Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.18576/msl/060212

The Hyper-Zagreb Index of Four Operations on Graphs

B. Basavanagoud and Shreekant Patil*

Department of Mathematics, Karnatak University, Dharwad - 580 003, Karnataka, India

Received: 27 May 2016, Revised: 21 Nov. 2016, Accepted: 23 Nov. 2016
Published online: 1 May 2017

Abstract: The hyper-Zagreb index of a connected graph G, denoted by $HM(G)$, is defined as $HM(G) = \sum_{uv\in E(G)} [d_G(u) + d_G(v)]^2$ where $d_G(z)$ is the degree of a vertex z in G. In this paper, we study the hyper-Zagreb index of four operations on graphs.

Keywords: Degree, Zagreb index, hyper-Zagreb index, operation on graphs

1 Introduction

In this paper, we are concerned with simple connected graphs. Let G be such a graph with vertex set $V(G)$, $|V(G)| = n$, and edge set $E(G)$, $|E(G)| = m$. As usual, n is order and m is size of G. If u and v are two adjacent vertices of G, then the edge connecting them will be denoted by uv. The degree of a vertex $w \in V(G)$ is the number of vertices adjacent to w and is denoted by $d_G(w)$. *Line graph* $L = L(G)$; $V(L) = E(G)$ and two vertices of L are adjacent if the corresponding edges of G are incident with a common vertex. We refer to [11] for unexplained terminology and notation.

A graphical invariant is a number related to a graph, in other words, it is a fixed number under graph automorphisms. In chemical graph theory, these invariants are also called the topological indices. The first and second Zagreb indices are defined as

$$M_1(G) = \sum_{u\in V(G)} d_G(u)^2 \text{ and } M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v)$$

respectively. The first Zagreb index can also expressed as [7]

$$M_1(G) = \sum_{uv\in E(G)} [d_G(u) + d_G(v)].$$

The vertex-degree-based graph invariant

$$F(G) = \sum_{v\in V(G)} d_G(v)^3 = \sum_{uv\in E(G)} [d_G(u)^2 + d_G(v)^2]$$

was encountered in [10]. Recently there has been some interest to F, called "forgotten topological index"[9].

Shirdel et al.[15] introduced a new Zagreb index of a graph G named hyper-Zagreb index and is defined as:

$$HM(G) = \sum_{uv\in E(G)} (d_G(u) + d_G(v))^2.$$

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, there are four related graphs as follows:
• *Subdivision graph* $S = S(G)$ [11]; $V(S) = V(G) \cup E(G)$ and the vertex of S corresponding to the edge uv of G is inserted in the edge uv of G;
• *Semitotal-point graph* $T_2 = T_2(G)$ [14]; $V(T_2) = V(G) \cup E(G)$ and $E(T_2) = E(S) \cup E(G)$;
• *Semitotal-line graph* $T_1 = T_1(G)$ [14]; $V(T_1) = V(G) \cup E(G)$ and $E(T_1) = E(S) \cup E(L)$;
• *Total graph* $T = T(G)$ [4]; $V(T) = V(G) \cup E(G)$ and $E(T) = E(S) \cup E(G) \cup E(L)$.

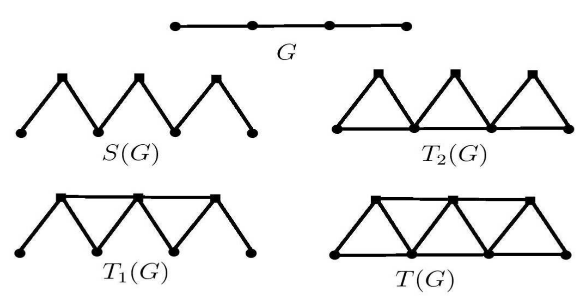

Figure 1: Graph G and its $S(G)$, $T_2(G)$, $T_1(G)$ and $T(G)$.

In Fig. 1. The vertices of transformation graphs $S(G)$, $T_2(G)$, $T_1(G)$, $T(G)$ corresponding to the vertices of the parent graph G, are indicated by circles. The vertices of

* Corresponding author e-mail: b.basavanagoud@gmail.com

these graphs corresponding to the edges of the parent graph G are indicated by squares.
For a given graph G_i, its vertex and edge sets will be denoted by $V(G_i)$ and $E(G_i)$, respectively, and their cardinalities by n_i and m_i, respectively, where $i = 1, 2$.

The Cartesian product $G_1 \times G_2$ of graphs G_1 and G_2 has the vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $(u_1, v_1)(u_2, v_2)$ is an edge of $G_1 \times G_2$ if and only if $[u_1 = u_2$ and $v_1v_2 \in E(G_2)]$ or $[v_1 = v_2$ and $u_1u_2 \in E(G_1)]$. In the recent paper [8], Eliasi and Taeri introduced four new operations on graphs as follows:
Let $F \in \{S, T_2, T_1, T\}$. The F-sum of G_1 and G_2, denoted by $G_1 +_F G_2$, is a graph with the set of vertices $V(G_1 +_F G_2) = (V(G_1) \cup E(G_1)) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 +_F G_2$ are adjacent if and only if $[u_1 = v_1 \in V(G_1)$ and $u_2v_2 \in E(G_2)]$ or $[u_2 = v_2 \in V(G_2)$ and $u_1v_1 \in E(F(G_1))]$.
Thus, they obtained four new operations as $G_1 +_S G_2$, $G_1 +_{T_2} G_2$, $G_1 +_{T_1} G_2$ and $G_1 +_T G_2$ and studied the Wiener indices of these graphs. In [6], Deng et al. gave the expressions for first and second Zagreb indices of these new graphs.

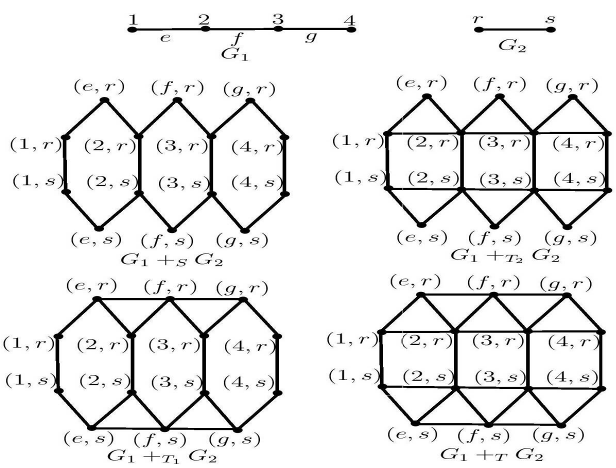

Figure 2: Graphs G_1 and G_2 and $G_1 +_F G_2$.

In this paper, we study the hyper-Zagreb index of $G_1 +_S G_2$, $G_1 +_{T_2} G_2$, $G_1 +_{T_1} G_2$ and $G_1 +_T G_2$. Readers interested in more information on computing topological indices of graph operations can be referred to [1, 2, 3, 5, 12, 13].

2 Main results

Theorem 2.1. Let G_1 and G_2 be the graphs. Then $HM(G_1 +_S G_2) = 4(n_2 + 2m_2)M_1(G_1) + 10m_1M_1(G_2) + n_1HM(G_2) + n_2F(G_1) + 8n_2m_1 + 16m_1m_2$.

Proof. By definition of hyper-Zagreb index, we have

$$\begin{aligned}
&HM(G_1 +_S G_2)\\
&= \sum_{(u_1,v_1)(u_2,v_2)\in E(G_1+_S G_2)} [d_{G_1+_S G_2}(u_1,v_1) + d_{G_1+_S G_2}(u_2,v_2)]^2\\
&= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [d_{G_1+_S G_2}(u,v_1) + d_{G_1+_S G_2}(u,v_2)]^2\\
&+ \sum_{v\in V(G_2)} \sum_{u_1u_2\in E(S(G_1))} [d_{G_1+_S G_2}(u_1,v) + d_{G_1+_S G_2}(u_2,v)]^2\\
&= A_1 + A_2
\end{aligned}$$

where A_1, A_2 are the sums of the above terms, in order.

$$\begin{aligned}
A_1 &= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [d_{G_1}(u) + d_{G_2}(v_1) + d_{G_1}(u) + d_{G_2}(v_2)]^2\\
&= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [4d_{G_1}^2(u) + (d_{G_2}(v_1) + d_{G_2}(v_2))^2\\
&+ 4d_{G_1}(u)(d_{G_2}(v_1) + d_{G_2}(v_2))]\\
&= 4m_2M_1(G_1) + n_1HM(G_2) + 8m_1M_1(G_2).
\end{aligned}$$

$$\begin{aligned}
A_2 &= \sum_{v\in V(G_2)} \sum_{u_1u_2\in E(S(G_1))} [d_{S(G_1)}(u_1) + d_{S(G_1)}(u_2) + d_{G_2}(v)]^2\\
&= \sum_{v\in V(G_2)} \sum_{u_1u_2\in E(S(G_1))} [(d_{S(G_1)}(u_1) + d_{S(G_1)}(u_2))^2 + d_{G_2}^2(v)\\
&+ 2d_{G_2}(v)(d_{S(G_1)}(u_1) + d_{S(G_1)}(u_2))]\\
&= n_2HM(S(G_1)) + 2m_1M_1(G_2) + 4m_2M_1(S(G_1)).
\end{aligned}$$

Note that $M_1(S(G_1)) = M_1(G_1) + 4m_1$ and $HM(S(G_1)) = 4M_1(G_1) + F(G_1) + 8m_1$.

$$\begin{aligned}
\therefore A_2 &= (4n_2 + 4m_2)M_1(G_1) + 2m_1M_1(G_2) + n_2F(G_1)\\
&+ 8n_2m_1 + 16m_1m_2.
\end{aligned}$$

Adding A_1 and A_2, we get the desired result. □

Theorem 2.2. Let G_1 and G_2 be the graphs. Then $HM(G_1 +_{T_2} G_2) = 8(5m_2 + n_2)M_1(G_1) + 22m_1M_1(G_2) + n_1HM(G_2) + 4n_2HM(G_2) + 4n_2F(G_1) + 8m_1n_2 + 16m_1m_2$.

Proof. By definition of hyper-Zagreb index, we have

$$\begin{aligned}
&HM(G_1 +_{T_2} G_2)\\
&= \sum_{(u_1,v_1)(u_2,v_2)\in E(G_1+_{T_2} G_2)} [d_{G_1+_{T_2} G_2}(u_1,v_1) + d_{G_1+_{T_2} G_2}(u_2,v_2)]^2\\
&= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [d_{G_1+_{T_2} G_2}(u,v_1) + d_{G_1+_{T_2} G_2}(u,v_2)]^2\\
&+ \sum_{v\in V(G_2)} \sum_{u_1u_2\in E(T_2(G_1))} [d_{G_1+_{T_2} G_2}(u_1,v) + d_{G_1+_{T_2} G_2}(u_2,v)]^2.
\end{aligned}$$

Note that $E(T_2(G_1)) = E(G_1) \cup E(S(G_1))$.

$$\begin{aligned}
&HM(G_1 +_{T_2} G_2)\\
&= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [d_{G_1+_{T_2} G_2}(u,v_1) + d_{G_1+_{T_2} G_2}(u,v_2)]^2\\
&+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T_2(G_1))\\ u_1,u_2\in V(G_1)}} [d_{G_1+_{T_2} G_2}(u_1,v) + d_{G_1+_{T_2} G_2}(u_2,v)]^2\\
&+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T_2(G_1))\\ u_1\in V(G_1), u_2\in V(T_2(G_1))\setminus V(G_1)}} [d_{G_1+_{T_2} G_2}(u_1,v)\\
&+ d_{G_1+_{T_2} G_2}(u_2,v)]^2\\
&= B_1 + B_2 + B_3
\end{aligned}$$

where B_1, B_2 and B_3 are the sums of the above terms, in order.

$$\begin{aligned}
B_1 &= \sum_{u\in V(G_1)}\sum_{v_1v_2\in E(G_2)}[2d_{T_2(G_1)}(u)+d_{G_2}(v_1)+d_{G_2}(v_2)]^2\\
&= \sum_{u\in V(G_1)}\sum_{v_1v_2\in E(G_2)}[4d_{G_1}(u)+d_{G_2}(v_1)+d_{G_2}(v_2)]^2\\
&= \sum_{u\in V(G_1)}\sum_{v_1v_2\in E(G_2)}[16d^2_{G_1}(u)+(d_{G_2}(v_1)+d_{G_2}(v_2))^2\\
&+ 8d_{G_1}(u)(d_{G_2}(v_1)+d_{G_2}(v_2))]\\
&= 16m_2M_1(G_1)+n_1HM(G_2)+16m_1M_1(G_2).
\end{aligned}$$

$$\begin{aligned}
B_2 &= \sum_{v\in V(G_2)}\sum_{\substack{u_1u_2\in E(T_2(G_1))\\ u_1,u_2\in V(G_1)}}[2d_{G_2}(v)+d_{T_2(G_1)}(u_1)\\
&+ d_{T_2(G_1)}(u_2)]^2\\
&= \sum_{v\in V(G_2)}\sum_{u_1u_2\in E(G_1)}[2d_{G_2}(v)+2d_{G_1}(u_1)+2d_{G_1}(u_2)]^2\\
&= \sum_{v\in V(G_2)}\sum_{u_1u_2\in E(G_1)}[4d^2_{G_2}(v)+4(d_{G_1}(u_1)+d_{G_1}(u_2))^2\\
&+ 8d_{G_2}(v)(d_{G_1}(u_1)+d_{G_1}(u_2))]\\
&= 4m_1M_1(G_2)+4n_2HM(G_1)+16m_2M_1(G_1).
\end{aligned}$$

$$\begin{aligned}
B_3 &= \sum_{v\in V(G_2)}\sum_{\substack{u_1u_2\in E(T_2(G_1))\\ u_1\in V(G_1),u_2\in V(T_2(G_1))\setminus V(G_1)}}[d_{T_2(G_1)}(u_1)+d_{G_2}(v)\\
&+ d_{T_2(G_1)}(u_2)]^2\\
&= \sum_{v\in V(G_2)}\sum_{\substack{u_1u_2\in E(T_2(G_1))\\ u_1\in V(G_1),u_2\in V(T_2(G_1))\setminus V(G_1)}}[2d_{G_1}(u_1)+d_{G_2}(v)+2]^2.
\end{aligned}$$

The quantity $2d_{G_1}(u_1)+d_{G_2}(v)+2$ appears $d_{G_1}(u_1)$ times. Hence,

$$\begin{aligned}
B_3 &= \sum_{v\in V(G_2)}\sum_{u\in V(G_1)}d_{G_1}(u)[4d^2_{G_1}(u)+d^2_{G_2}(v)+4\\
&+ 4d_{G_2}(v)+4d_{G_1}(u)(d_{G_2}(v)+2)]\\
&= 4n_2F(G_1)+2m_1M_1(G_2)+8m_1n_2+16m_1m_2\\
&+ 8m_2M_1(G_1)+8n_2M_1(G_1).
\end{aligned}$$

Adding B_1, B_2 and B_3, we get the desired result. □

Theorem 2.3. Let G_1 and G_2 be the graphs. Then $HM(G_1+_{T_1}G_2) = 16m_2M_1(G_1)+10m_1M_1(G_2)+n_1HM(G_2)+4n_2HM(G_1)+n_2F(G_1)+n_2[\sum_{w_iw_j\in E(G_1),w_jw_k\in E(G_1)}[d_{G_1}(w_i)+2d_{G_1}(w_j)+d_{G_1}(w_k)]^2]$.

Proof. By definition of hyper-Zagreb index, we have

$$\begin{aligned}
&HM(G_1+_{T_1}G_2)\\
&= \sum_{(u_1,v_1)(u_2,v_2)\in E(G_1+_{T_1}G_2)}[d_{G_1+_{T_1}G_2}(u_1,v_1)+d_{G_1+_{T_1}G_2}(u_2,v_2)]^2\\
&= \sum_{u\in V(G_1)}\sum_{v_1v_2\in E(G_2)}[d_{G_1+_{T_1}G_2}(u,v_1)+d_{G_1+_{T_1}G_2}(u,v_2)]^2\\
&+ \sum_{v\in V(G_2)}\sum_{u_1u_2\in E(T_1(G_1))}[d_{G_1+_{T_1}G_2}(u_1,v)+d_{G_1+_{T_1}G_2}(u_2,v)]^2.
\end{aligned}$$

Partition the edge set $E(T_1(G_1))$ into $E(S(G_1))$ and $E(L(G_1))$.

$$\begin{aligned}
&HM(G_1+_{T_1}G_2)\\
&= \sum_{u\in V(G_1)}\sum_{v_1v_2\in E(G_2)}[d_{G_1+_{T_1}G_2}(u,v_1)+d_{G_1+_{T_1}G_2}(u,v_2)]^2\\
&+ \sum_{v\in V(G_2)}\sum_{\substack{u_1u_2\in E(T_1(G_1))\\ u_1\in V(G_1),u_2\in V(T_1(G_1))\setminus V(G_1)}}[d_{G_1+_{T_1}G_2}(u_1,v)\\
&+ d_{G_1+_{T_1}G_2}(u_2,v)]^2\\
&+ \sum_{v\in V(G_2)}\sum_{\substack{u_1u_2\in E(T_1(G_1))\\ u_1,u_2\in V(T_1(G_1))\setminus V(G_1)}}[d_{G_1+_{T_1}G_2}(u_1,v)\\
&+ d_{G_1+_{T_1}G_2}(u_2,v)]^2\\
&= C_1+C_2+C_3
\end{aligned}$$

where C_1, C_2 and C_3 are the sums of the above terms, in order.

$$\begin{aligned}
C_1 &= \sum_{u\in V(G_1)}\sum_{v_1v_2\in E(G_2)}[2d_{T_1(G_1)}(u)+d_{G_2}(v_1)+d_{G_2}(v_2)]^2\\
&= \sum_{u\in V(G_1)}\sum_{v_1v_2\in E(G_2)}[2d_{G_1}(u)+d_{G_2}(v_1)+d_{G_2}(v_2)]^2\\
&= \sum_{u\in V(G_1)}\sum_{v_1v_2\in E(G_2)}[4d^2_{G_1}(u)+(d_{G_2}(v_1)+d_{G_2}(v_2))^2\\
&+ 4d_{G_1}(u)(d_{G_2}(v_1)+d_{G_2}(v_2))]\\
&= 4m_2M_1(G_1)+n_1HM(G_2)+8m_1M_1(G_2).
\end{aligned}$$

$$\begin{aligned}
C_2 &= \sum_{v\in V(G_2)}\sum_{\substack{u_1u_2\in E(T_1(G_1))\\ u_1\in V(G_1),u_2\in V(T_1(G_1))\setminus V(G_1)}}[d_{T_1(G_1)}(u_1)+d_{G_2}(v)\\
&+ d_{T_1(G_1)}(u_2)]^2\\
&= \sum_{v\in V(G_2)}\sum_{\substack{u_1u_2\in E(T_1(G_1))\\ u_1\in V(G_1),u_2\in V(T_1(G_1))\setminus V(G_1)}}[d_{G_1}(u_1)+d_{G_2}(v)\\
&+ d_{T_1(G_1)}(u_2)]^2
\end{aligned}$$

$$\begin{aligned}
&= \sum_{v\in V(G_2)}\sum_{\substack{u_1u_2\in E(T_1(G_1))\\ u_1\in V(G_1),u_2\in V(T_1(G_1))\setminus V(G_1)}}[d^2_{G_1}(u_1)+d^2_{G_2}(v)\\
&+ 2d_{G_1}(u_1)d_{G_2}(v)+d^2_{T_1(G_1)}(u_2)\\
&+ 2d_{T_1(G_1)}(u_2)(d_{G_1}(u_1)+d_{G_2}(v))]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{v\in V(G_2)}\sum_{u\in V(G_1)}d_{G_1}(u)[d^2_{G_1}(u)+d^2_{G_2}(v)+2d_{G_1}(u)d_{G_2}(v)]\\
&+ \sum_{v\in V(G_2)}\sum_{\substack{u_1u_2\in E(T_1(G_1))\\ u_1\in V(G_1),u_2\in V(T_1(G_1))\setminus V(G_1)}}[d^2_{T_1(G_1)}(u_2)\\
&+ 2d_{T_1(G_1)}(u_2)(d_{G_1}(u_1)+d_{G_2}(v))].
\end{aligned}$$

Note that for $u_2 \in V(T_1(G_1))\backslash V(G_1)$, $d_{T_1(G_1)}(u_2) = d_{G_1}(u) + d_{G_1}(w)$ where $u_2 = uw \in E(G_1)$.

$$\begin{aligned} C_2 &= n_2F(G_1) + 2m_1M_1(G_2) + 4m_2M_1(G_1) \\ &+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T_1(G_1)) \\ u_1\in V(G_1), u_2\in V(T_1(G_1))\backslash V(G_1)}} [(d_{G_1}(u) \\ &+ d_{G_1}(w))^2 + 2(d_{G_1}(u) + d_{G_1}(w))d_{G_1}(u_1) \\ &+ 2(d_{G_1}(u) + d_{G_1}(w))d_{G_2}(v)] \\ &= n_2F(G_1) + 2m_1M_1(G_2) + 4m_2M_1(G_1) + 2n_2HM(G_1) \\ &+ 2n_2[F(G_1) + 2M_2(G_1)] + 8m_2M_1(G_1) \\ &= n_2F(G_1) + 2m_1M_1(G_2) + 4n_2HM(G_1) + 12m_2M_1(G_1). \end{aligned}$$

$$\begin{aligned} C_3 &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T_1(G_1)) \\ u_1,u_2\in V(T_1(G_1))\backslash V(G_1)}} [d_{G_1+_{T_1}G_2}(u_1,v) \\ &+ d_{G_1+_{T_1}G_2}(u_2,v)]^2 \\ &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T_1(G_1)) \\ u_1,u_2\in V(T_1(G_1))\backslash V(G_1)}} [d_{T_1(G_1)}(u_1) + d_{T_1(G_1)}(u_2)]^2 \\ &= n_2[\sum_{w_iw_j\in E(G_1), w_jw_k\in E(G_1)} [d_{G_1}(w_i) + d_{G_1}(w_j) + d_{G_1}(w_j) \\ &+ d_{G_1}(w_k)]^2]. \end{aligned}$$

Adding C_1, C_2 and C_3, we get the desired result. □

Theorem 2.4. Let G_1 and G_2 be the graphs. Then $HM(G_1 +_T G_2) = 48m_2M_1(G_1) + 22m_1M_1(G_2) + 10n_2HM(G_1) + n_1HM(G_2) + 4n_2F(G_1) + n_2[\sum_{w_iw_j\in E(G_1), w_jw_k\in E(G_1)} [d_{G_1}(w_i) + 2d_{G_1}(w_j) + d_{G_1}(w_k)]^2]$.

Proof. By definition of hyper-Zagreb index, we have

$$\begin{aligned} & HM(G_1 +_T G_2) \\ &= \sum_{(u_1,v_1)(u_2,v_2)\in E(G_1+_TG_2)} [d_{G_1+_TG_2}(u_1,v_1) + d_{G_1+_TG_2}(u_2,v_2)]^2 \\ &= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [d_{G_1+_TG_2}(u,v_1) + d_{G_1+_TG_2}(u,v_2)]^2 \\ &+ \sum_{v\in V(G_2)} \sum_{u_1u_2\in E(T(G_1))} [d_{G_1+_TG_2}(u_1,v) + d_{G_1+_TG_2}(u_2,v)]^2. \end{aligned}$$

Note that $E(T(G_1)) = E(G_1) \cup E(S(G_1)) \cup E(L(G_1))$.

$$\begin{aligned} & HM(G_1 +_T G_2) \\ &= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [d_{G_1+_TG_2}(u,v_1) + d_{G_1+_TG_2}(u,v_2)]^2 \\ &+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)) \\ u_1,u_2\in V(G_1)}} [d_{G_1+_TG_2}(u_1,v) + d_{G_1+_TG_2}(u_2,v)]^2 \\ &+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)) \\ u_1\in V(G_1), u_2\in V(T(G_1))\backslash V(G_1)}} [d_{G_1+_TG_2}(u_1,v) \\ &+ d_{G_1+_TG_2}(u_2,v)]^2 \\ &+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T_1(G_1)) \\ u_1,u_2\in V(T_1(G_1))\backslash V(G_1)}} [d_{G_1+_{T_1}G_2}(u_1,v) \\ &+ d_{G_1+_{T_1}G_2}(u_2,v)]^2 \\ &= D_1 + D_2 + D_3 + D_4 \end{aligned}$$

where D_1, D_2, D_3 and D_4 are the sums of the above terms, in order.

$$\begin{aligned} D_1 &= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [2d_{T(G_1)}(u) + d_{G_2}(v_1) + d_{G_2}(v_2)]^2 \\ &= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [4d_{G_1}(u) + d_{G_2}(v_1) + d_{G_2}(v_2)]^2 \\ &= 16m_2M_1(G_1) + n_1HM(G_2) + 16m_1M_1(G_2). \end{aligned}$$

$$\begin{aligned} D_2 &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)) \\ u_1,u_2\in V(G_1)}} [2d_{G_2}(v) + d_{T(G_1)}(u_1) + d_{T(G_1)}(u_2)]^2 \\ &= \sum_{v\in V(G_2)} \sum_{u_1u_2\in E(G_1)} [2d_{G_2}(v) + 2d_{G_1}(u_1) + 2d_{G_1}(u_2)]^2 \\ &= 4m_1M_1(G_2) + 4n_2HM(G_1) + 16m_2M_1(G_1). \end{aligned}$$

$$\begin{aligned} D_3 &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)) \\ u_1\in V(G_1), u_2\in V(T(G_1))\backslash V(G_1)}} [d_{T(G_1)}(u_1) + d_{G_2}(v) \\ &+ d_{T(G_1)}(u_2)]^2 \\ &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)) \\ u_1\in V(G_1), u_2\in V(T(G_1))\backslash V(G_1)}} [2d_{G_1}(u_1) + d_{G_2}(v) \\ &+ d_{T(G_1)}(u_2)]^2 \end{aligned}$$

$$\begin{aligned} &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)) \\ u_1\in V(G_1), u_2\in V(T(G_1))\backslash V(G_1)}} [4d^2_{G_1}(u_1) + d^2_{G_2}(v) \\ &+ 4d_{G_1}(u_1)d_{G_2}(v) + d^2_{T(G_1)}(u_2) + 2d_{T(G_1)}(u_2)(2d_{G_1}(u_1) \\ &+ d_{G_2}(v))] \end{aligned}$$

$$\begin{aligned} &= \sum_{v\in V(G_2)} \sum_{u\in V(G_1)} d_{G_1}(u)[4d^2_{G_1}(u) + d^2_{G_2}(v) + 4d_{G_1}(u)d_{G_2}(v)] \\ &+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)) \\ u_1\in V(G_1), u_2\in V(T(G_1))\backslash V(G_1)}} [d^2_{T(G_1)}(u_2) \\ &+ 2d_{T(G_1)}(u_2)(2d_{G_1}(u_1) + d_{G_2}(v))]. \end{aligned}$$

Note that for $u_2 \in V(T(G_1))\backslash V(G_1)$, $d_{T(G_1)}(u_2) = d_{G_1}(u) + d_{G_1}(w)$ where $u_2 = uw \in E(G_1)$.

$$\begin{aligned} D_3 &= 4n_2F(G_1) + 2m_1M_1(G_2) + 8m_2M_1(G_1) \\ &+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1)) \\ u_1\in V(G_1), u_2\in V(T(G_1))\backslash V(G_1)}} [(d_{G_1}(u) + d_{G_1}(w))^2 \\ &+ 4(d_{G_1}(u) + d_{G_1}(w))d_{G_1}(u_1) + 2(d_{G_1}(u) \\ &+ d_{G_1}(w))d_{G_2}(v)] \\ &= 4n_2F(G_1) + 2m_1M_1(G_2) + 8m_2M_1(G_1) + 2n_2HM(G_1) \\ &+ 4n_2[F(G_1) + 2M_2(G_1)] + 8m_2M_1(G_1) \\ &= 4n_2F(G_1) + 2m_1M_1(G_2) + 16m_2M_1(G_1) + 6n_2HM(G_1). \end{aligned}$$

$$D_4 = \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1))\\ u_1,u_2\in V(T(G_1))\setminus V(G_1)}} [d_{G_1+_T G_2}(u_1,v) + d_{G_1+_T G_2}(u_2,v)]^2$$
$$= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1))\\ u_1,u_2\in V(T(G_1))\setminus V(G_1)}} [d_{T(G_1)}(u_1) + d_{T(G_1)}(u_2)]^2$$
$$= n_2[\sum_{w_iw_j\in E(G_1),w_jw_k\in E(G_1)} [d_{G_1}(w_i) + d_{G_1}(w_j) + d_{G_1}(w_j) + d_{G_1}(w_k)]^2].$$

Adding D_1, D_2, D_3 and D_4, we get the desired result. □

Applying the above four theorems to the graphs $G_1 = P_r$ and $G_2 = P_q$, we have

(i)$HM(P_r +_S P_q) = 136rq - 138r - 150q + 124, q > 2$;
(ii)$HM(P_r +_{T_2} P_q) = 416rq - 338r - 576q + 388, r, q > 2$;
(iii)$HM(P_r +_{T_1} P_q)$
$$= \begin{cases} 192rq - 154r - 234q + 156 \ if\ r = 3,\ q > 2; \\ 256rq - 154r - 428q + 156 \ if\ r > 3,\ q > 2; \end{cases}$$
(iv)$HM(P_r +_T P_q)$
$$= \begin{cases} 488rq - 354r - 696q + 420 \ if\ r = 3,\ q > 2; \\ 552rq - 354r - 890q + 420 \ if\ r > 3,\ q > 2. \end{cases}$$

3 Conclusion

In this paper, we have studied the hyper Zagreb index of new four sums of graphs. Also we apply our results to compute the hyper Zagreb index of $P_r +_S P_q$, $P_r +_{T_2} P_q$, $P_r +_{T_1} P_q$ and $P_r +_T P_q$. For further research, one can study the other topological indices of these new operations.

Acknowledgement

The first author acknowledges the financial support by UGC-SAP DRS-III, New Delhi, India for 2016-2021: F.510/3/DRS-III/2016(SAP-I) Dated: 29th Feb. 2016.

References

[1] A. R. Ashrafi, T. Došlić, A. Hamzeh, The Zagreb coindices of graph operations, *Discrete Appl. Math.* **158** (2010) 1571–1578.
[2] B. Basavanagoud, S. Patil, A note on hyper-Zagreb index of graph operations, *Iranian J. Math. Chem.* **7**(1) (2016) 89–92.
[3] B. Basavanagoud, S. Patil, A note on hyper-Zagreb coindex of graph operations, *J. Appl. Math. Comput.* (2016) doi:10.1007/s12190-016-0986-y.
[4] M. Behzad, A criterion for the planarity of a total graph, Pro. Cambridge Philos. Soc. **63** (1967) 697–681.
[5] K. C. Das, A. Yurttas, M. Togan, A. S. Cevik, I. N. Cangul, The multiplicative Zagreb indices of graph operations, *J. Inequal. Appl.* **90** (2013) 1–14.
[6] H. Deng, D. Sarala, S. K. Ayyaswamy, S. Balachandran, The Zagreb indices of four operations on graphs, *Appl. Math. Comput.* **275** (2016) 422–431.
[7] T. Došlić, B. Furtula, A. Graovac, I. Gutman, S. Moradi, Z. Yarahmadi, On vertex-degree-based molecular structure descriptors, *MATCH Commun. Math. Comput. Chem.* **66** (2011) 613–626.
[8] M. Eliasi, B. Taeri, Four new sums of graphs and their Wiener indices, Discrete Appl. Math. **157** (2009) 794–803.
[9] F. Furtula, I. Gutman, A forgotten topological index, *J. Math. Chem.* **53** (2015) 1184–1190.
[10] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* **17** (1972) 535–538.
[11] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Mass 1969.
[12] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, The first and second Zagreb indices of some graph operations, *Discrete Appl. Math.* **157** (2009) 804–811.
[13] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, The hyper-Wiener index of graph operations, *Comput. Math. Appl.* **56** (2008) 1402–1407.
[14] E. Sampathkumar, S. B. Chikkodimath, Semitotal graphs of a graph-I, J. Karnatak Univ Sci. **18** (1973) 274–280.
[15] G. H. Shirdel, H. Rezapour, A.M. Sayadi, The hyper-Zagreb index of graph operations, *Iranian J. Math. Chem.* **4**(2) (2013) 213–220.

B. Basavanagoud received his both MSc and PhD degree in Mathematics from Gulbarga University, Gulbarga, Karnataka state, India. Currently he is working as Professor and Chairman, Department of Mathematics, Karnatak University, Dharwad, Karnataka state, India. He has published many research papers in reputed national and international journals. His area of interest lies mainly in Graph Theory. He is also life member of The Indian Mathematical Society, The Indian Science Congress Association and International Journal of Mathematical Sciences and Engineering Applications. He chaired the session in the International Congress of Mathematicians(ICM-2010) and delivered a contributed talk at the same.

Shreekant Patil received his MSc degree in Mathematics from Karnatak University, Dharwad, Karnataka state, India. He is currently a research student in the Department of Mathematics, Karnatak University, Dharwad working under the guidance of Dr. B. Basavanagoud.