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A Numerical Study for Fractional Problems with Nonlinear Phenomena in Physics

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Abstract: In this paper, we investigate the solution of fractional Duffing equation. This problem is important since it appears in a variety of science models, including engineering, biology, and physics. The fractional derivative will give us the chance to consider the history of the displacement function in the interval $[0,t]$. A numerical solution of fractional problems with strongly oscillators is investigated. The spline spaces are used to approximate the solution. It is worth mentioning that the standard basis such as polynomials will not work with this type of problems since there are strong oscillators. To show the validity of our results, we compare them with four different methods which are HPM, MHPM, SHPM, and collocation method using polynomials as basis for the approximate solution. The error in our approximation is 10^{-10} comparing with other methods which are of 10^{-6} or more. The numerical results reveal that our results are accurate and the proposed method can be used for other physical problems.

Keywords: Approximate solution, fractional problems, oscillators, spline spaces.

1 Introduction

The history of fractional calculus in continuous settings is old, almost as old as ordinary integer order calculus. Recently, fractional derivatives have been studied in both aspects, not only for their theoretical interest, but also for their applications [1,2].

The applications of fractional calculus in various disciplines such as Kadomtsov-Petviashili-Benjamin-Bona-Mahony model [3], and continuum mechanics [4] are of great interest to researchers despite the physical or geometrical challenges involved in understanding the meaning of fractional operators [5]. It is crucial to learn the methods and techniques that can be used to solve fractional differential equations due to the complexity of fractional calculus and its applications.

Comparatively speaking to classical calculus, there are numerous non-equivalent definitions for both fractional derivative and integral operators. The most established and frequently employed definition throughout history is the Riemann-Liouville fractional derivative. However, it has some inherent drawbacks: for fractional differential equations in this model, the necessary initial conditions are intrinsically fractional, which makes the model less useful for applications. Due to its requirement for initial conditions to be in the classical form, the Caputo fractional derivative emerged as a rival to Riemann-Liouville in the late 20th century [6]. This made it more appropriate for modeling physical phenomena.

The Duffing equation is employed in a variety of science models, including engineering, biology, and physics. It was named for Ger-Man Duffing, who found it in 1918. This equation is a good example of strong nonlinear model. This problem is used in Van der Pol's equation as a nonlinear oscillation example. This equation is important since it is used in many models such as mechanical oscillators [7], vibration beams [8], and disease prediction [9].
After several simplifications, Duffing modeled the motion of special types of pendulum problems as

$$w''(t)+a_1w(t)+a_2w^3(t)=0 \tag{1}$$

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with the following initial conditions

$$w(0) = \eta, w'(0) = 0. \tag{2}$$

The Duffing equation describes the motion of a classical particle in a double well potential. To derive the Duffing equation, we start with Cauchy problem with offset [10]. Here, $w(t)$ is the displacement function which is the solution of Problem (1)-(2). To consider the history of the displacement function on the interval $[0,t]$, we use the fractional derivative which will give better representation to the model. For this reason, in this article, we study the generalization form of Problem (1)-(2) which is given by

$$D^{2\alpha}w(t) + a_1 w(t) + a_2 w^3(t) = 0, 0.5 < \alpha \leq 1 \tag{3}$$

with

$$w(0) = \eta, D^{\alpha}w(0) = 0. \tag{4}$$

Several researchers investigated the numerical solution of Problem (3)-(4) using different methods such as HPM [11], Series solution [12], variational method [13], operational matrix method [14], ADM method [15], and Broyden method [16]. More details can be found in [17, 18, 19, 20].
We organize this paper as follows. In Section 2, we write some definitions and theorems which we use in this article. In Sections 3 and 4, we present the method of the solution and some theoretical results while in Section 5, we give some numerical results and comparisons with other researchers. Discussion for the results is given at the end of the article.

2 Preliminaries

In this paper, we use the Caputo derivatives which is defined as follow.

Definition 1. *Let $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. Define the spaces C_α and C_α^k as*

$$C_\alpha = \{f : (0,\infty) \to \mathbb{R} : f(t) = t^j f_1(t), f_1 \in C[0,\infty), j > \alpha\}$$

and

$$C_\alpha^k = \{f : (0,\infty) \to \mathbb{R} : f^{(k)} \in C_\alpha\}.$$

If $k-1 < \alpha < k$, $\alpha > 0$, $t > 0$, $f \in C_{-1}^k$, then the Caputo derivative is given by

$$D^{\alpha}r(t) = \frac{1}{\Gamma(k-\alpha)} \int_0^t (t-s)^{k-1-\alpha} r^{(k)}(s)ds$$

where Γ is the Gamma function.
A direct result from Definition 1 is the functional power rule which is given by

$$D^{\alpha}t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}, \mu \geq \alpha > 0$$

and $D^{\alpha}\, a = 0$, when a is constant.
The notation of functional B-splines (FBS) and their attributes are introduced in this section. Let us start by defining fractional truncated power functions (FTPFs).

Definition 2. *For $\alpha \in \mathbb{R}^+$, the FTPF is defined by*

$$w_+^{\alpha} = \begin{cases} w^{\alpha} & w \geq 0 \\ 0 & x < 0 \end{cases}$$

and the FBS is defined by

$$S_{\alpha}(w) = \frac{1}{\Gamma(\alpha+1)} \sum_{j=0}^{\infty} \frac{(-1)^j\, \Gamma(\alpha+2)}{j!\, \Gamma(\alpha-j+2)} (w-j)_+^{\alpha}. \tag{5}$$

Using the power rule of Caputo derivative, one has

$$D^{B} w_+^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)} w_+^{\alpha-\beta}, 0 < \beta \leq \alpha$$

and

$$D^{\beta}S_{\alpha}(w) = \sum_{j=0}^{\infty} \frac{(-1)^k \Gamma(\alpha+2)}{j!\,\Gamma(\alpha-j+2)\,\Gamma(\alpha-\beta+1)} (w-j)_+^{\alpha-\beta}.$$

To get an idea about FBS function, we plot $S_{\alpha}(w)$ for $\alpha = \frac{j}{5}, j = 0:10$ in Fig. 1. From Figure 1, we note that $S_{\alpha}(w)$ decays very fast when w becomes large.

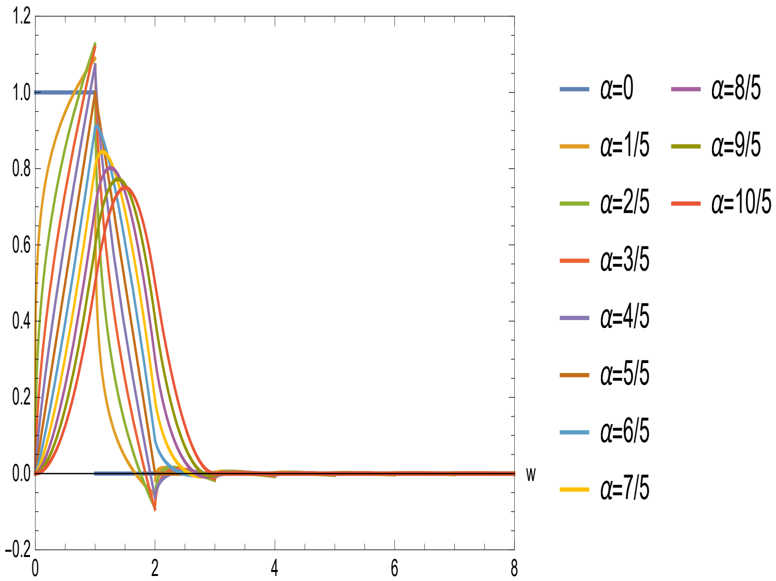

Fig. 1: $S_{\alpha}(w)$ for $\alpha = \frac{k}{5}, k = 0, 1, ..., 10$.

Also, we plot $D^{\beta}S_6(w)$, $D^{\beta}S_{\frac{31}{5}}(w)$, and $D^{\beta}S_{\frac{32}{5}}(w)$ for $\beta = \frac{1}{5}, \frac{3}{5}, 1$ in Figures Fig.2, Fig.3, and Fig.4, respectively. Also, we notice that $D^{\beta}S_{\alpha}(w)$ decays to zero when w becomes large.

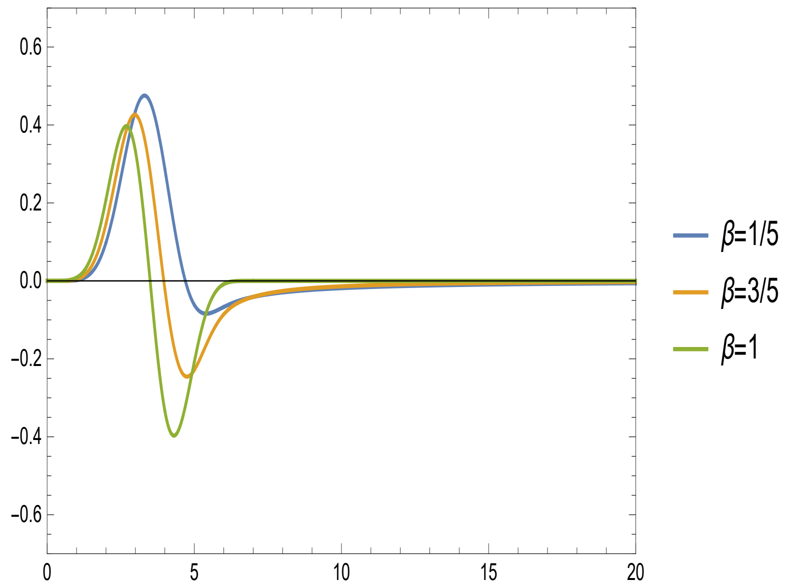

Fig. 2: $D^{\beta}S_6(w)$ for $\beta = \frac{1}{5}, \frac{3}{5}, 1$.

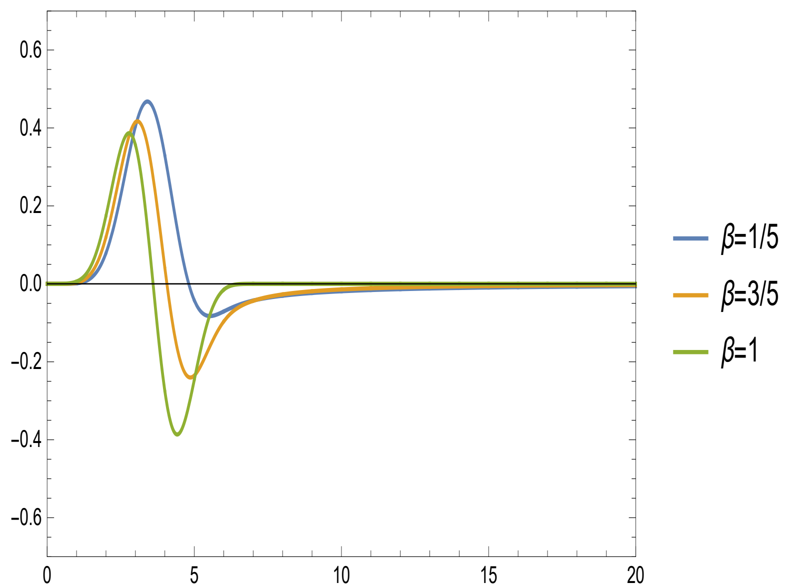

Fig. 3: $D^{\beta}S_{\frac{31}{5}}(w)$ for $\beta = \frac{1}{5}, \frac{3}{5}, 1$.

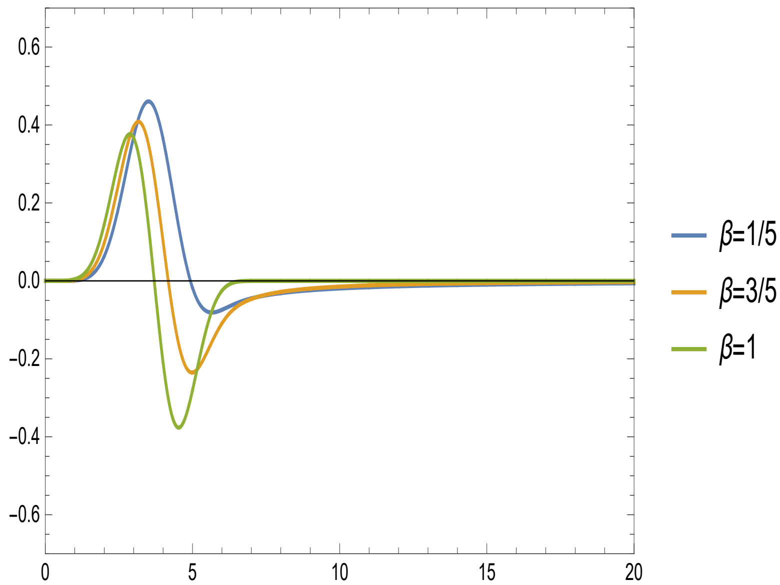

Fig. 4: $D^{\beta} S_{\frac{32}{5}}(w)$ for $\beta = \frac{1}{5}, \frac{3}{5}, 1$.

3 Method of solution

This section is devoted for deriving the numerical method for solving the following problem

$$D^{2\alpha} w(t) + a_1 w(t) + a_2 w^3(t) = 0, \ \frac{1}{2} < \alpha \leq 1, 0 < t < t_{max} \tag{6}$$

with

$$w(0) = \eta, D^{\alpha} w(0) = 0. \tag{7}$$

Let $t_j = j\Delta$, for $j = 0, 1, ..., k$ and $\Delta = \frac{t_{max}}{k}$. From Figure 1, one can see that $S_\alpha(t)$ decay to zero when t becomes large but it does not have a compact support. This fact suggests that to treat the closed interval $[0, M]$ as a compact support for $S_\alpha(t)$ where $M \in \mathbb{N}$. Define the set

$$S_{\alpha,\Delta} = \{S_{\alpha,j}(t) = S_\alpha(\frac{t}{\Delta} - j) : j \in I_\Delta, t \in [0, t_{max}]\}$$

where $I_\Delta = \{-M, -M+1, ..., K\}$. Since $S_{\alpha,j}(t) \in [0, t_{max}]$, we assume that $k = M - 1$.

Theorem 1. *The functions $S_{\alpha,j}(t), j \in I_\Delta = \{-M, -M+1, ..., M-1\}$, are linearly independent functions on $[0, t_{max}]$.*

Proof. For any $-M \leq i \leq M-1$, we have

$$S_\alpha(\frac{t}{\Delta} - i) = \frac{1}{\Gamma(\alpha+1)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\alpha+2)}{j! \Gamma(\alpha - j + 2)} (\frac{t}{\Delta} - i - j)_+^\alpha. \tag{8}$$

Now,

$$(\frac{t}{\Delta} - i - j)_+^\alpha = \begin{cases} (\frac{t}{\Delta} - i - j)^\alpha, & \frac{t}{\Delta} - i - j \geq 0 \\ 0, & otherwise \end{cases}$$
$$= \begin{cases} \frac{(M-1)t}{t_{max}} - i - j, & \frac{(M-1)t}{t_{max}} - i \geq j \\ 0, & otherwise \end{cases}.$$

Thus, if $j > \frac{(M-1)t_{max}}{t_{max}} - i = (M-1) - i \geq \frac{(M-1)t}{t_{max}} - i$, for all $t \in [0, t_{max}]$, then

$$(\frac{t}{\Delta} - i - j)_+^\alpha = 0.$$

Thus,

$$S_\alpha(\frac{t}{\Delta} - i) = \frac{1}{\Gamma(\alpha+1)} \sum_{j=0}^{M-1-j} \frac{(-1)^j \Gamma(\alpha+2)}{j!\,\Gamma(\alpha-j+2)} (\frac{t}{\Delta} - i - j)_+^\alpha. \tag{9}$$

Then, $S_\alpha(\frac{t}{\Delta} - i)$ is a piece-wise function with $M-i$ terms. Thus, the result follows directly from the last fact.

Let the spline space be defined by

$$S_\alpha = \text{span}(S_\alpha, \Delta).$$

Then, the dimension of S_α is $2M$. Now, we want to solve Equations (6)-(7) in the Space S_α. Let

$$w_M(t) = \sum_{j=-M}^{M-1} w_j\, S_{\alpha,j}(t) = \sum_{j=-M}^{M-1} w_j\, S_\alpha(\frac{t}{\Delta} - j) \tag{10}$$

be the approximate solution to Equations (6)-(7). Let

$$\Lambda = \{t_j = j\lambda : j = 1, 2, ..., 2M-2\} \tag{11}$$

where $\lambda = \frac{t_{max}}{2M}$.

Substituting the collocation points in Equation (6), we get the following system

$$D^{2\alpha}\, w_M(t_j) + a_1\, w_M(t_j) + a_2\, w_M^3(t_j) = 0 \quad \text{for} \quad j = 1, 2, .., 2M-2.$$

Thus,

$$\sum_{j=-M}^{M-1} w_j\, D^{2\alpha}\, S_\alpha(\frac{t_i}{\Delta} - j) + a_1 \sum_{j=-M}^{M} w_j\, S_\alpha(\frac{t_i}{\Delta} - j) + a_2 \left(\sum_{j=-M}^{M} w_j\, S_\alpha(\frac{t_i}{\Delta} - j) \right)^3 = 0 \tag{12}$$

for $i = 1, 2, , ..., 2M-2$. Using the first initial condition, we get

$$\eta = w_M(0) = \sum_{j=-M}^{M-1} w_j\, S_\alpha(-j). \tag{13}$$

Theorem 2. *For $j \in \{-M, -M+1, ..., M+1\}$, we have*

$$S_\alpha(-j) = \begin{cases} \frac{1}{\Gamma(\alpha+1)} \sum_{i=0}^{-j-1} \frac{(-1)^i \Gamma(\alpha+2)}{i! \Gamma(\alpha-i+2)} (-i-j)^\alpha, & if\ j \in \{-M, -M+1, ..., -1\} \\ 0, & otherwise \end{cases}.$$

*Proof.*Since

$$(-j-i)_+^\alpha = \begin{cases} (-j-i)^\alpha, & -j > i \\ 0, & -j \leq i \end{cases}.$$

Then, we have two cases:

1.If $j \in \{0, 1, 2, ..., M-1\}$, then $-j \leq i$ for any $i \in \{0, 1, 2, ...\}$. Thus, $(-j-i)_+^\alpha = 0$ for all $i \in \{0, 1, 2, ...\}$. Hence, $S_\alpha(-j) = 0$.
2.If $j \in \{-M, -M+1, ..., -1\}$, then $-j > i$ which implies that $i \in \{0, 1, ..., -j-1\}$. Thus,

$$(-j-i)_t^\alpha = \begin{cases} (-j-i)^\alpha, & if i \in \{0, 1, ..., -j-1\} \\ 0, & otherwise \end{cases}.$$

Hence

$$S_\alpha(-j) = \frac{1}{\Gamma(\alpha+1)} \sum_{i=0}^{-j-1} \frac{(-1)^i\,\Gamma(\alpha+2)}{i!\,\Gamma(\alpha-i+2)}(-j-i)^\alpha.$$

From Equation (7), we get

$$\eta = w_{-M}S_\alpha(M) + w_{-M+1}\,S_\alpha(M-1) + \ldots + w_{-1}S_\alpha(1). \tag{14}$$

Now

$$0 = D^\alpha w_M(0) = \sum_{j=-M}^{M-1} w_j\, D^\alpha\, S_\alpha(-j). \tag{15}$$

From Equations (12) and (14)-(15), we get a nonlinear system of $2M$ equation in $2M$ unknowns. We use Mathematica to solve it and generate w_i for $i \in \{-M, -M+1, \ldots, M-1\}$.

4 Stability of the B-spline basis

In this section, we show that the B-Spline functions are stable.

Definition 3. *A basis(S_j) of a normed linear space W is said to be stable with respect to the norm $\|.\|$ if there exists to small positive numbers β_1 and β_2 such that*

$$\frac{\|(w_i)\|}{\beta_1} \leq \|\sum_i w_i S_i\| \leq \beta_2 \|(w_i)\|. \tag{16}$$

Let β_1^* and β_2^* be the smallest values that satisfy Equation (16), then the condition number of this basis is

$$k(S_j) = \beta_1^* \beta_2^*.$$

The first result is given in the following theorem.

Theorem 3. *If the basis (S_j) of a normed linear space W is stable with respect to the norm $\|.\|$ and*

$$f_1 = \sum_i w_i S_i(t), f_2 = \sum_i v_i S_i(t),$$

then

$$\frac{\|f_1 - f_2\|}{\|f_1\|} \leq k(S_j) \frac{(w_i - v_i)\|}{\|(w_i)\|}. \tag{17}$$

Proof. Since (S_j) is stable basis, then by equation (16), we get

$$\frac{\|(w_i - v_i)\|}{\beta_1^*} \leq \|f_1 - f_2\| \leq \beta_2^* \|(w_i - v_i)\|, \frac{\|(w_i)\|}{\beta_1^*} \leq \|f_1\| \leq \beta_2^* \|(w_i)\|. \tag{18}$$

Then,

$$\frac{\|f_1 - f_2\|}{\|f_1\|} \leq \beta_1^* \beta_2^* \frac{\|(w_i - v_i)\|}{\|(w_i)\|} \leq k(S_j) \frac{\|(w_i - v_i)\|}{\|(w_i)\|}. \tag{19}$$

Let us define the space $V_A^k[0,1]$ for non-negative integer k. Then, $f \in V_A^k[0,1]$ if $f \in C^{k-1}[0,1]$ and $f^{(k)}$ is continuous on $[0,1]$ except on the set $A \subset [0,1]$. We define the distance between S_α and f by

$$dis(f, S_\alpha) = \inf_{g \in S_\alpha} \|f - g\|.$$

We define $G_d f \in S_\alpha$ for $f \in V_A^0[0,1]$ by

$$G_d f(x) = \sum_i \lambda_i(f) S_i(x),$$

where

$$\lambda_i(f) = \sum_{k=0}^{d} a_{i,k} f(x_{i,k})$$

Table 1: The absolute errors for $\eta = a_1 = a_2 = \alpha = 1$

t	e_1	e_2	e_3	e_4	e_5
$\frac{1}{2}$	0.006	0.001	0.00004	1×10^{-6}	1×10^{-9}
1	0.06	0.00005	0.00001	1×10^{-6}	1×10^{-9}
2	0.2	0.032	0.00003	1×10^{-6}	1×10^{-9}
$\frac{7}{2}$	0.37	0.014	0.00002	1×10^{-6}	1×10^{-9}
5	1.1	0.854	0.00002	1×10^{-6}	1×10^{-9}

and $(x_{i,k})$ are points in $[0,1]$. Simple calculations implies that

$$|\lambda_i(f)| \leq \sum_{k=0}^{d} a_{i,k}\|f\| \leq K_d\|f\|, \tag{20}$$

where K_d depends on d only. Finally, we prove that (S_j) is stable.

Theorem 4. *Let*

$$f = \sum_{j=-\mu}^{m-1} w_j S_{\alpha j}.$$

Then,

$$\frac{\|(w_j)\|}{\beta_d} \leq \|f\| \leq 2M\mu\|(w_j)\|,$$

where β_d is the degree of $S_{\alpha j}$ and $S_{\alpha j}$ is bounded by $\mu = \sup|S_{\alpha j}|$.

Proof. Since $S_{\alpha j}$ has compact support and bounded by μ, then

$$\|f\| = max\left|\sum_{j=-\mu}^{m-1} w_j S_{\alpha j}\right| \leq \mu max\left|\sum_{j=-\mu}^{m-1} w_j\right| \leq 2M\mu\|(w_i)\|.$$

Now, since $f \in S_\alpha$, then $G_d f = f$ and $\lambda_i(f) = w_i$. Then, using Equation (22), we get

$$|w_i| \leq \beta_d\|f\|$$

which implies that

$$\frac{\|(w_i)\|}{\beta_d} \leq \|f\|.$$

Therefore, the basis $(S_{\alpha j)}$ is stable basis.

5 Numerical results

To compare our results with Sibauda and Khider[19] and Syam [20], we solve the problem when $\eta = a_1 = a_2 = \alpha = 1$. The absolute errors are reported in Table 1. For $\eta = \frac{3}{4}$, $\alpha = 1$, $a_1 = a_2 = \frac{3}{2}$, the absolute errors are reported in Table 2. Since the exact solution is unknown in such cases, then the full explicit RK methods built-in file in Mathematica is used as exact solution. Define the absolute errors e_1, e_2, e_3 to be the absolute errors using HPM, MHPM, SHPM as reported in Sibanda and Khider [19], respectively. Also, we defined e_4 to be the absolute error as reported in Syam [20]. Finally, e_5 is the absolute error using the proposed method in this paper.

Table 2: The absolute errors for $\eta = \frac{3}{4}, \alpha = 1, a_1 = a_2 = \frac{3}{2}$

t	e_1	e_2	e_3	e_4	e_5
1	0.024	0.0004	8×10^{-6}	1×10^{-6}	1×10^{-10}
2	0.079	0.0102	2×10^{-5}	1×10^{-6}	1×10^{-10}
3	0.101	0.0008	6×10^{-6}	1×10^{-6}	1×10^{-10}
4	0.223	0.0388	7×10^{-5}	1×10^{-6}	1×10^{-10}
5	0.505	0.0078	1×10^{-6}	1×10^{-6}	1×10^{-10}

Figures 5 and 6 present the exact and the approximate solutions for $\alpha = 1, \eta = 1.5, a_1 = \frac{1}{2}, a_2 = \frac{3}{2}$ and $\alpha = 1, a_1 = 1, a_2 = 1, \eta = 2$, respectively.

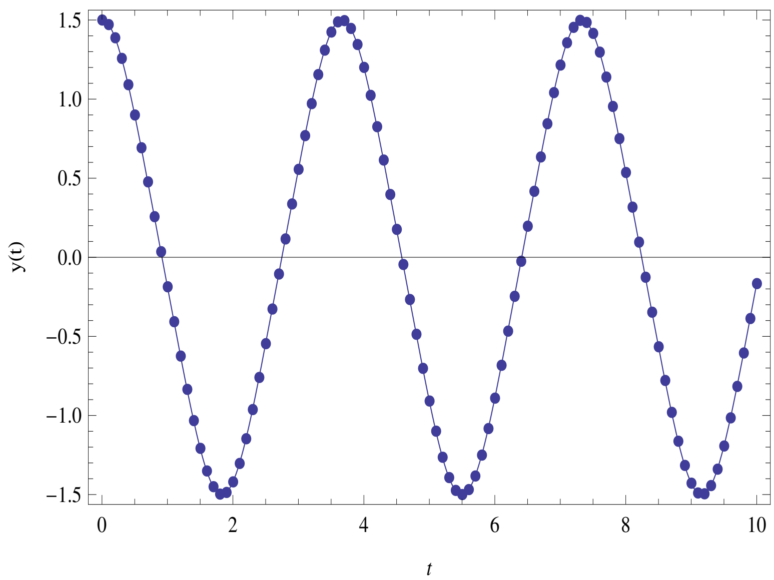

Fig. 5: $\alpha = 1, \eta = 1.5, a_1 = 1.5, a_2 = 0.5$

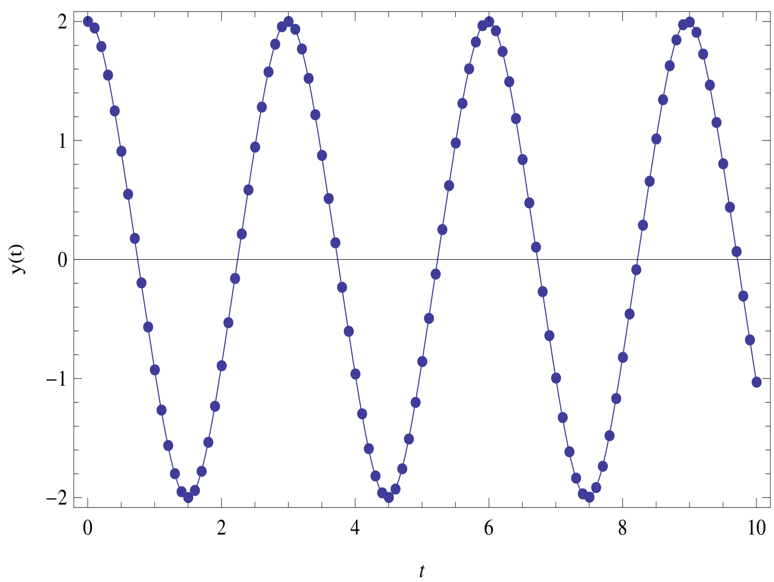

Fig. 6: $\alpha = 1, \eta = 2, a_1 = 1.5, a_2 = 1$

6 Conclusion

In this paper, we investigate the solution of fractional Duffing equation. This problem is important since it appears in a variety of science models, including engineering, biology, and physics. The fractional derivative will give us the chance to consider the history of the displacement function on the interval $[0,t]$. A numerical solution of fractional problems with strongly oscillators is investigated. The spline spaces are used to approximate the solution. It is worth mentioning that the standard basis such as polynomials will not work with this type of problems since there are strong oscillators. To show the validity of our results, we compare them with four different methods which are HPM, MHPM, SHPM, and collocation method using polynomials as basis for the approximate solution. The error in our approximation is 10^{-10} comparing with other methods which are of 10^{-6} or less. The numerical results reveal that our results are accurate and the proposed method can be used for other physical problems. From the previous results, we note the following.

1. The error in our approximation is 10^{-10} comparing with other methods which are of 10^{-6} or less.
2. There are clear agreements between the approximate and the exact solutions as in Tables (1)-(2).
3. From Figure 5 and 6, we notice that there are agreements between the approximate and the exact solutions.
4. We can implement the proposed method to other problems in physics with nonlinear phenomena.

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