\Gamma(\beta+q_1)}\int_a^T\left(\ln\frac{T}{s}\right)^{\beta+q_1-1}g(s,u(s),v(s))\frac{ds}{s}$$

$$-\frac{\phi_3(t)}{\Gamma(\beta-q_2)}\int_a^T\left(\ln\frac{T}{s}\right)^{\beta-q_2-1}g(s,u(s),v(s))\frac{ds}{s}$$

$$+\frac{\phi_4(t)}{\Gamma(\beta)}\int_a^\eta\int_a^s\left(\ln\frac{s}{\tau}\right)^{\beta-1}g(\tau,u(\tau),v(\tau))\frac{d\tau}{\tau}ds.$$

For computational convenience we get

$$Q_1=\frac{1}{\Gamma(\alpha)}sup_{a\le t\le T}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}+sup_{a\le t\le T}\frac{|\lambda_2(t)|}{\Gamma(\alpha+p_1)}\int_a^T\left(\ln\frac{T}{s}\right)^{\alpha+p_1-1}\frac{ds}{s}$$

$$+sup_{a\le t\le T}\frac{|\lambda_3(t)|}{\Gamma(\alpha-p_2)}\int_a^T\left(\ln\frac{T}{s}\right)^{\alpha-p_2-1}\frac{ds}{s}+sup_{a\le t\le T}\frac{|\lambda_4(t)|}{\Gamma(\alpha)}\int_a^\xi\int_a^s\left(\ln\frac{s}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau}ds,$$

$$Q_2=sup_{a\le t\le T}\frac{1}{\Gamma(\beta)}\int_a^t\left(\ln\frac{t}{s}\right)^{\beta-1}\frac{ds}{s}+sup_{a\le t\le T}\frac{|\phi_2(t)|}{\Gamma(\beta+q_1)}\int_a^T\left(\ln\frac{T}{s}\right)^{\beta+q_1-1}\frac{ds}{s}$$

$$+sup_{a\le t\le T}\frac{|\phi_3(t)|}{\Gamma(\beta-q_2)}\int_a^T\left(\ln\frac{T}{s}\right)^{\beta-q_2-1}\frac{ds}{s}+sup_{a\le t\le T}\frac{|\phi_4(t)|}{\Gamma(\beta)}\int_a^\eta\int_a^s\left(\ln\frac{s}{\tau}\right)^{\beta-1}\frac{d\tau}{\tau}ds.$$

Theorem 3.1 Assume $f,g\in[a,T]\times\mathbb{R}^2\to\mathbb{R}$ are jointly continuous and assume that there exist $L_{f_1},L_{f_2},L_{g_1},L_{g_2}>0$ such that $\forall t\in[a,T],\forall\, u_1,u_2,v_1,v_2\varepsilon\,\mathbb{R}$,
such that

$$|f(t,u_1,v_1)-f(t,u_2,v_2)|\le L_{f_1}|u_1-u_2|+L_{f_2}|v_1-v_2|,$$

$$|g(t,u_1,v_1)-g(t,u_2,v_2)|\le L_{g_1}|u_1-u_2|+L_{g_2}|v_1-v_2|,$$

if $(L_{f_1}+L_{f_2})Q_1+(L_{g_1}+L_{g_2})Q_2<1$ then the BVP (1) has a unique solution on $[a,T]$.

Proof: Define $R_1 = sup_{a\le t\le T} f(t,0,0)$ and $R_2 = sup_{a\le t\le T} g(t,0,0)$ such that

$$r \ge max\left\{ \frac{R_1Q_1+|\lambda_1(t)|}{1-(L_{f_1}+L_{f_2})Q_1}, \frac{R_2Q_2+|\phi_1(t)|}{1-(L_{g_1}+L_{g_2})Q_2}. \right\}$$

We show that $TS_r \subset S_r$ where $S_r = \{(u,v)\varepsilon U\times V, |(u,v)| \le r\}, \forall (u,v)\varepsilon S_r, \forall t \in [a,T]$ we have

$$\begin{aligned}
|(T_1(u,v))(t)| \le |\lambda_1(t)| &+ \frac{1}{\blacksquare(\alpha)} \int_a^t \left(ln\frac{t}{s}\right)^{\alpha-1} |f(s,u(s),v(s))| \frac{ds}{s} \\
&+ \frac{|\lambda_2(t)|}{\blacksquare(\alpha+p_1)} \int_a^T \left(ln\frac{T}{s}\right)^{\alpha+p_1-1} |f(s,u(s),v(s))| \frac{ds}{s} \\
&+ \frac{|\lambda_3(t)|}{\blacksquare(\alpha-p_2)} \int_a^T \left(ln\frac{T}{s}\right)^{\alpha-p_2-1} |f(s,u(s),v(s))| \frac{ds}{s} \\
&+ \frac{|\lambda_4(t)|}{\blacksquare(\alpha)} \int_a^\xi \int_a^s \left(ln\frac{s}{\tau}\right)^{\alpha-1} |f(\tau,u(\tau),v(\tau))| \frac{d\tau}{\tau} ds,
\end{aligned}$$

but

$$\begin{aligned}
|f(t,u(t),v(t))| &\le |f(t,u(t),v(t)) - f(t,0,0)| + |f(t,0,0)| \\
&\le L_{f_1}|u| + L_{f_2}|v| + sup_{a\le t\le T} f(t,0,0) \\
&\le L_{f_1}|u| + L_{f_2}|v| + R_1,
\end{aligned}$$

then

Similarly

$$|(T_2(u,v))(t)| \le |\phi_1| + [(L_{g_1}+L_{g_2})r + R_2]\, Q_2 \le r.$$

Consequently

$$|T(u,v)(t)| \le r,$$

which implies $T(u,v)\varepsilon S_r$ that is
Next ,we show that the operator T is a contraction

$$\begin{aligned}
|T_1(u_1,v_1) - T_1(u_2,v_2)| \le &\frac{1}{\Gamma(\alpha)} sup_{a\le t\le T} \int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} |f(s,u_1(s),v_1(s)) - f(s,u_2(s),v_2(s))| \frac{ds}{s} \\
&+ \frac{|\lambda_2(t)|}{\Gamma(\alpha+p_1)} \int_a^T \left(\ln\frac{T}{s}\right)^{\alpha+p_1-1} |f(s,u_1(s),v_1(s)) - f(s,u_2(s),v_2(s))| \frac{ds}{s} \\
&+ \frac{|\lambda_3(t)|}{\Gamma(\alpha-p_2)} \int_a^T \left(\ln\frac{T}{s}\right)^{\alpha-p_2-1} |f(s,u_1(s),v_1(s)) - f(s,u_2(s),v_2(s))| \frac{ds}{s} \\
&+ \frac{|\lambda_4(t)|}{\Gamma(\alpha)} \int_a^\xi \int_a^s \left(\ln\frac{s}{\tau}\right)^{\alpha-1} |f(\tau,u_1(\tau),v_1(\tau)) - f(\tau,u_2(\tau),v_2(\tau))| \frac{d\tau}{\tau} ds \\
&\le \left[L_{f_1}|u_1-u_2| + L_{f_2}|v_1-v_2|\right] Q_1 \\
&= L_{f_1}|u_1-u_2|Q_1 + L_{f_2}|v_1-v_2|Q_1 \\
&\le \left(L_{f_1}Q_1 + L_{f_2}Q_1\right)(|u_1-u_2| + |v_1-v_2|).
\end{aligned}$$

Similarly

$$|T_2(u_1,v_1) - T_2(u_2,v_2)| \le (L_{g_1}Q_2 + L_{g_2}Q_2)(|u_1-u_2| + |v_1-v_2|).$$

Consequently

$$|T(u_1,v_1) - T(u_2,v_2)| \le \left(\left[(L_{f_1}+L_{f_2})\right] Q_1 + \left[L_{g_1}+L_{g_2}\right] Q_2 \right)(|u_1-u_2| + |v_1-v_2|),$$

hence T is a contraction. Based on the Banach contraction mapping theorem the BVP has a unique solution on $[a,T]$ which complete the proof.

Theorem3.2 : Assume $f,g \in [a,T] \times \mathbb{R}^2 \to \mathbb{R}$ are jointly continuous and assume there exist $a_1,a_2,b_1,b_2,r_1,r_2 \;\varepsilon\; \mathbb{R}$ such that

$$|f(t,u,v)| \le a_1 + b_1|u| + r_1|v|,$$
$$|g(t,u,v)| \le a_2 + b_2|u| + r_2|v|, \forall (t,u,v)\varepsilon [a,T] \times \mathbb{R}^2$$

if $(b_1Q_1 + b_2Q_2) < 1$ and $(r_1Q_1 + r_2Q_2) < 1$ then the BVP (1) has at least one solution on $[a,T]$.
Proof. Step1. We show that the operator $T : U \times V \to U \times V$ is completely continuous
It's clear that T is continuous as both f and g are continuous.
Let A be a bounded set in $U \times V$ then the exist positive constants γ_1, γ_2 such that
$|f(t,u(t),v(t))| \le \gamma_1$ and $|g(t,u(t),v(t))| \le \gamma_2$, $\forall (u,v)\varepsilon A$
then for any $(u,v)\varepsilon A$, it follows that $|T_1(u,v)| \le |\lambda_1| + \gamma_1 Q_1$ and $|T_2(u,v)| \le |\phi_1| + \gamma_2 Q_2$.
Consequently
$|T(u,v)| \le |\lambda_1| + |Q_1| + \gamma_1 Q_1 + \gamma_2 Q_2$, that is T is uniformly bounded.
Step 2. We show that T is equicontinuous for this we let $t_1,t_2 \varepsilon [a,T]$ with $t_1 < t_2$, then similarly

$$|T_2(u,v)(t_2) - T_2(u,v)(t_1)| \to 0 \; as \; t_1 \to t_2.$$

Therefore, the operator T is equicontinuous and hence it is completely continuous.
Finally, it will be verified that the set

$$\beta = \{(u,v)\,\varepsilon\, u \times v : (u,v) = \mu T(u,v), \mu \in [0,1]\}$$

is bounded
$\forall\, t\; \varepsilon\, [a,T]$, $\forall\; (u,v)\,\varepsilon\beta$ we have

$$u(t) = \mu T_1(u,v) \quad , \quad v(t) = \mu T_2(u,v),$$

then
implying that

$$|u| + |v| \le |\lambda_1| + |\phi_1| + (a_1Q_1 + a_2Q_2) + (b_1Q_1 + b_2Q_2)|u| + (r_1Q_1 + r_2Q_2)|v|$$

. Consequently

$$|(u,v)| \le \frac{|\lambda_1| + |\phi_1| + (a_1Q_1 + a_2Q_2)}{min\{1 - (b_1Q_1 + b_2Q_2),\, 1 - (r_1Q_1 + r_2Q_2)\}},$$

which proves that β is bounded by Leray-schauder Alterrature the operator T has at least one fixed point. ¡hence BVP (1) has at least one solution.

Example
Consider the following system of fractional differential equations

$$\begin{cases} {}_HD^{\frac{3}{2}}_{1^+}u(t) = \frac{lnt}{e^t(t+2)^5}\left(\frac{|u(t)|}{1+|u(t)|}\right) + \frac{1}{20}cosv(t)^2 \;,\; 1 \le t \le e \\ {}_HD^{\frac{4}{3}}_{1^+}v(t) = \frac{t}{(t^2+1)}\left(\frac{1}{12}cosu(t) + \frac{1}{13}sinv(t)\right) \;,\; 1 \le t \le e \\ u(1) = 0,\; {}_HI^{\frac{1}{2}}_{1^+}u(e) - {}_HD^{\frac{1}{4}}_{1^+}u(e) = 2\int_1^2 u(s)\,ds \\ v(1) = 0,\; 2{}_HI^{\frac{2}{5}}_{1^+}v(e) + {}_HD^{\frac{3}{5}}_{1^+}v(e) = 3\int_1^{\frac{5}{2}} v(s)\,ds + 1, \end{cases} \tag{9}$$

with

$$\alpha = \tfrac{3}{2}, a = 1, \xi = 2, , T = e \,, \delta_1 = 2,\; \varepsilon_1 = 0, , k_1 = 1, p_1 = \tfrac{1}{2}, M_1 = -1,$$
$$\beta = \tfrac{4}{3}, k_2 = 2, p_2 = \tfrac{1}{4}, q_1 = \tfrac{2}{5}, q_2 = \tfrac{3}{5}, \delta_2 = 3, \eta = \tfrac{5}{2}, M_2 = 1, \varepsilon_2 = 1,$$

and

$$f(t,x,y) = \frac{lnt}{e^t(t+2)^5}\left(\frac{|u(t)|}{1+|u(t)|}\right) + \frac{1}{20}cosv(t)^2,$$
$$g(t,x,y) = \frac{t}{(t^2+1)}\left(\frac{1}{12}cosu(t) + \frac{1}{13}sinv(t)\right),$$

it is clear that the functions f an g are continuous and Lipschitzian with $L_{f_1} = \frac{1}{243e}, L_{f_2} = \frac{1}{20}$ and $L_{g_1} = \frac{1}{12}, L_{g_2} = \frac{1}{13}$,

$$\begin{aligned} \Delta_1 &= k_1\frac{\Gamma(p_1)}{\Gamma(p_1+\alpha)}\left(ln\tfrac{T}{a}\right)^{\alpha+p_1-1} + M_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-p_2)}\left(ln\tfrac{T}{a}\right)^{\alpha-p_2-1} - \delta_1\int_a^{\xi}\left(\ln\tfrac{s}{a}\right)^{\alpha-1}ds \\ &= (1)\frac{\Gamma(\frac{1}{2})}{\Gamma(2)}(\ln\tfrac{e}{1})^1 + (-1)\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{4})}(ln\tfrac{e}{1})^{\frac{1}{4}} - 2\int_1^2(\ln\tfrac{s}{1})^{\frac{1}{2}}ds \\ &\ge 1.7725 - \frac{0.8862}{0.906} - 1.67 = -0.870, \end{aligned}$$

note that $\int_1^2 (\ln\frac{s}{1})^{\frac{1}{2}} ds \leq \sqrt{\ln 2}$. In similar manner one can find $\Delta_2 \geq 3.62$,

$$|\lambda_2| = max\left\{0, \frac{1}{0.870}(lne)^{\frac{1}{2}}\right\} = 1.148,$$
$$|\lambda_3| = max\left\{0, \frac{1}{0.870}(lne)^{\frac{1}{2}}\right\} = 1.148,$$
$$|\lambda_4| = max\left\{0, \frac{2}{0.870}(lne)^{\frac{1}{2}}\right\} = 2.296,$$

now $Q_1 \leq 6.13$ *and* $Q_2 \leq 2.65$,
with the given values it is found that the condition $(L_{f_1} + L_{f_2})Q_1 + (L_{g_1} + L_{g_2})Q_2 = \left(\frac{1}{243e} + \frac{1}{20}\right)6.13 + \left(\frac{1}{12} + \frac{1}{13}\right)2.65 = 0.798 < 1$. All the condition of theorem 1 satisfied, that is problem (9) has unique solution on $[1, e]$.

4 Conclusion

In this paper, we have discussed the existence and uniqueness of solutions for a new class of boundary value problems consisting of a system of fractional differential equations involving Hadamard fractional derivative and supplemented with nonlocal mixed boundary conditions with multiple orders. It should be stressed that, similarly problems via different fractional derivatives such as Katogampola and Atangana-Baleanu can be investigated. So the present work is a useful contribution to the existing literature on the topic.

References

[1] I. Podlubny, Fractional differential equations, Academic Press, San Diego, USA, 1999.
[2] I. Area, H. Batar, J.Losada, J. J. Nieto, W. Shammakh and A. Torres, On a fractional order Ebola epidemic model, *Adv. Differ. Equ.* **2015**, (2015).
[3] Y. Sofuoglu and N. Ozalp, A fractional order model on bilingualism, *Commun. Fac. Sci. Univ. Ank. Ser. A* **2014**, 81–89 (2014)
[4] Y. Sofuoglu and N. Ozalp, Fractional order bilingualism model without conversion from dominant unilingual group to bilingual group, *Differ. Equ. Dyn. Syst.* **2015**, 1–9 (2015).
[5] R. Hilfer, Applications of fractional calculus in physics, World Scientific, Hackensack, NJ, USA, 2001.
[6] F. Mainardi, Fractional relaxation-oscillation and fractional diffusion-wave phenomena, *Chaos Solit. Fract.* **7**, 1461–1477 (1996).
[7] R. Almeida, N. Bastos and M. T. T. Monteiro, Modeling some real phenomena by fractional differential equations, *Math. Meth. Appl. Sci.* 2015
[8] P. L. Butzer, A. A. Kilbas and J. J. Trujillo, Compositions of Hadamard-type fractional integration operators and the semigroup property, *J. Math. Anal. Appl.* **269**, 387–400 (2002).
[9] P. L. Butzer, A. A. Kilbas and J. J. Trujillo, Fractional calculus in the Mellin setting and Hadamard-type fractional integrals, *J. Math. Anal. Appl.* **269**, 1–27 (2002).
[10] P. L. Butzer, A. A. Kilbas and J. J. Trujillo, Mellin transform analysis and integration by parts for Hadamard-type fractional integrals, *J. Math. Anal. Appl.* **270**, 1-15 (2002).
[11] A. A. Kilbas, H. M. Srivastava and J.J. Trujillo, Theory and applications of fractional differential equations, Vol. 204, North-Holland Mathematics Studies, Elsevier Science B.V., Amsterdam, 2006.
[12] A. A. Kilbas, Hadamard-type fractional calculus, *J. Korean Math. Soc*,**38**, 1191-1204 (2001).
[13] A. A. Kilbas and J. J. Trujillo, Hadamard-type integrals as G-transforms, *Integr. Transf. Spec. Funct.* **14**, 413-427 (2003).
[14] M. M. Matar and O. A. Al-Salmy, Existence and uniqueness of solution for Hadamard fractional sequential differential equations, *IUG J. Nat. Stud.* 141–147 (2017).
[15] S. K. Ntouyas, J. Tariboon and P. Thiramanus, Mixed problems of fractionalcoupled systems of Riemann-Liouville differential equations and Hadamard integral conditions, *J. Comput. Anal. Appl.* **21**, 813{H828 (2016).
[16] S. K. Ntouyas, J. Tariboon and C. Thaiprayoon, Nonlocal boundary value problems for Riemann-Liouville fractional diffeerential inclusions withHadamard fractional integral boundary conditions, *Taiwan. J. Math.* **20** 91{107 (2016).
[17] J. Tariboon, S. K. Ntouyas and C. Thaiprayoon, Nonlinear Langevin equation of Hadamard-Caputo type fractional derivatives with nonlocal fractional integral conditions, *Adv. Math. Phys.* Art. ID 372749, 1-15 (2014).
[18] W. Yukunthorn, S. Suantai, S. K. Ntouyas and J. Tariboon, Boundary value problems for impulsive multi-order Hadamard fractional differential equations, *Bound. Val. Probl.* **148** 1–13 (2015).
[19] X. Zhang, T. Shu, H. Cao, Z. Liu and W. Ding, The general solution for impulsive differential equations with Hadamard fractional derivative of order _ 2(1,2), *Adv. Differ. Equ.* **2016**, 1–36 (2016).
[20] X. Zhang, On impulsive partial differential equations with Caputo-Hadamard fractional derivatives, *Adv.Differ. Equ.* **28**, 1-21 (2016).
[21] N. Mahmudov, M. Awadalla and K. Abuassba, Nonlinear sequential fractional differential equations with nonlocal boundary conditions, *Adv. Differ. Equ.* **2017**:319 (2017).
[22] N. I. Mahmudov, M. Awadalla and K. Abuassba, Hadamard and Caputo-Hadamard FDE's with three point integral boundary conditions, *Nonlin. Anal. Differ. Equ.* **5**(6), 271–282 (2017).