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# Fractional Differential Inclusions in the Almgren Sense with Riemann-Liouville Derivative

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**Abstract:** In this work, the authors study the existence of solutions for fractional differential inclusions in the sense of Almgren with Riemann-Liouville derivative. They also show the compactness of the solution set. A Peano type existence theorem is also proved.

**Keywords:** Differential inclusions, multiple-functions in the Almgren sense, fractional derivative, Riemann-Liouville derivative, Peano theorem, local existence, compactness

## 1 Introduction

The theory of multiple-valued functions in the sense of Almgren [1] has several applications in the framework of geometric measure theory. It gives a very useful tool to approximate some abstract objects arising from geometric measure theory. For example, Almgren [1] used multiple-valued functions to approximate mass minimizing rectifiable currents, and hence successfully obtained their partial interior regularity. Solomon [2] succeeded in giving proofs of the closure theorem without using the structure theorem. His proofs rely on various facts about multiple-valued functions. There are also other objects similar to these functions, such as the union of graphs of Sobolev's functions introduced by Ambrosio, Gobbino and Pallara (see [3]).

In complex function theory one often speaks of the two-valued function $f(z) = \sqrt{z}$. This can be considered as a function from $\mathbb{C} \to \mathscr{A}_2(\mathbb{C})$. Almgren [4] introduced $\mathscr{A}_Q(\mathbb{R}^n)$-valued functions to address the problem of estimating the size of the singular set of mass-minimizing integral currents (see [1] for a summary). Almgren's multiple-valued functions are a fundamental tool for understanding geometric variational problems in codimension higher than 1. The success of Almgren's regularity theory raises the need for further studying multiple-valued functions. For additional information concerning multiple-valued functions, we suggest the references [5,6,7,8,9,10].

Differential equations of fractional order recently have proved to be valuable tools in the modeling of many physical phenomena [11,12,13,14,15,16,17,18]. There have also been significant theoretical developments in fractional differential equations in recent years; see the monographs of Kilbas *et al.* [19], Miller and Ross [20], Oustaloup [21], Podlubny [22], and Samko *et al.* [23]. For details on geometric and physical interpretations of fractional derivatives of the Riemann-Liouville type, see [22].

In this paper we consider the problem

$$\begin{cases} D^{\alpha}_{0^+}y(t) \in \{f_1(t,t^{1-\alpha}y),\dots,f_k(t,t^{1-\alpha}y)\}, \quad t\in(0,b],\\ \lim\limits_{t\to 0^+} t^{1-\alpha}y(t) = c, \end{cases} \tag{1}$$

where each $f_i\colon [0,b]\times\mathbb{R}^n \to \mathbb{R}^n$ is a single-valued function, and $D^{\alpha}_{0^+}$ denotes the usual Riemann-Liouville fractional derivative of order $\alpha \in (0,1]$.

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For some existence results for differential inclusions and fractional differential inclusions in the sense of Almgren, see Goblet [7]. For the existence and structure of the solution sets of some classes of differential inclusions and fractional differential equations and inclusions in the usual sense, we suggest [24,25,26,27,28,29] and the references therein.

The goal of this note is to provide an existence result for solution sets of fractional differential inclusions in the sense of Almgren.

## 2 Preliminaries

In this section, we recall from the literature some notations, definitions, and auxiliary results that will be used throughout this paper.

*Notation:* We denote by $[[p_i]]$ the Dirac mass in $p_i \in \mathbb{R}^n$.

**Definition 1.** *For every $T_1, T_2 \in \mathscr{A}_k(\mathbb{R}^n)$, with $T_1 = \sum_i [[p_i]]$ and $T_2 = \sum_i [[s_i]]$, we define $d_{\mathscr{A}}(T_1, T_2)$ by either*

$$d_{\mathscr{A}}(T_1, T_2) := \min_{\sigma \in \mathscr{P}_k} \sqrt{\sum_{i=1}^{k} |p_i - s_{\sigma(i)}|^2},$$

$$d_{\mathscr{A}}(T_1, T_2) := \min_{\sigma \in \mathscr{P}_k} \sum_{i=1}^{k} |p_i - s_{\sigma(i)}|,$$

*or*

$$d_{\mathscr{A}}(T_1, T_2) := \min_{\sigma \in \mathscr{P}_k} \{\max |p_i - s_{\sigma(i)}| : i = 1, \dots, k\},$$

*where $\mathscr{P}_k$ denotes the group of permutations of $\{1, \dots, k\}$.*

*Remark.* In the above definition, we designated each expression with the same symbol because the results in this paper are independent of the form of $d_{\mathscr{A}}$ used.

**Definition 2.** *A multiple-valued function in the sense of Almgren is a map $T : \Omega \to \mathscr{A}_k(\mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^n$ and*

$$\mathscr{A}_k(\mathbb{R}^n) = \left\{ \sum_{i=1}^{k} [[p_i]] : p_i \in \mathbb{R}^n \text{ for every } i = 1, \dots, k \right\}$$

*equipped with the metric $d_{\mathscr{A}}$.*

Next, we define what we mean by a selection function and give some of their useful properties.

**Definition 3.** *Let $\Omega \subset \mathbb{R}^m$ and $f : \Omega \to \mathscr{A}_k(\mathbb{R}^n)$ be a $k-$valued function. If there exist single-valued maps $g_i : \Omega \to \mathbb{R}^m$, $i = 1, \dots, k$, such that*

$$f(x) = \sum_{i=1}^{k} [[g_i(x)]] \quad \text{for each } x \in \mathbb{R}^m,$$

*then we say that the vector $(g_1, \dots, g_k)$ is a selection for $f$.*

**Theorem 1.** ([4,5]) *Let $f : [0, b] \to \mathscr{A}_k(\mathbb{R}^n)$ be a continuous multiple-valued function. Then there are continuous functions $f_1, \dots, f_k : [0, b] \to \mathbb{R}^n$ such that*

$$f = \sum_{i=1}^{k} f_i.$$

*Remark.* If for each $i \in \{1, \dots, k\}$, $f_i$ is continuous, then $f$ has a continuous selection.

**Lemma 1.** ([7]) *Let $f : \mathbb{R} \to \mathscr{A}_k(\mathbb{R}^n)$ be a continuous multiple-valued function and $g : \mathbb{R} \to \mathbb{R}^n$ be a continuous function. If $h : [0, b] \times \mathbb{R} \to \mathscr{A}_{k-1}(\mathbb{R}^n)$ satisfies*

$$f = [[g]] + h,$$

*then $h$ is a continuous function.*

*Remark.* An $\mathscr{A}_k(\mathbb{R}^n)$-valued function is essentially a rule assigning $k$ unordered and not necessarily distinct elements of $\mathbb{R}^n$ to each element of its domain.

**Lemma 2.** ([7]) *Let $(f_i) : [0,b] \to \mathscr{A}_k(\mathbb{R}^n)$ be a sequence of multiple-valued functions converging pointwise to $f$, and let $(g_i) : [0,b] \to \mathbb{R}^n$ be a sequence of functions converging pointwise to $g$ such that $g_i$ is a selection of $f_i$ for each $i \in \mathbb{N}$. Then $g$ is a selection of $f$.*

**Theorem 2.** ([4]) *Suppose $f_1,\dots,f_k : [0,b] \to \mathbb{R}^n$ are continuous functions and $f = \sum_{i=1}^{k}[[f_i]] : [0,b] \to \mathscr{A}_k(\mathbb{R}^n)$. Then there exists a constant $C_{n,k} > 0$, depending only on $n$ and $k$, such that*

$$\omega_{f_i} \leq C_{n,k}\omega_f, \text{ for each } i = 1,\dots,k,$$

*where $\omega_f$ is the modulus of continuity of $f$, i.e.,*

$$\omega_f(\delta) = \sup\{d_{\mathscr{A}}(f(s_1), f(s_2)) : s_1,s_2 \in [0,b] \text{ and } |s_1 - s_2| \leq \delta\},$$

*and*

$$\omega_{f_i}(\delta) = \sup\{|f_i(s_1) - f_i(s_2)| : s_1,s_2 \in [0,b] \text{ and } |s_1 - s_2| \leq \delta\}.$$

## 3 Fractional calculus

According to the Riemann-Liouville approach to fractional calculus, the notation of fractional integral of order $\alpha > 0$ is a natural consequence of the well known formula (usually attributed to Cauchy), that reduces the calculation of the $n-$fold primitive of a function $f(t)$ to a single integral of convolution type. The Cauchy formula is

$$I^n f(t) := \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f(s)ds, \; t > 0, \; n \in \mathbb{N}.$$

Recall that Euler's Gamma function is defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt, \; \alpha > 0.$$

**Definition 4.** *The fractional integral of order $\alpha > 0$ of a function $f \in L^1([a,b],\mathbb{R})$ is defined by*

$$I^\alpha_{a^+} f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds.$$

*If $a = 0$, we write $I^\alpha f(t) = f(t) * \phi_\alpha(t)$ with $\phi_\alpha(t) : \mathbb{R} \to \mathbb{R}$ defined by*

$$\phi_\alpha(t) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0, \end{cases}$$

*and $\phi_\alpha(t) \to \delta(t)$ as $\alpha \to 0$, where $\delta$ is the delta function.*

*Remark.* For consistency, we take $I^0$ to be the identity operator, i.e., $I^0 f(t) = f(t)$. Furthermore, by $I^\alpha f(0^+)$ we mean the limit (if it exists) of $I^\alpha f(t)$ as $t \to 0^+$ (this limit may be infinite).

After the notion of fractional integral, that of fractional derivative of order $\alpha > 0$ becomes a natural requirement. It is tempting to replace $\alpha$ with $-\alpha$ in the above formulas, however, this generalization needs some care in order to guarantee the convergence of the integral and preserve the well known properties of the ordinary derivative of integer order. Denoting by $D^n$, with $n \in \mathbb{N}$, the derivative operator, we first note that

$$D^n I^n = I^0, \quad I^n D^n \neq I^0, \quad n \in \mathbb{N},$$

i.e., $D^n$ is the left-inverse (and not the right-inverse) of the corresponding integral operator $I^n$. It is easily proved that

$$I^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(a^+)\frac{(t-a)^k}{k!}, \; t > 0.$$

As a consequence, we expect that $D^\alpha$ is defined as the left-inverse to $I^\alpha$. For this purpose, introducing the positive integer $n$ such that $n-1 < \alpha \leq n$, we define the fractional derivative of order $\alpha > 0$ as follows.

**Definition 5.** *For a function $f$ given on interval $[a,b]$, the $\alpha$-th Riemann-Liouville fractional order derivative of $f$ is defined by*

$$D^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1}f(s)ds,$$

*where $n=[\alpha]+1$ and $[\alpha]$ is the integer part of $\alpha$.*

We also need the following generalization of Gronwall's lemma for singular kernels; its proof can be found in [30, Lemma 7.1.1].

**Lemma 3.** *Let $v:[0,b]\to[0,\infty)$ be a real function, $w:[0,b]\to[0,\infty)$ be a locally integrable function on $[0,b]$, and assume that there are constants $a>0$ and $0<\alpha<1$ such that*

$$v(t) \le w(t) + a\int_0^t \frac{v(s)}{(t-s)^{\alpha}}ds.$$

*Then, there exists a constant $K=K(\alpha)$ such that*

$$v(t) \le w(t) + Ka\int_0^t \frac{w(s)}{(t-s)^{\alpha}}ds,$$

*for every $t\in[0,b]$.*

## 4 Main results

We consider the Banach space of continuous functions

$$C_*([0,b],\mathbb{R}^n) = \{y\in C((0,b],\mathbb{R}^n) : \lim_{t\to 0^+} t^{1-\alpha}y(t) \text{ exists}\}$$

with the norm

$$\|y\|_* = \sup\{|t^{1-\alpha}y(t)| : t\in(0,b]\}.$$

For a subset $\mathscr{A}$ of the space $C_*([0,b],\mathbb{R})$, define $\mathscr{A}_\alpha$ by

$$\mathscr{A}_\alpha = \{y_\alpha : y\in\mathscr{A}\},$$

where

$$y_\alpha(t) = \begin{cases} t^{1-\alpha}y(t), & t\in(0,b],\\ \lim\limits_{t\to 0^+} t^{1-\alpha}y(t), & t=0.\end{cases}$$

The following theorem is a simple variant of the classical Arzelà-Ascoli theorem.

**Theorem 3.** ([31]) *Let $\mathscr{A}$ be a bounded set in $C_*([0,b],\mathbb{R})$. Assume that $\mathscr{A}_\alpha$ is equicontinuous on $[0,b]$. Then $\mathscr{A}$ is relatively compact in $C_*([0,b],\mathbb{R})$.*

Now, we present our main results.

**Theorem 4.** *Let $f_i:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$, $i=1,\dots,k$, be single-valued functions such that the associated multiple-valued function in the sense of Almgren*

$$f = \sum_{i=1}^{k} [[f_i]] : [0,b]\times\mathbb{R}\to\mathscr{A}_k(\mathbb{R}^n)$$

*is continuous. Assume that there exists $M>0$ such that*

$$d_{\mathscr{A}}(f(t,x),k[[0]]) \le M, \quad \text{for all } x\in\mathbb{R} \text{ and } t\in(0,b]. \tag{2}$$

*Then the problem (1) has at least one solution. Moreover, the solution set of problem (1) is compact in $C_*([0,b],\mathbb{R}^n)$.*

*Proof.* First we construct a solution on the interval $(0,b]$. We form two sequences $\{y_i\}_{i=1}^{i=\infty}$ and $\{g_i\}_{i=1}^{i=\infty}$ in the spaces $C_*([0,b],\mathbb{R}^n)$ and $C([0,b],\mathbb{R}^n)$, respectively, as follows. Let

$$y_\alpha^1(t) = \lim_{s\to t^+} s^{1-\alpha}y_1(s) = c, \quad t\in[0,b],$$

where

$$y_1(t) = ct^{\alpha-1}, \quad t\in(0,b].$$

Let

$$y_\alpha^2(t) = \begin{cases} c, & t\in[0,\frac{b}{2}], \\ t^{1-\alpha}y_2(t), & t\in(\frac{b}{2},b], \end{cases}$$

where

$$y_2(t) = \begin{cases} y_1(t), & t\in(0,\frac{b}{2}], \\ ct^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^{t-\frac{b}{2}}(t-s)^{\alpha-1}g_{2,1}(s)ds, & t\in(\frac{b}{2},b], \end{cases}$$

and $g_{2,1}:[0,\frac{b}{2}]\to\mathbb{R}^n$ is a continuous selection of $f(\cdot,y_\alpha^2(\cdot)):[0,\frac{b}{2}]\times\mathbb{R}^n\to\mathscr{A}_k(\mathbb{R}^n)$. From Theorem 1, we can find a continuous selection $g_2:[0,b]\to\mathbb{R}^n$ of $f(\cdot,y_\alpha^2(\cdot)):[0,b]\times\mathbb{R}^n\to\mathscr{A}_k(\mathbb{R}^n)$ such that $g_2(\cdot)=g_{2,1}(\cdot)$ on $[0,\frac{b}{2}]$. We define

$$y_\alpha^3(t) = \begin{cases} c, & t\in[0,\frac{b}{3}], \\ t^{1-\alpha}y_3(t), & t\in(\frac{b}{3},b], \end{cases}$$

where

$$y_3(t) = \begin{cases} y_2(t), & t\in(0,\frac{b}{3}], \\ ct^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^{t-\frac{b}{3}}(t-s)^{\alpha-1}g_{3,1}(s)ds, & t\in(\frac{b}{3},\frac{2b}{3}], \\ ct^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^{t-\frac{2b}{3}}(t-s)^{\alpha-1}g_{3,2}(s)ds, & t\in(\frac{2b}{3},b], \end{cases}$$

$g_{3,1}:[0,\frac{b}{3}]\to\mathbb{R}^n$ is a continuous selection of $f(\cdot,y_\alpha^3(\cdot)):[0,\frac{b}{3}]\to\mathscr{A}_k(\mathbb{R}^n)$, and $g_{3,2}:[0,\frac{2b}{3}]\to\mathbb{R}^n$ is a continuous selection of $f(\cdot,y_\alpha^2(\cdot)):[0,\frac{2b}{3}]\to\mathscr{A}_k(\mathbb{R}^n)$ such that $g_{3,1}(\cdot)=g_{3,2}(\cdot)$ on $[0,\frac{b}{3}]$. By Theorem 1, we can choose a continuous selection $g_3:[0,b]\to\mathbb{R}^n$ of $f(\cdot,y_\alpha^3(\cdot)):[0,b]\times\mathbb{R}^n\to\mathscr{A}_k(\mathbb{R}^n)$ such that $g_3(\cdot)=g_{3,2}(\cdot)$ on $[0,\frac{2b}{3}]$. By induction, for $j\in\mathbb{N}$,

$$y_\alpha^j(t) = \begin{cases} c, & t\in[0,\frac{b}{j}], \\ t^{1-\alpha}y_j(t), & t\in(\frac{b}{j},b], \end{cases}$$

where

$$y_j(t) = \begin{cases} y_{j-1}(t), & t\in(0,\frac{b}{j}], \\ ct^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^{t-\frac{b}{j}}(t-s)^{\alpha-1}g_{j,1}(s)ds, & t\in(\frac{b}{j},\frac{2b}{j}], \\ ct^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^{t-\frac{2b}{j}}(t-s)^{\alpha-1}g_{j,2}(s)ds, & t\in(\frac{2b}{j},\frac{3b}{j}], \\ \cdots\cdots\cdots \\ ct^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^{t-\frac{(j-1)b}{j}}(t-s)^{\alpha-1}g_{j,j-1}(s)ds, & t\in(\frac{(j-1)b}{j},b], \end{cases}$$

$g_{j,1}:[0,\frac{b}{j}]\to\mathbb{R}^n$ is a continuous selection of $f(\cdot,y_\alpha^j(\cdot)):[0,\frac{b}{j}]\to\mathscr{A}_k(\mathbb{R}^n)$, and $g_{j,2}:[0,\frac{2b}{j}]\to\mathbb{R}^n$ is a continuous selection of $f(\cdot,y_\alpha^j(\cdot)):[0,\frac{2b}{j}]\to\mathscr{A}_k(\mathbb{R}^n)$ such that $g_{j,1}(\cdot)=g_{j,2}(\cdot)$ on $[0,\frac{b}{j}],\ldots,g_{j,j-2}:[0,\frac{(j-2)b}{j}]\to\mathbb{R}^n$ is a continuous selection of $f(\cdot,y_\alpha^j(\cdot)):[0,\frac{(j-2)b}{j}]\to\mathscr{A}_k(\mathbb{R}^n)$, and $g_{j,j-1}:[0,\frac{(j-1)b}{j}]\to\mathbb{R}^n$ is a continuous selection of $f(\cdot,y_\alpha^j(\cdot)):[0,\frac{(j-1)b}{j}]\to\mathscr{A}_k(\mathbb{R}^n)$ such that $g_{j,j-2}(\cdot)=g_{j,j-1}(\cdot)$ on $[0,\frac{(j-2)b}{j}]$. By Theorem 1, we can choose a continuous selection $g_j:[0,b]\to\mathbb{R}^n$ of $f(\cdot,y_\alpha^j(\cdot)):[0,b]\times\mathbb{R}^n\to\mathscr{A}_k(\mathbb{R}^n)$ such that $g_j(\cdot)=g_{j-1,j-2}(\cdot)$ on $[0,\frac{(j-2)b}{j}]$.

We have $\{g_j\}\in C([0,b],\mathbb{R}^n)$. Also, there exists a constant $M>0$ such that

$$|g_j(t)|\le M, \qquad t\in[0,b].$$

We observe that if $\tau_1,\ \tau_2 \in (0, \frac{b}{j}]$, then

$$|y^j_\alpha(\tau_2) - y^j_\alpha(\tau_1)| = 0.$$

If $J$ is an integer with $0 \le J < j$ and if $0 \le \tau_1 \le \frac{Jb}{j} \le \tau_2 \le b$, then

$$\begin{aligned}
|y^J_\alpha(\tau_2) - y^J_\alpha(\tau_1)| &\le \frac{\tau_2^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\tau_2 - \frac{Jb}{j}} |(\tau_2 - s)^{\alpha-1} g_j(s)| ds \\
&\le \frac{\tau_2^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\tau_2-\tau_1} (\tau_2 - s)^{\alpha-1} |g_j(s)| ds \\
&\le \frac{M\tau_2^{1-\alpha}}{\Gamma(\alpha+1)} (\tau_2^\alpha - \tau_1^\alpha),
\end{aligned}$$

and if $\frac{Jb}{j} < \tau_1 < \tau_2 < b$,

$$\begin{aligned}
|y^J_\alpha(\tau_2) - y^J_\alpha(\tau_1)| &\le \frac{|\tau_2^{1-\alpha} - \tau_1^{1-\alpha}|}{\Gamma(\alpha)} \int_0^{\tau_1} (\tau_2 - s)^{\alpha-1} |g_j(s)| ds + \frac{\tau_1^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\tau_1} [(\tau_1 - s)^{\alpha-1} - (\tau_2 - s)^{\alpha-1}] |g_j(s)| ds \\
&\quad + \frac{\tau_2^{1-\alpha}}{\Gamma(\alpha)} \int_{\tau_1 - \frac{Jb}{j}}^{\tau_2 - \frac{Jb}{j}} (\tau_2 - s)^{\alpha-1} |g_j(s)| ds \\
&\le \frac{M|\tau_2^{1-\alpha} - \tau_1^{1-\alpha}|}{\Gamma(\alpha)} \int_0^{\tau_1} (\tau_2 - s)^{\alpha-1} ds + \frac{M\tau_1^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\tau_1} [(\tau_1 - s)^{\alpha-1} - (\tau_2 - s)^{\alpha-1}] ds \\
&\quad + \frac{M\tau_2^{1-\alpha}}{\Gamma(\alpha)} \int_{\tau_1 - \frac{Jb}{j}}^{\tau_2 - \frac{Jb}{j}} \left( \frac{\tau_2 - s - \frac{Jb}{j}}{\tau_2 - s} \right)^{1-\alpha} \left( \tau_2 - s - \frac{Jb}{j} \right)^{\alpha-1} ds \\
&\le \frac{M|\tau_2^{1-\alpha} - \tau_1^{1-\alpha}|}{\Gamma(\alpha)} \int_0^{\tau_1} (\tau_2 - s)^{\alpha-1} ds + \frac{M\tau_1^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\tau_1} [(\tau_1 - s)^{\alpha-1} - (\tau_2 - s)^{\alpha-1}] ds \\
&\quad + \frac{M\tau_2^{1-\alpha}}{\Gamma(\alpha)} \int_{\tau_1 - \frac{Jb}{j}}^{\tau_2 - \frac{Jb}{j}} \left( \tau_2 - s - \frac{Jb}{j} \right)^{\alpha-1} ds \\
&\le \frac{M|\tau_2^{1-\alpha} - \tau_1^{1-\alpha}|}{\Gamma(\alpha+1)} \tau_2^\alpha + \frac{M\tau_1^{1-\alpha}}{\Gamma(\alpha+1)} |\tau_2 - \tau_1|^\alpha + \frac{M\tau_1^{1-\alpha}}{\Gamma(\alpha+1)} [\tau_1^\alpha - \tau_2^\alpha] + \frac{M\tau_2^{1-\alpha}}{\Gamma(\alpha+1)} |\tau_2 - \tau_1|^{\alpha-1} \\
&\le \frac{M|\tau_2^{1-\alpha} - \tau_1^{1-\alpha}|}{\Gamma(\alpha+1)} \tau_2^\alpha + \frac{M\tau_1^{1-\alpha}}{\Gamma(\alpha+1)} |\tau_2 - \tau_1|^\alpha + \frac{M\tau_2^{1-\alpha}}{\Gamma(\alpha+1)} |\tau_2 - \tau_1|^{\alpha-1}.
\end{aligned}$$

As $\tau_2 \to \tau_1$ the right-hand side of the above inequality tends to zero. Consequently the sequence $\{y^i_\alpha\}$ is equicontinous.

Next, note that

$$\begin{aligned}
|t^{1-\alpha} y_i(t)| &\le |c| + \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t - \frac{b}{j}} (t - s)^{\alpha-1} |g_j(s)| ds \\
&\le |c| + \frac{b^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t - \frac{b}{j}} (t - s)^{\alpha-1} |g_j(s)| ds \\
&\le |c| + \frac{Mb^{1-\alpha}}{\Gamma(\alpha)} \left[ -\left( \frac{b}{j} \right)^\alpha + t^\alpha \right] \\
&\le |c| + \frac{Mb}{\Gamma(\alpha+1)} := M_*.
\end{aligned}$$

Hence, the sequence $\{y^j_\alpha\}$ is uniformly bounded and equicontinuous. By the Arzelá-Ascoli theorem, there exists a subsequence, still denoted as $\{y^j_\alpha\}_{j=1}^{j=\infty}$, converging uniformly to some function $y$ in $C_*(([0,b], \mathbb{R}^n)$.

Let $K = [0,b] \times B(c, M)$, and

$$\begin{aligned}
\omega|_{f|_K}(\delta) = \sup \{ & d_{\mathscr{A}}(f(\tau_2, y_2), f(\tau_1, y_1)) : |(\tau_2, y_2) - (\tau_1, y_1)| \le \delta \\
& \text{where } (\tau_1, y_1),\ (\tau_2, y_2) \in K \}
\end{aligned}$$

be a modulus of continuity of $f$ restricted to $K$. For each $\varepsilon > 0$ there exists $\delta_1 > 0$ such that for every $|\tau_2 - \tau_1| \le \delta_1$,

$$\frac{M|\tau_2^{1-\alpha} - \tau_1^{1-\alpha}|}{\Gamma(\alpha+1)}\tau_2^{\alpha-1} + \frac{M\left(\tau_1^{1-\alpha} + \tau_2^{1-\alpha}\right)}{\Gamma(\alpha+1)}|\tau_2 - \tau_1|^{\alpha} \le \varepsilon,$$

and

$$\frac{M\tau_2^{1-\alpha}}{\Gamma(\alpha+1)}(\tau_2^{\alpha} - \tau_1^{\alpha}) \le \varepsilon.$$

Hence, for each $j \in \mathbb{N}$, we have

$$|\tau_2^{1-\alpha}y_j(\tau_2) - \tau_1^{1-\alpha}y_j(\tau_1)| \le \varepsilon, \text{ for all } \tau_1, \tau_2 \in [0,b] \text{ and } |\tau_2 - \tau_1| \le \delta_1.$$

This implies

$$\begin{aligned}
\omega|_{f(\cdot,y_\alpha^j(\cdot))}(\delta_2) &= \sup\left\{d_{\mathscr{A}}(f(\tau_2,y_\alpha^j(\tau_2)), f(\tau_1,y_\alpha^j(\tau_1))) : |\tau_2 - \tau_1| \le \delta_1, \text{ and } \tau_1, \tau_2 \in [0,b]\right\} \\
&\le \sup\{d_{\mathscr{A}}(f(\tau_2,y_1), f(\tau_1,y_2)) : |\tau_2 - \tau_1| \le \delta_2, |y_1 - y_2| \le \delta_2, \text{ and } (\tau_2,y_2), (\tau_1,y_1) \in K\} \\
&\le \omega|_{f_K}(\delta_2),
\end{aligned}$$

where $\delta_2 = \max(\delta_1, \varepsilon)$. It clear that $f(\cdot,y_\alpha^j(\cdot)) - [[g_j(.)]] : [0,b] \to \mathscr{A}_{k-1}(\mathbb{R}^n)$ is a continuous multiple-valued function, so there exist continuous functions $h_1^j, \ldots, h_{k-1}^j : [0,b] \to \mathbb{R}^n$ such that

$$f(\cdot,y_\alpha^j(\cdot)) = [[g_i(\cdot)]] + \sum_{i=1}^{k-1}[[h_i^j(\cdot)]].$$

Then,

$$|g_i| \le M \;\text{ for each }\; n \in \mathbb{N}$$

and

$$\omega|_{g_i} \le \omega|_{f_K}(\delta_2) \;\text{ for every }\; n \in \mathbb{N}.$$

Consequently, $\{g_i\}_{i=1}^{i=\infty}$ is bounded and equicontinuous. From the Arzelá-Ascoli theorem (Theorem 3), we can conclude that $\{g_i\}_{i=1}^{i=\infty}$ is compact in $C([0,b],\mathbb{R}^n)$. Hence, there exists a subsequence, again denoted by $\{g_i\}_{i=1}^{i=\infty}$, converging uniformly to $g$.

Now set

$$z(t) = ct^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds := y(t), \;\; t \in (0,b].$$

Then we see that

$$\begin{aligned}
|t^{1-\alpha}y_j(t) - t^{1-\alpha}z(t)| &\le \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|g_j(s) - g(s)|ds \\
&\le \frac{t}{\Gamma(\alpha+1)}\|g_j - g\|_\infty \\
&\le \frac{b}{\Gamma(\alpha+1)}\|g_j - g\|_\infty \to 0 \text{ as } j \to \infty,
\end{aligned}$$

and so $\{y_j(t)\} \to z(t)$.

Set

$$y_\alpha(t) = \begin{cases} c, & t = 0, \\ t^{1-\alpha}y(t), & t \in (0,b]. \end{cases}$$

By Lemma 2, we conclude that $g$ is a continuous selection of $f(\cdot,y_\alpha(\cdot))$ on $[0,b]$.

Next, we want to show that the set

$$S = \{y \in C_*([0,b],\mathbb{R}^n) \mid y \text{ is a solution of } (1)\}$$

is compact. Let $\{y_i\}_{i=1}^{i=\infty}$ be a sequence in $S$. Then,

$$y_i(t) = ct^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g_i(s)ds, \; t \in [0,b],$$

where $g_i(\cdot)$ is a continuous selection of $f(\cdot,y_i(\cdot))$. As in the first part of the proof, we conclude there exists a subsequence, still denoted as $\{y_i\}_{i=1}^{i=\infty}$, that converges to a continuous function $y:[0,b]\to\mathbb{R}^n$. It clear that $f(\cdot,y_\alpha^i(\cdot))-[[g_i(\cdot)]]:[0,b]\to\mathscr{A}_{k-1}(\mathbb{R}^n)$ is a continuous multiple-valued function. Then, there exist continuous functions $h_1^j,\dots,h_{k-1}^j:[0,b]\to\mathbb{R}^n$ such that

$$f(\cdot,y_\alpha^j(\cdot))=[[g_i(\cdot)]]+\sum_{i=1}^{k-1}[[h_i^j(\cdot)]].$$

Again as in the first part of the proof, we can show that $\{g_i\}$ is bounded and equicontinuous. Then from the Arzelá-Ascoli theorem, Theorem 3, we have that $\{g_i : i\in\mathbb{N}\}$ is compact in $C_*([0,b],\mathbb{R}^n)$, and so there is a subsequence, again denoted by $\{g_i\}_{i\in\mathbb{N}}$, converging uniformly to $g$. Setting

$$z(t)=ct^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s)ds:=y(t),\ \ t\in(0,b],$$

we have

$$\|y_i-z\|_*\le\frac{b}{\Gamma(\alpha+1)}\|g_i-g\|_*\to 0 \text{ as } i\to\infty.$$

By Lemma 2, we conclude that $g$ is a continuous selection of $f(\cdot,y_\alpha(\cdot))$ on $[0,b]$. This completes the proof of the theorem.

*Remark.* We can replace the condition (2) by

$$d_{\mathscr{A}}(f(t,x),k[[0]])\le M_1|x|+M_2,\ \ \text{for all}\ \ x\in\mathbb{R},\ t\in[0,b] \tag{3}$$

and problem (1) still has at least one solution.

*Proof.* Let $\{y_j\}$ be a sequence defined as in the proof of Theorem 4, i.e., for each $j\in\mathbb{N}$ we have

$$y_j(t)=\begin{cases}ct^{\alpha-1}, & t\in(0,\frac{b}{j}],\\ ct^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^{t-\frac{b}{j}}(t-s)^{\alpha-1}g_j(s)ds, & t\in(\frac{b}{j},b],\end{cases}$$

where $g_j:[0,b]\to\mathbb{R}^n$ is a continuous selection of $f(\cdot,y_\alpha^j(\cdot)):[0,b]\times\mathbb{R}\to\mathscr{A}_k(\mathbb{R}^n)$. We have $\{g_j\}\in C([0,b],\mathbb{R}^n)$, and from (3),

$$|g_j(t)|\le M_1t^{1-\alpha}|y(t)|+M_2 \text{ for each } j\in\mathbb{N}.$$

Then,

$$\begin{aligned}|t^{1-\alpha}y_i(t)|&\le|c|+\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|g_i(s)|ds\\&\le|a|+\frac{b^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}(M_1s^{1-\alpha}|y_i(s)|+M_2)ds,\end{aligned}$$

and from Gronwall's Lemma 3, there exists $M>0$ such that

$$\|y_i\|_*\le M\ \ \text{for each}\ j\in\mathbb{N}.$$

Finally, as in Theorem 4, we can prove that $\{y_\alpha^i\}$ converges to a solution of problem (1). This proves the remark.

We conclude this paper by proving a Peano type existence result for (1).

**Theorem 5.** *Let $\Omega\subset\mathbb{R}\times\mathbb{R}^n$ be an open set, $f:\Omega\to\mathscr{A}_k(\mathbb{R}^n)$ be a continuous function, and $(0,c)\in\Omega$. Then there exist $h>0$, $\eta>0$, and $y:(0,h]\to\mathbb{R}^n$ such that*

$$t^{1-\alpha}y(t)\in\overline{B}(c,\eta),\ \ \textit{for all}\ t\in[0,h],$$

*and $y$ is a solution of problem (1).*

*Proof.* Let $\varepsilon > 0$ be given by the continuity of $f$, if there exists $\eta > 0$ such that

$$|t| \leq \eta \quad \text{and} \quad \|y - c\| \leq \eta$$

then

$$d_{\mathscr{A}}(f(t,y), f(0,c)) \leq \varepsilon.$$

Since the set $G = [-\eta, \eta] \times \overline{B}(c, \eta) \subset \Omega$ is compact, the continuity of f implies there exists a constant $M_1 > 0$ such that

$$d_{\mathscr{A}}(f(t,x), k[[0]]) < M_1, \quad \text{for all} \quad (t,x) \in G.$$

Consider the Cauchy problem

$$\begin{cases} D^{\alpha}_{0+}y(t) \in \{f_1(t, t^{1-\alpha}y), \dots f_k(t, t^{1-\alpha}y)\}, & t \in [0,h], \\ \lim\limits_{t \to 0^+} t^{1-\alpha}y(t) = c, \end{cases} \tag{4}$$

where $h \leq \min\left(\eta, \left(\frac{\eta\Gamma(\alpha+1)}{M_1}\right)^{\frac{1}{\alpha}}\right)$. By the same method used in the proof of Theorem 4, we define the sequences

$$y_i(t) = \begin{cases} ct^{\alpha-1}, & t \in (0, \frac{h}{i}], \\ ct^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \displaystyle\int_0^{t-\frac{h}{i}} (t-s)^{\alpha-1} g_i(s) ds, & t \in (\dfrac{h}{i}, h], \end{cases}$$

where $i \in \mathbb{N}$ and $g_i : [0,h] \to \mathbb{R}^n$ is a continuous selection of $f(\cdot, y^i_\alpha(\cdot)) : [0,h] \times \mathbb{R} \to \mathscr{A}_k(\mathbb{R}^n)$. Then,

$$|t^{1-\alpha}y_i(t) - c| \leq \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g_i(s)| ds \leq \frac{M_1 h^{\alpha}}{\Gamma(\alpha+1)} \leq \eta,$$

so

$$t^{1-\alpha}y_i(t) \in \overline{B}(c, \eta), \quad \text{for all } t \in [0,h].$$

As in the proof of Theorem 4, we can show that $\{y_i\}$ and $\{g_i\}$ are relatively compact in $C_*([0,h], \overline{B}(c,\eta))$ and $C([0,h], \overline{B}(c,\eta))$, respectively. Thus, there exist subsequences of $\{y_i\}$ and $\{g_i\}$ converging uniformly to $y$ and $g$, respectively, and from Lemma 2 we conclude that

$$y(t) = ct^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^{t-\frac{h}{i}} (t-s)^{\alpha-1} g(s) ds, \text{ for } t \in (0,h],$$

where $g$ a continuous selection of $f(\cdot, y_\alpha(\cdot))$ and $y$ is a solution of problem (1).

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