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Positive Integer Powers for One Type of Skew Circulant Matrices

*Fikri KÖKEN**

Department of Mathematics, Konya Necmettin Erbakan University, Konya, Turkey

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Abstract: In this study, we derive the general expression for the entries of the q^{th} integer powers for one type of complex skew circulant matrices scirc_n(0,a,0,...,-b).

Keywords: Matrix power, skew-circulant matrix, chebyshev polynomial

1 Introduction

Pentadiagonal and skew circulant matrices as well as tridiagonal and circulant matrices have a wide number of applications in various fields of science such as mechanics, image processing, mathematical chemistry, etc.. In [1], there is an example about how we use pentadiagonal and tridiagonal matrices in fluid mechanics. The author says, "For example, in fluid mechanics which is a commonly used subject, the number of meshes necessary to obtain reasonably good results is at times expressible in millions. Powerful techniques were developed to solve such systems. In the most common of these methods, inverses of tridiagonal and pentadiagonal matrices are encountered". Circulant and skew circulant matrices also arise in applications involving the discrete Fourier transform and the study of cyclic codes for error correction [2]. Solving some difference, differential and delay differential equations, authors need to compute the arbitrary positive integer powers of some special square matrix. In [4-8], some authors have studied positive integer powers for some types of circulant and skew circulant matrices.

In this study, we derive the general expression of q^{th} $(q \in \mathbb{Z}^+)$ power for one type of complex skew circulant matrices S_n where $n \in \mathbb{N}$ and S_n having the form

$$S_n = scirc_n(0,a,0,...0,-b), \tag{1}$$
$$= \begin{pmatrix} 0 & a & 0 & \cdots & -b \\ b & 0 & a & \ddots & 0 \\ \ddots & b & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & 0 & a \\ -a & 0 & \cdots & b & 0 \end{pmatrix}$$

for any complex numbers a and b.

For the solution of this problem, we will use eigenvalue decomposition;

$$S_n = G_n D_n G_n^*,$$

where * denotes conjugate transpose, G_n are $n \times n$ matrices which have entries

$$[G_n]_{kj} = \frac{1}{\sqrt{n}} e^{\frac{\pi(2j-1)(k-1)}{n} i}, \quad 1 \le k, \ j \le n,$$

and D_n are $n \times n$ matrices that can be written by $D_n = diag(\mu_1 \ \mu_2 \ ... \ \mu_n)$ where

$$\mu_j = \sum_{k=1}^{n} c_k e^{\frac{\pi(2j-1)(k-1)}{n} i}, \ j = 1,2, \ ..., \ n.$$

In this formula, the constants c_k $(k = 1, \ ..., \ n)$ are elements of skew circulant matrices S_n [1]. Also, in [3] the h^{th} degree Chebyshev polynomials of the second kind and the first kind $\{U_h(x)\}_{h \ge -1}$ and $\{T_h(x)\}_{h \ge 0}$ are given

* Corresponding author e-mail: kokenfikri@gmail.com

as follows, respectively:

$$U_h(x) = \frac{\sin((h+1)\arccos x)}{\sin(\arccos x)}, \quad (2)$$

$$T_h(x) = \cos(h(\arccos x)). \quad (3)$$

2 General expression for the entries of S_n^q

In this part we give the general expression for entries of S_n^q and we say that it is based on the eigenvalue decomposition of S_n for all $n \in \mathbb{N}$. Theorem 2.1 and Theorem 2.2 provide the general expression of q^{th} power $(q \in \mathbb{Z}^+)$ for one type of even and odd order $n \times n$ the skew circulant matrices in (1), in terms of the Chebyshev polynomials. Finally, two numerical examples are given in section 3 for odd and even n.

Theorem 2.1. Let S_n be an $n \times n$ complex skew circulant matrix having the form $scirc_n(0,a,0,...,0,-b)$, $n \geq 4$ $(n = 2t,\ t \in \mathbb{N})$. Fix $q \in \mathbb{N}$ and $1 \leq j,k \leq n$, then

$$[S_n^q]_{kj} = \frac{\left(1+(-1)^{q+k-j}\right)}{n} \sum_{m=1}^{\frac{n}{2}} \left[((a+b)\lambda_m + i(a-b)\beta)^q \times \left(T_{|k-j|}(\lambda_m) + sign(k-j) i\beta U_{|k-j|-1}(\lambda_m)\right)\right],$$

where $\lambda_m = \cos\frac{\pi(2m-1)}{n}$, $\beta_m = \sqrt{1-\lambda_m^2}$ $1 \leq m \leq \frac{n}{2}$ and $sign(x)$ denotes the signum function

$$sign(x) = \begin{cases} 1 & if\ x > 0, \\ 0 & if\ x = 0, \\ -1 & if\ x < 0, \end{cases}.$$

Proof. By using the *eigenvalue decomposition of* S_n, *the entries of* S_n^q are written by

$$[S_n^q]_{kj} = [G_n D_n^q G_n^*]_{kj} = \sum_{m=1}^{n} [G_n]_{km} \mu_m^q \left[\overline{G}_n\right]_{jm}$$
$$= \frac{1}{n} \sum_{m=1}^{n} \mu_m^q e^{\frac{\pi(2m-1)(k-j)}{n} i}, \quad (4)$$

with

$$\mu_m = (a+b)\cos\frac{\pi(2m-1)}{n} + (a-b)i\sin\frac{\pi(2m-1)}{n}. \quad (5)$$

Also, it is seen that the eigenvalues of the skew circulant matrix S_n provide $\mu_m = -\mu_{\frac{n}{2}+m}$, $m = 1,2,3,...,\frac{n}{2}$ equations. Therefore, the diagonal matrices D_n are

$$D_n = diag\left(\mu_1, \mu_2,\ ..., \mu_{\frac{n}{2}}, -\mu_1, -\mu_2, ..., -\mu_{\frac{n}{2}}\right).$$

Furthermore, $e^{\frac{\pi(2(\frac{n}{2}+m)-1)(k-j)}{n} i} = (-1)^{k-j} e^{\frac{\pi(2m-1)(k-j)}{n} i}$ for all $1 \leq m \leq \frac{n}{2}$. Consequently, $\left[S_n^q\right]_{kj}$ can be expressed as

$$[S_n^q]_{kj} = \frac{1}{n} \sum_{m=1}^{\frac{n}{2}} \mu_m^q e^{\frac{\pi(2m-1)(k-j)}{n} i} + \frac{1}{n} \sum_{m=1}^{\frac{n}{2}} \mu_{\frac{n}{2}+m}^q e^{\frac{\pi(2(\frac{n}{2}+m)-1)(k-j)}{n} i}$$
$$= \frac{1}{n} \sum_{m=1}^{\frac{n}{2}} \mu_m^q e^{\frac{\pi(2m-1)(k-j)}{n} i} + \frac{1}{n} \sum_{m=1}^{\frac{n}{2}} (-\mu_m)^{slq} (-1)^{k-j} e^{\frac{\pi(2m-1)(k-j)}{n} i}$$
$$= \frac{\left(1+(-1)^{q+k-j}\right)}{n} \sum_{m=1}^{\frac{n}{2}} \left((a+b)\cos\frac{\pi(2m-1)}{n} + (a-b)i\sin\frac{\pi(2m-1)}{n}\right)^q e^{\frac{\pi(2m-1)(k-j)}{n} i}.$$

If $\lambda_m = \cos\frac{\pi(2m-1)}{n}$, so $\lambda_{\frac{n}{2}+m} = \cos\frac{(2(\frac{n}{2}+m)-1)\pi}{n} = -\cos\frac{(2m-1)\pi}{n} = -\lambda_m$, substituting these formulas at the last equation, the entries of S_n^q are given as follows

$$[S_n^q]_{kj} = \frac{\left(1+(-1)^{q+k-j}\right)}{n} \times \sum_{m=1}^{\frac{n}{2}} \left((a+b)\lambda_m + i(a-b)\sqrt{1-\lambda_m^2}\right)^q \times \left(\cos\frac{\pi(2m-1)(k-j)}{n} + i\sin\frac{\pi(2m-1)(k-j)}{n}\right). \quad (6)$$

Taking into λ_m account from the equations (2) and (3), following equations are obtained

$$T_{|k-j|}(\lambda_m) = T_{|k-j|}\left(\cos\frac{\pi(2m-1)}{n}\right) = \cos\frac{\pi(2m-1)|k-j|}{n} = \cos\frac{\pi(2m-1)(k-j)}{n},$$

$$U_{|k-j|-1}(\lambda_m) = \frac{\sin\frac{\pi(2m-1)|k-j|}{n}}{\sin\frac{\pi(2m-1)}{n}} = sign(k-j)\frac{\sin\frac{\pi(2m-1)(k-j)}{n}}{\sin\frac{\pi(2m-1)}{n}}.$$

If we say $\beta_m = \sqrt{1-\lambda_m^2}$ for convenience and replace these last values into (6), we devise

$$[S_n^q]_{kj} = \frac{\left(1+(-1)^{q+k-j}\right)}{n} \sum_{m=1}^{\frac{n}{2}} \left[((a+b)\lambda_m + i(a-b)\beta_m)^q \left(T_{|k-j|}(\lambda_m) + sign(k-j) i\beta_m U_{|k-j|-1}(\lambda_m)\right)\right].$$

This is desired.□

Theorem 2.2. Let $S_n = scirc_n(0,a,0,...,0,-b)$ be an $n \times n$ $(n = 2t+1, t \in \mathbb{N})$ the complex skew circulant matrix,

$\lambda_s = \cos\frac{(2s-1)\pi}{n}$ and $1 \le s \le \frac{n-1}{2}$. Then, for all $q \in \mathbb{N}$, and $1 \le j,k \le n$

$$[S_n^q]_{kj} = \frac{(-1)^{q+k-j}\ (a+b)^q}{n} + \frac{1}{n}\sum_{m=1}^{\frac{n-1}{2}} [((a+b)\lambda + i\beta(a-b))^q \times T_{|k-j|}(\lambda) + i\beta sign(k-j)U_{|k-j|-1}(\lambda) + ((a+b)\lambda - i\beta(a-b))^q \times \left(T_{|k-j|}(\lambda) - i\beta sign(k-j)U_{|k-j|-1}(\lambda)\right)]$$

where $\lambda = \lambda_{n+1-m}$ and $\beta = \sqrt{1-\lambda^2}$, $sign(x)$ denotes the signum function

$$sign(x) = \begin{cases} 1 & if\ x > 0, \\ 0 & if\ x = 0, \\ -1 & if\ x < 0. \end{cases}$$

Proof. It is performed similarly to proof of Theorem 2.1 by using the eigenvalue decomposition of S_n

$$[S_n^q]_{kj} = \frac{1}{n}\sum_{m=1}^{n} \mu_m^q e^{\frac{\pi(2m-1)(k-j)}{n}i} \quad (7)$$

with $\mu_m = (a+b)\cos\frac{\pi(2m-1)}{n} + (a-b)i\sin\frac{\pi(2m-1)}{n}$, $m = 1,2,...,n$. But, we have used a different symmetry for eigenvalues of the skew circulant matrices in (1). They can be noted that

$$\overline{\mu}_m = \mu_{n+1-m}, \quad m = 1,2,\ldots,\frac{n-1}{2}$$

and $\mu_{\frac{n+1}{2}} = -(a+b)$ for all odd numbers n. Thus, the diagonal matrices D_n can be defined by

$$D_n = diag(\mu_1,\ldots,\mu_{\frac{n-1}{2}},\mu_{\frac{n+1}{2}},\overline{\mu}_{\frac{n-1}{2}},\ldots,\overline{\mu}_1).$$

Furthermore, $e^{\frac{\pi(2(n+1-m)-1)(k-j)}{n}i} = e^{\frac{-\pi(2m-1)(k-j)}{n}i}$ for all $m = 1,2,\ldots,\frac{n-1}{2}$. Consequently, $\left[S_n^q\right]_{kj}$ can be expressed as

$$\begin{aligned}[S_n^q]_{kj} &= \frac{1}{n}\sum_{m=1}^{\frac{n-1}{2}} \mu_m^q e^{\frac{\pi(2m-1)(k-j)}{n}i} + \\ &\frac{1}{n}\mu_{\frac{n+1}{2}}^q e^{\frac{\pi(2(\frac{n+1}{2})-1)(k-j)}{n}i} \\ &+ \frac{1}{n}\sum_{m=\frac{n+3}{2}}^{n} \mu_m^q e^{\frac{\pi(2m-1)(k-j)}{n}i} \\ &= \frac{1}{n}\sum_{m=1}^{\frac{n-1}{2}} \mu_m^q e^{\frac{\pi(2m-1)(k-j)}{n}i} \\ &+ \frac{1}{n}\sum_{m=1}^{\frac{n-1}{2}} \overline{\mu}_m^q e^{\frac{-\pi(2m-1)(k-j)}{n}i} \\ &+ \frac{1}{n}(-1)^{q+k-j}(a+b)^q \\ &= \frac{(-1)^{q+k-j}(a+b)^q}{n} + L_1 + L_2 \end{aligned} \quad (8)$$

where

$$L_1 = \frac{1}{n}\sum_{m=1}^{\frac{n-1}{2}} \mu_m^q e^{\frac{\pi(2m-1)(k-j)}{n}i},$$

$$L_2 = \frac{1}{n}\sum_{m=1}^{\frac{n-1}{2}} \overline{\mu}_m^q e^{-\frac{\pi(2m-1)(k-j)}{n}i}.$$

Since $\lambda_s = \cos\frac{(2s-1)\pi}{n}$ that is $\lambda_{n+1-m} = \cos\frac{(2(n+1-m)-1)\pi}{n} = \cos\frac{(2m-1)\pi}{n} = \lambda_m$, by substituting these formulas at the last two equations

$$\begin{aligned} L_1 &= \frac{1}{n}\sum_{m=1}^{\frac{n-1}{2}} ((a+b)\lambda + i(a-b)\beta)^q \\ &\times \left(\cos\frac{\pi(2m-1)(k-j)}{n} + i\sin\frac{\pi(2m-1)(k-j)}{n}\right), \\ L_2 &= \frac{1}{n}\sum_{m=1}^{\frac{n-1}{2}} ((a+b)\lambda - i(a-b)\beta)^q \\ &\times \left(\cos\frac{\pi(2m-1)(k-j)}{n} - i\sin\frac{\pi(2m-1)(k-j)}{n}\right) \end{aligned}$$

where λ is used instead of λ_m, and $\beta = \sqrt{1-\lambda^2}$ for simplicity. Taking into λ_m account from the equations (2) and (3), they are as follows

$$T_{|k-j|}(\lambda_m) = \cos\frac{\pi(2m-1)(k-j)}{n},$$

$$U_{|k-j|-1}(\lambda_m) = sign(k-j)\frac{\sin\frac{\pi(2m-1)(k-j)}{n}}{\sin\frac{\pi(2m-1)}{n}}.$$

Taking these expressions into account, they

$$\begin{aligned} L_1 &= \frac{1}{n}\sum_{m=1}^{\frac{n-1}{2}} ((a+b)\lambda + i(a-b)\beta)^q \\ &\times \left(T_{|k-j|}(\lambda) + i\beta sign(k-j)U_{|k-j|-1}(\lambda)\right), \\ L_2 &= \frac{1}{n}\sum_{m=1}^{\frac{n-1}{2}} ((a+b)\lambda - i(a-b)\beta)^q \\ &\times \left(T_{|k-j|}(\lambda) - i\beta sign(k-j)U_{|k-j|-1}(\lambda)\right), \end{aligned}$$

where $\lambda = \lambda_{n+1-m} = \cos\frac{(2m-1)\pi}{n}$ and $\beta = \sqrt{1-\lambda^2}$. If they are written in equation (8), the result which we are looking for are obtained.□

3 Numerical example

Let S_n be an $n \times n$ complex skew circulant matrix and so S_n^q is also skew circulant with $q \in \mathbb{N}$ (see [2]). As an example, we can find any q^{th} positive integer power of S_n with $n \in \mathbb{N}$.

In Theorem 2.1, if we take $n = 4$ then we attain

$$
\begin{aligned}
S_4^q &= (scirc_4(0,a,0,-b))^q \\
&= scirc_4(\gamma_1,\gamma_2,\gamma_3,\gamma_4), \\
&= \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ -\gamma_4 & \gamma_1 & \gamma_2 & \gamma_3 \\ -\gamma_3 & -\gamma_4 & \gamma_1 & \gamma_2 \\ -\gamma_2 & -\gamma_3 & -\gamma_4 & \gamma_1 \end{pmatrix}
\end{aligned}
$$

with

$$
\begin{aligned}
\gamma_1 &= \tfrac{1+(-1)^q}{4}\left[v^q + (-1)^q \overline{v}^q\right], \\
\gamma_2 &= \tfrac{1+(-1)^{q-1}}{4}\left[\tfrac{\sqrt{2}}{2}(1-i)v^q + (-1)^{q+1}\tfrac{\sqrt{2}}{2}(1+i)\overline{v}^q\right], \\
\gamma_3 &= \tfrac{1+(-1)^q}{4}\left[-i.v^q + (-1)^q i.\overline{v}^q\right], \\
\gamma_4 &= \tfrac{1+(-1)^{q-1}}{4}\left[\tfrac{-\sqrt{2}}{2}(1+i)v^q + (-1)^q\tfrac{\sqrt{2}}{2}(1-i)\overline{v}^q\right],
\end{aligned}
$$

where $v = \frac{\sqrt{2}}{2}\left((a+b)+i(a-b)\right)$.

In Theorem 2.2, if we take $n = 3$ then we obtain

$$
\begin{aligned}
S_3^q &= (scirc_3(0,a,-b))^q \\
&= scirc_3(\gamma_1,\gamma_2,\gamma_3), \\
&= \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ -\gamma_2 & \gamma_1 & \gamma_2 \\ -\gamma_3 & -\gamma_2 & \gamma_1 \end{pmatrix}
\end{aligned}
$$

with

$$
\begin{aligned}
\gamma_1 &= \tfrac{1}{3}\left[(-1)^q(a+b) + w^q + \overline{w}^q\right], \\
\gamma_2 &= \tfrac{1}{3}\left[(-1)^{q+1}(a+b) + \tfrac{1}{2}(w^q+\overline{w}^q) - \tfrac{i\sqrt{3}}{2}(w^q-\overline{w}^q)\right], \\
\gamma_3 &= \tfrac{1}{3}\left[(-1)^q(a+b) - \tfrac{1}{2}(w^q+\overline{w}^q) - \tfrac{i\sqrt{3}}{2}(w^q-\overline{w}^q)\right],
\end{aligned}
$$

where $w = \frac{(a+b)}{2} + \frac{\sqrt{3}i(a-b)}{2}$.

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References

[1] N. A. Baykara, On the Inversion of Adjacent Tridiagonal and Pentadiagonal Matrices, WSEAS International Conference, Kos Island, Greece July 14-17, 2012.

[2] P. J. Davis, Circulant Matrices, Chelsea Publishing, New York, 1994.

[3] J. C. Mason and D. Handscomb, Chebyshev Polynomials, Chapman & Hall, 2003.

[4] J. Gutiérrez-Gutiérrez, Positive integer powers of complex skew symmetric circulant matrices, Appl. Math. Comput. 202 (2008) 798–802.

[5] J. Gutiérrez-Gutiérrez, Positive integer powers of complex symmetric circulant matrices, Appl. Math. Comput. 202 (2008) 877–881.

[6] F. Koken, D. Bozkurt, Positive integer powers for one type of odd order circulant matrices, Appl. Math. Comput. (2010), doi:10.1016/j.amc.2010.10.030.

[7] S. Arslan, F. Koken and D. Bozkurt, Positive integer powers and inverse for one type of even order symmetric pentadiagonal matrices, Appl. Math. and Comput. 219 (2013) 5241–5248.

[8] A. Öteles and M. Akbulak, Positive integer powers of certain complex tridiagonal matrices, Math. Sci. Lett. 2, No.1, 63-72 (2013).

Fikri KÖKEN is Assistant Professor of Department of Mathematics Konya Necmettin Erbakan University. His main research interests are: Combinatorics, Group Theory, Calculus, Graph Theory, Matrix Theory, Powers of Matrices , Integer Sequences. He has published research articles in international journals of mathematical. He is referee of mathematical journals.