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New Properties for Conformable Fractional Derivative and Applications

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Abstract: The fractional derivative (FD) has recently captured the minds of scientists. The most common are Riemann-Liouville (RL) and Caputo (C). These fractional derivatives have been used to successfully model many real-world problems due to their physical properties. In 2014, Khalil et al introduced a new definition of an FD called the conformable FD (CFD). In this work, we introduce new properties and theorems related to this new derivative, such as the CFD of the reciprocal function, power of function, exponential of function, and the ω-Leibniz integral rule used to solve fractional differential equations as in the applications section.

Keywords: FD, Fractional integral, ω-Leibniz integral rule, exponential of function, reciprocal function.

1 Introduction

While the concept of FD dates back to L'Hospital in the 17th century, extensive investigations into FD's have been conducted in recent centuries. Many researchers have proposed integral-based definitions for FDs, with the RL and C definitions being the most widely adopted. For comprehensive discussions on these definitions and their characteristics, interested readers are directed to [1,2,3,4]. The RL fractional derivative is defined as follows:

$$D_{\kappa}^{\omega}\Theta(\kappa)=\frac{1}{\Gamma(n-\omega)}(\frac{d}{d\kappa})^{n}\int_{0}^{\kappa}\frac{\Theta(t)}{(\kappa-t)^{\omega+1-n}}dt,$$

where $n-1<\omega\leq n$. The C fractional derivative is defined by:

$$D_{\kappa}^{\omega}\Theta(\kappa)=\frac{1}{\Gamma(n-\omega)}\int_{0}^{\kappa}\frac{1}{(\kappa-t)^{\omega+1-n}}(\frac{d}{d\kappa})^{n}\Theta(t)dt,$$

where $n-1<\omega\leq n\in\mathbb{N}$.

In 2014, Khalil et al. introduced a novel definition of FD's and partial integrals [5], offering a specific form akin to conventional derivatives. Subsequently, in 2015, Abdeljawad derived a new theorem encompassing Taylor Power series representation and Laplace transformation for certain functions, alongside providing formulas for partial integration by parts, the chain rule, and Gronwall inequality [6]. In the same year, Atangana et al. presented novel properties of the CFD [7].

In 2018, Hashemi developed invariant subspace methods to obtain exact solutions for various conformable differential equations, extending this theory to coupled systems of conformable differential equations [8]. Touchent et al., in 2019, investigated new solutions to conformable Boiti-Leon Pempinelli equations, employing an expansion technique based on the Sinh-Gordon equation, resulting in solutions expressed in trigonometric, complex, and hyperbolic functions [9].

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Additionally, Abdeljawad et al. studied different types of conformable fractional logistic models [10], while Birgani et al. discussed, improved, and supplemented recent results concerning the CFD [11].

In 2019, Balci et al. delved into the dynamical behavior of a conformable fractional tumor model [12]. Touchent et al. in 2020, employed three efficient integration algorithms to extract solutions for the optical soliton space-time fractional nonlinear equation, crucial for understanding microtubule dynamics in cellular processes biology [13]. Similarly, Xie et al. in 2020 examined a continuous grey model utilizing CFD [14]. Chaudhary et al., also in 2020, investigated the Fractional convection-dispersion equation using the CFD [15]. In 2021, Ulutas et al. explored traveling wave and optical soliton solutions of the Wick-type stochastic NLSE employing CFD [16]. Al-Zhour, in 2022, scrutinized the controllability and observability behaviors of a non-homogeneous conformable fractional dynamical system [17]. Meanwhile, Sadek et al., in 2022, delved into the controllability, observability, and fractional linear-quadratic problem for fractional linear systems utilizing CFD, alongside showcasing applications [18]. Additionally, Teodoro et al., in 2019, revisited the definitions of fractional derivatives and related factors [19]. Numerous works employ such derivations; for a comprehensive overview, refer to [20,21,22,23,24] and the references therein.

Definition 1.[5] *Let $0<\omega\leq 1$ and $\Theta:[0,\infty)\times[0,\infty)\to\mathbb{R}$. The CFD of a function Θ of order ω of κ we denote ${}_{\kappa}T_{\omega}\Theta$ defined by:*

$$ {}_{\kappa}T_{\omega}\Theta(\kappa,t)=\lim_{\varepsilon\to 0}\frac{\Theta(\kappa+\varepsilon\kappa^{1-\omega},t)-\Theta(\kappa,t)}{\varepsilon}, \tag{1}$$

for all $\kappa>0$. If $\lim_{\kappa\to 0^+}{}_{\kappa}T_{\omega}\Theta(\kappa,t)$ exists, then define ${}_{\kappa}T_{\omega}\Theta(0,t)=\lim_{\kappa\to 0^+}{}_{\kappa}T_{\omega}\Theta(\kappa,t)$. Every real function that satisfies the equation (1) and the limit exists, is called the ω-differentiable function.

If Θ is differentiable in κ, then

$$ {}_{\kappa}T_{\omega}\Theta(\kappa,t)=\kappa^{1-\omega}\frac{\partial}{\partial\kappa}\Theta(\kappa,t).$$

If $\omega=1$, we have

$$ {}_{\kappa}T_{1}=\frac{\partial}{\partial\kappa}.$$

Theorem 1.[5] *Let $\omega\in(0,1]$ and Θ,g be ω-differentiable at a point $\kappa>0$. We have*

1. ${}_{\kappa}T_{\omega}(a\Theta+bg)=a\,{}_{\kappa}T_{\omega}\Theta+b\,{}_{\kappa}T_{\omega}g$, *for all* $a,b\in\mathbb{R}$.
2. ${}_{\kappa}T_{\omega}(\kappa^{p})=p\kappa^{p-\omega}$.
3. ${}_{\kappa}T_{\omega}(\Theta(\kappa))=0$, *for all* $\Theta(\kappa)=\lambda$.
4. ${}_{\kappa}T_{\omega}(\Theta g)={}_{\kappa}T_{\omega}\Theta g+\Theta\,{}_{\kappa}T_{\omega}g$.
5. ${}_{\kappa}T_{\omega}(\Theta/g)=\frac{{}_{\kappa}T_{\omega}g\Theta-\Theta\,{}_{\kappa}T_{\omega}g}{g^{2}}$.

CFD of certain functions

Lemma 1.[5]

1. ${}_{\kappa}T_{\omega}e^{c\kappa}=c\kappa^{1-\omega}e^{c\kappa}, c\in\mathbb{R}$.
2. ${}_{\kappa}T_{\omega}\sin(b\kappa)=b\kappa^{1-\omega}\cos b\kappa, b\in\mathbb{R}$.
3. ${}_{\kappa}T_{\omega}\cos(b\kappa)=-b\kappa^{1-\omega}\sin b\kappa, b\in\mathbb{R}$.
4. ${}_{\kappa}T_{\omega}\frac{\kappa^{\omega}}{\omega}=1$.
5. ${}_{\kappa}T_{\omega}\sin(\frac{1}{\omega}\kappa^{\omega})=\cos(\frac{1}{\omega}\kappa^{\omega})$.
6. ${}_{\kappa}T_{\omega}\cos(\frac{1}{\omega}\kappa^{\omega})=-\sin(\frac{1}{\omega}\kappa^{\omega})$.
7. ${}_{\kappa}T_{\omega}e^{\lambda\frac{1}{\omega}\kappa^{\omega}}=\lambda e^{\lambda\frac{1}{\omega}\kappa^{\omega}}$.

Theorem 2.[5] *Let $\omega\in(0,1]$ and the function $\Theta:[0,\infty)\longrightarrow\mathbb{R}$ ω-differentiable at $\kappa_0>0$, so Θ is continuous at κ_0.*

Notation.

$$ I_{\omega}\Theta(\kappa)=\int_{0}^{\kappa}\Theta(s)d\omega(s)=\int_{0}^{\kappa}s^{\omega-1}\Theta(s)ds.$$

The operator I_{ω} is called conformable left fractional integrals of order $0<\omega\leq 1$.

Lemma 2.[5] *Let $\Theta:[0,\infty)\to\mathbb{R}$ is continuous and $0<\omega\leq 1$. Then, for all $\kappa>0$ we have*

$$ {}_{\kappa}T_{\omega}I_{\omega}\Theta(\kappa)=\Theta(\kappa).$$

2 The CFD for reciprocal function

Theorem 3.[6] *Let $0 < \omega \leq 1$, $\Theta, g : [0,\infty) \to \mathbb{R}$ be ω-differentiable functions, and $h(\kappa) = \Theta(g(\kappa))$. So $h(\kappa)$ is ω-differentiable and for all κ with $\kappa \neq 0$ and $g(\kappa) \neq 0$, we have*

$$ {}_{\kappa}T_{\omega}h(\kappa) = {}_{\kappa}T_{\omega}\Theta(g(\kappa)) \cdot {}_{\kappa}T_{\omega}g(\kappa) \cdot g(\kappa)^{\omega-1}. \tag{2} $$

If $\kappa = 0$ we have

$$ {}_{\kappa}T_{\omega}h(a) = \lim_{\kappa \to 0^+} {}_{\kappa}T_{\omega}\Theta(g(\kappa)) \cdot {}_{\kappa}T_{\omega}g(\kappa) \cdot g(\kappa)^{\omega-1}. $$

Corollary 1.*Let $g : [0,\infty) \to \mathbb{R}$ be ω-differentiable functions, where $0 < \omega \leq 1$. So $e^{g(\kappa)}$ is ω-differentiable and for all κ with $\kappa \neq 0$, we have*

$$ {}_{\kappa}T_{\omega}e^{g(\kappa)} = e^{g(\kappa)} \cdot {}_{\kappa}T_{\omega}g(\kappa). $$

If $\kappa = 0$ we have

$$ {}_{\kappa}T_{\omega}e^{g(\kappa)} = \lim_{\kappa \to 0^+} e^{g(\kappa)} \cdot {}_{\kappa}T_{\omega}g(\kappa). $$

*Proof.*From (2) we have

$$ {}_{\kappa}T_{\omega}e^{g(\kappa)} = ({}_{\kappa}T_{\omega}e^{\kappa})(g(\kappa)) \cdot {}_{\kappa}T_{\omega}g(\kappa) \cdot g(\kappa)^{\omega-1}, $$

From 1 in Lemma 1, we have

$$ \begin{aligned} {}_{\kappa}T_{\omega}e^{g(\kappa)} &= (\kappa^{1-\omega}e^{\kappa})(g(\kappa)) \cdot {}_{\kappa}T_{\omega}g(\kappa) \cdot g(\kappa)^{\omega-1} \\ &= g(\kappa)^{1-\omega}e^{g(\kappa)} \cdot {}_{\kappa}T_{\omega}g(\kappa) \cdot g(\kappa)^{\omega-1} \\ &= e^{g(\kappa)} \cdot {}_{\kappa}T_{\omega}g(\kappa). \end{aligned} $$

Corollary 2.*Let $0 < \omega \leq 1$ and $g : [0,\infty) \to \mathbb{R}$ be ω-differentiable functions. Then $g(\kappa)^n$ is ω-differentiable and for all κ with $\kappa \neq 0$ we have*

$$ {}_{\kappa}T_{\omega}g(\kappa)^n = n.g(\kappa)^{n-1}.{}_{\kappa}T_{\omega}g(\kappa). $$

If $\kappa = 0$ we have

$$ {}_{\kappa}T_{\omega}g(\kappa)^n = \lim_{\kappa \to 0^+} g(\kappa)^{n-1}.n.{}_{\kappa}T_{\omega}g(\kappa). $$

*Proof.*From (2) we have

$$ {}_{\kappa}T_{\omega}e^{g(\kappa)} = ({}_{\kappa}T_{\omega}e^{\kappa})(g(\kappa)) \cdot {}_{\kappa}T_{\omega}g(\kappa) \cdot g(\kappa)^{\omega-1}, $$

From 1 in Lemma 1, we have

$$ \begin{aligned} {}_{\kappa}T_{\omega}e^{g(\kappa)} &= (\kappa^{1-\omega}e^{\kappa})(g(\kappa)) \cdot {}_{\kappa}T_{\omega}g(\kappa) \cdot g(\kappa)^{\omega-1} \\ &= g(\kappa)^{1-\omega}e^{g(\kappa)} \cdot {}_{\kappa}T_{\omega}g(\kappa) \cdot g(\kappa)^{\omega-1} \\ &= e^{g(\kappa)} \cdot {}_{\kappa}T_{\omega}g(\kappa). \end{aligned} $$

Theorem 4.*Let $\Theta : [a,b] \to [a',b']$ ω-differentiable and bijective from which we denote by $\Theta^{-1} : [a',b'] \to [a,b]$ the reciprocal bijection with $a,a' \geq 0$. If ${}_{\kappa}T^{a}_{\omega}\Theta(y) \neq 0$, $\forall y \in [a,b]$, then Θ^{-1} is ω-differentiable and we have for all $\kappa \in [a',b']$:*

$$ {}_{\kappa}T_{\omega}\Theta^{-1}(\kappa) = \frac{\kappa^{1-\omega}(\Theta^{-1}(\kappa))^{1-\omega}}{{}_{\kappa}T_{\omega}\Theta(\Theta^{-1}(\kappa))}. $$

*Proof.*We have

$$ {}_{\kappa}T_{\omega}(\Theta(\Theta^{-1}(\kappa))) = {}_{\kappa}T_{\omega}\kappa, $$

from Theorem 3

$$ {}_{\kappa}T_{\omega}\Theta(\Theta^{-1}(\kappa)) \cdot {}_{\kappa}T_{\omega}\Theta^{-1}(\kappa) \cdot (\Theta^{-1}(\kappa))^{\omega-1} = \kappa^{1-\omega}, $$

so

$$ {}_{\kappa}T_{\omega}\Theta^{-1}(\kappa) = \frac{\kappa^{1-\omega}(\Theta^{-1}(\kappa))^{1-\omega}}{{}_{\kappa}T_{\omega}\Theta(\Theta^{-1}(\kappa))}. $$

And ${}_{\kappa}T_{\omega}\Theta^{-1}(0) = 0$.

*Example 1.*Let $\Theta : [0,+\infty) \to [1,+\infty)$ be the function defined by $\Theta(\kappa) = e^{\kappa}$. Let's study Θ in detail. First of all:

1.Θ is ω-differentiable. In particular, Θ is continuous.
2.Θ is strictly increasing.
3.Θ is a bijective.
4.${}_{\kappa}T_{\omega}\Theta(\kappa) = \kappa^{1-\omega}e^{\kappa} \neq 0$, $\kappa \in \mathbb{R} - \{0\}$.

By the above Theorem 4, Θ^{-1} is ω-differentiable and

$$\begin{aligned} {}_{\kappa}T_{\omega}\Theta^{-1}(\kappa) &= \frac{\kappa^{1-\omega}(\Theta^{-1}(\kappa))^{1-\omega}}{(\Theta^{-1}(\kappa))^{1-\omega}e^{\Theta^{-1}(\kappa)}} \\ &= \frac{\kappa^{1-\omega}(\ln(\kappa))^{1-\omega}}{(\ln(\kappa))^{1-\omega}e^{\ln(\kappa)}} \\ &= \frac{1}{\kappa^{\omega}}, \end{aligned}$$

and we have $\Theta^{-1}(\kappa) = \ln(\kappa)$, so

$$\begin{aligned} {}_{\kappa}T_{\omega}(\Theta^{-1}(\kappa)) &= {}_{\kappa}T_{\omega}(\ln(\kappa)) \\ &= \lim_{\varepsilon\to 0} \frac{\ln(\kappa+\varepsilon\kappa^{1-\omega}) - \ln(\kappa)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{\ln(\frac{\kappa+\varepsilon\kappa^{1-\omega}}{\kappa})}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{\ln(1+\varepsilon\kappa^{-\omega})}{\varepsilon\kappa^{-\omega}}\kappa^{-\omega} \\ &= \kappa^{-\omega}. \end{aligned}$$

3 CFD of functions defined as integrals

There are two main types of functions defined as integral. These functions are often found in analysis and in mathematical physics.

Type I:The functions of the form

$$G(\kappa) = \int_{u(\kappa)}^{v(\kappa)} \Theta(t)d\omega(t),$$

where κ is in the terminals.

Type II:The functions of the form

$$H(\kappa) = \int_{u(\kappa)}^{v(\kappa)} \Theta(\kappa,s)d\omega(s).$$

3.1 CFD for type I

A slightly different category of integrals is when the bounds are the parameters of the function:

$$G(\kappa) = \int_{u(\kappa)}^{v(\kappa)} \Theta(t)d\omega(t),$$

where u, v are functions of κ.

Theorem 5.*Let Θ be a continuous function over an interval $[a,b]$ of values in $\mathbb{R}$ and $v : I \to [a,b]$ of class functions $\mathscr{C}^1$, with $a \geq 0$. Then the function H defined on the interval I by*

$$H(\kappa) = \int_{0}^{v(\kappa)} \Theta(t)d\omega(t), \tag{3}$$

is of class $\mathscr{C}^1$ and

$${}_{\kappa}T_{\omega}H(\kappa) = {}_{\kappa}T_{\omega}v(\kappa)v(\kappa)^{\omega-1}\Theta(v(\kappa)).$$

*Proof.*This function H is composed of two functions:

$$H(\kappa) = F(v(\kappa)) = (F \circ v)(\kappa),$$

where F is

$$F(\kappa) = \int_0^{\kappa} \Theta(t) d\omega(t),$$

As F and v are of class $\mathscr{C}^1$ then H is of class $\mathscr{C}^1$ and by the derivative formula of a composition :

$${}_{\kappa}T_{\omega}H(\kappa) = {}_{\kappa}T_{\omega}v(\kappa)v(\kappa)^{\omega-1}{}_{\kappa}T_{\omega}F(v(\kappa)).$$

But since ${}_{\kappa}T_{\omega}F(\kappa) = \Theta(\kappa)$ then

$${}_{\kappa}T_{\omega}H(\kappa) = {}_{\kappa}T_{\omega}v(\kappa)v(\kappa)^{\omega-1}\Theta(v(\kappa)).$$

Theorem 6.*Let Θ be a continuous function over an interval $[a,b]$ of values in $\mathbb{R}$. Let I be an interval of $\mathbb{R}$ et $u, v : I \to [a,b]$ of class functions $\mathscr{C}^1$. Then the function G defined on the interval I by*

$$G(\kappa) = \int_{u(\kappa)}^{v(\kappa)} \Theta(t) d\omega(t),$$

is of class $\mathscr{C}^1$ and

$${}_{\kappa}T_{\omega}G(\kappa) = {}_{\kappa}T_{\omega}v(\kappa)v(\kappa)^{\omega-1}\Theta(v(\kappa)) - {}_{\kappa}T_{\omega}u(\kappa)u(\kappa)^{\omega-1}\Theta(u(\kappa)).$$

*Proof.*We have

$$\begin{aligned} G(\kappa) &= \int_{u(\kappa)}^{v(\kappa)} \Theta(t) d\omega(t) \\ &= \int_{u(\kappa)}^{0} \Theta(t) d\omega(t) + \int_0^{v(\kappa)} \Theta(t) d\omega(t) \\ &= -\int_0^{u(\kappa)} \Theta(t) d\omega(t) + \int_0^{v(\kappa)} \Theta(t) d\omega(t) \\ &= K(\kappa) - L(\kappa), \end{aligned}$$

where

$$\begin{cases} K(\kappa) = \int_0^{v(\kappa)} \Theta(t) d\omega(t), \\ L(\kappa) = \int_0^{u(\kappa)} \Theta(t) d\omega(t), \end{cases}$$

from (3), so

$${}_{\kappa}T_{\omega}G(\kappa) = {}_{\kappa}T_{\omega}K(\kappa) - {}_{\kappa}T_{\omega}L(\kappa),$$

then

$${}_{\kappa}T_{\omega}G(\kappa) = {}_{\kappa}T_{\omega}v(\kappa)v(\kappa)^{\omega-1}f(v(\kappa)) - {}_{\kappa}T_{\omega}u(\kappa)u(\kappa)^{\omega-1}\Theta(u(\kappa)).$$

*Example 2.*Let us calculate the CFD of

$$G(\kappa) = \int_{\kappa}^{\kappa^2} \frac{1}{\ln t} d\omega(t),$$

for $\kappa > 1$. To apply Theorem 6, we restrict ourselves to an interval $[a,b]$ such that, for κ fixed, $\kappa \in [a,b] \subset]1, +\infty[$. With $\Theta(t) = \frac{1}{\ln t}, u(\kappa) = \kappa, v(\kappa) = \kappa^2$, we have:

$$\begin{aligned} {}_{\kappa}T_{\omega}G(\kappa) &= {}_{\kappa}T_{\omega}v(\kappa) \cdot \Theta(v(\kappa)) - {}_{\kappa}T_{\omega}u(\kappa) \cdot \Theta(u(\kappa)) \\ &= 2\kappa^{2-\omega} \frac{1}{\ln(\kappa^2)} - \kappa^{1-\omega} \frac{1}{\ln \kappa} \\ &= \frac{\kappa^{2-\omega} - \kappa^{1-\omega}}{\ln \kappa}. \end{aligned}$$

3.2 CFD for type II

Definition 2. *The ω-Leibniz integral rule is:*

$$H(t) = \int_{u(t)}^{v(t)} \Theta(t,s)d\omega(s). \tag{4}$$

It is sometimes known as differentiation under the integral sign.

Theorem 7. *Let Θ be a continuous function over an interval $[0,b]\times[0,b]$ of values in $\mathbb{R}$. Then the function M defined by*

$$M(\kappa) = \int_0^b \Theta(\kappa,t)d\omega(t), \tag{5}$$

is of class $\mathscr{C}^1$ and

$$_{\kappa}T_{\omega}M(\kappa) = \int_0^b {_{\kappa}T_{\omega}}\Theta(\kappa,t)d\omega(t).$$

Proof. We have

$$\begin{aligned}\int_{\kappa}^{\kappa+\varepsilon\kappa^{1-\omega}} \int_0^b {_{\kappa}T_{\omega}}\Theta(\kappa,t)d\omega(t)d\omega(\kappa) &= \int_0^b \int_{\kappa}^{\kappa+\varepsilon\kappa^{1-\omega}} {_{\kappa}T_{\omega}}\Theta(\kappa,t)d\omega(\kappa)d\omega(t)\\ &= \int_0^b \Big(\Theta(\kappa+\varepsilon\kappa^{1-\omega},t) - \Theta(\kappa,t)\Big)d\omega(t)\\ &= \int_0^b \Theta(\kappa+\varepsilon\kappa^{1-\omega},t)d\omega(t) - \int_0^b \Theta(\kappa,t)d\omega(t),\end{aligned}$$

As a result

$$\begin{aligned}\frac{\int_0^b \Theta(\kappa+\varepsilon\kappa^{1-\omega},t)d\omega(t) - \int_0^b \Theta(\kappa,t)d\omega(t)}{\varepsilon} &= \frac{1}{\varepsilon}\int_{\kappa}^{\kappa+\varepsilon\kappa^{1-\omega}} \int_0^b {_{\kappa}T_{\omega}}\Theta(\kappa,t)d\omega(t)d\omega(\kappa)\\ &= \frac{F(\kappa+\varepsilon\kappa^{1-\omega}) - F(\kappa)}{\varepsilon},\end{aligned}$$

where we defined

$$F(u) = \int_0^u \int_0^b {_{\kappa}T_{\omega}}\Theta(\kappa,t)d\omega(t)d\omega(\kappa),$$

since F is ω-differentiable, so we can take the limit where ε approaches zero. For the left side, this limit is:

$$_{\kappa}T_{\omega}F(\kappa) = \int_a^b {_{\kappa}T_{\omega}}\Theta(\kappa,t)d\omega(t).$$

For the left side member, we get:

$$_{\kappa}T_{\omega}\int_0^b \Theta(\kappa,t)d\omega(t),$$

so

$$_{\kappa}T_{\omega}M(\kappa) = \int_0^b {_{\kappa}T_{\omega}}\Theta(\kappa,t)d\omega(t).$$

Example 3. Let's study

$$F(\kappa) = \int_0^1 \frac{1}{\kappa^2+t^2}d\omega(t),$$

for $\kappa \in]0,+\infty[$. Let's pose

$$\Theta(\kappa,t) = \frac{1}{\kappa^2+t^2}.$$

Then:

–Θ is continuous on $]0,+\infty[\times[0,1]$,

$-_{\kappa}T_{\omega}f(\kappa,t) = \frac{-2\kappa^{2-\omega}}{(\kappa^2+t^2)^2}$ is continuous on $]0,+\infty[\times[0,1]$.

So we will have

$$_{\kappa}T_{\omega}F(\kappa) = \int_0^1 \frac{-2\kappa^{2-\omega}}{(\kappa^2+t^2)^2} d\omega(t).$$

Theorem 8.*Let Θ be a continuous function over an interval $[0,b]$ of values in $\mathbb{R}$ and $v: I \to [0,b]$ of class functions $\mathscr{C}^1$. Then the function H defined on the interval I by*

$$H(\kappa) = \int_0^{v(\kappa)} \Theta(\kappa,t) d\omega(t), \tag{6}$$

is of class $\mathscr{C}^1$ and

$$_{\kappa}T_{\omega}H(\kappa) = \Theta(\kappa, v(\kappa))_{\kappa}T_{\omega}v(\kappa)v(\kappa)^{\omega-1} + \int_0^{v(\kappa)} {}_{\kappa}T_{\omega}f(\kappa,s)d\omega(s).$$

*Proof.*Let

$$H(\kappa) = M(\kappa, v(\kappa)),$$

where

$$M(m,y) = \int_0^y f(m,t)d\omega(t),$$

so

$$\begin{aligned}
_{\kappa}T_{\omega}H(\kappa) &= \lim_{\varepsilon\to 0} \frac{H(\kappa+\varepsilon\kappa^{1-\omega}) - H(\kappa)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{\int_0^{v(\kappa+\varepsilon\kappa^{1-\omega})} \Theta(\kappa+\varepsilon\kappa^{1-\omega},t)d\omega(t) - \int_0^{v(\kappa)} \Theta(\kappa,t)d\omega(t)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{\int_{v(\kappa)}^{v(\kappa+\varepsilon\kappa^{1-\omega})} \Theta(\kappa+\varepsilon\kappa^{1-\omega},t)d\omega(t)}{\varepsilon} + \int_0^{v(\kappa)} {}_{\kappa}T_{\omega}\Theta(\kappa,t)d\omega(t),
\end{aligned}$$

we but $u = \kappa + \varepsilon\kappa^{1-\omega}$ so

$$\begin{aligned}
\lim_{\varepsilon\to 0} \frac{\int_{v(\kappa)}^{v(\kappa+\varepsilon\kappa^{1-\omega})} \Theta(\kappa+\varepsilon\kappa^{1-\omega},t)d\omega(t)}{\varepsilon} &= \lim_{u\to\kappa} \frac{\kappa^{1-\omega}}{u-\kappa} \int_{v(\kappa)}^{v(u)} \Theta(u,t)d\omega(t) \\
&= \lim_{u\to\kappa} \frac{\kappa^{1-\omega}}{u-\kappa} (M(u,v(u)) - M(u,v(\kappa))) \\
&= \lim_{u\to\kappa} \frac{M(u,v(u)) - M(u,v(\kappa))}{v(u)-v(\kappa)} . \lim_{u\to\kappa} \frac{v(u)-v(\kappa)}{u-\kappa} .\kappa^{1-\omega} \\
&= \lim_{v(u)\to v(\kappa)} \frac{M(u,v(u)) - M(u,v(\kappa))}{v(u)-v(\kappa)} . {}_{\kappa}T_{\omega}v(\kappa).(v(\kappa))^{1-\omega} \\
&= \Theta(\kappa, v(\kappa))_{\kappa}T_{\omega}v(\kappa).(v(\kappa))^{1-\omega}.
\end{aligned}$$

Theorem 9.*Let Θ be a continuous function over an interval $[a,b]$ of values in $\mathbb{R}$ and $u,v: I \to [a,b]$ of class functions $\mathscr{C}^1$. Then the function H defined on the interval I by*

$$H(\kappa) = \int_{u(\kappa)}^{v(\kappa)} \Theta(\kappa,s)d\omega(s), \tag{7}$$

is of class $\mathscr{C}^1$ and

$$_{\kappa}T_{\omega}H(\kappa) = \Theta(\kappa, v(\kappa))_{\kappa}T_{\omega}v(\kappa).v(\kappa)^{\omega-1} - \Theta(t, u(\kappa))_{\kappa}T_{\omega}u(\kappa).u(t)^{\omega-1} + \int_{u(\kappa)}^{v(\kappa)} {}_{\kappa}T_{\omega}\Theta(\kappa,s)d\omega(s).$$

*Proof.*We have

$$\begin{aligned} H(\kappa) &= \int_{u(\kappa)}^{v(\kappa)} \Theta(\kappa,s)d\omega(s) \\ &= \int_{u(\kappa)}^{0} \Theta(\kappa,s)d\omega(s) + \int_{0}^{v(\kappa)} \Theta(\kappa,s)d\omega(s) \\ &= -\int_{0}^{u(\kappa)} \Theta(\kappa,s)d\omega(s) + \int_{0}^{v(\kappa)} \Theta(\kappa,s)d\omega(s) \\ &= K(\kappa) - L(\kappa), \end{aligned}$$

where

$$\begin{cases} K(\kappa) = \int_0^{v(\kappa)} \Theta(\kappa,s)d\omega(s), \\ L(\kappa) = \int_0^{u(\kappa)} \Theta(\kappa,s)d\omega(s), \end{cases}$$

from (6), so

$${}_{\kappa}T_{\omega}H(\kappa) = {}_{\kappa}T_{\omega}K(\kappa) - {}_{\kappa}T_{\omega}L(\kappa),$$

then

$$\begin{aligned} {}_{\kappa}T_{\omega}H(\kappa) = \Theta(\kappa,v(\kappa))_{\kappa}T_{\omega}v(\kappa)v(\kappa)^{\omega-1} + \int_0^{v(\kappa)} {}_{\kappa}T_{\omega}\Theta(\kappa,s)d\omega(s) \\ -\Theta(\kappa,u(\kappa))_{\kappa}T_{\omega}u(\kappa)u(\kappa)^{\omega-1} - \int_0^{u(\kappa)} {}_{\kappa}T_{\omega}\Theta(\kappa,s)d\omega(s). \end{aligned}$$

Corollary 3.*The function H defined on the interval I by*

$$H(t) = \int_0^t \Theta(t,s)d\omega(s), \tag{8}$$

is of class $\mathscr{C}^1$ and

$${}_{t}T_{\omega}H(t) = \Theta(t,t) + \int_0^t {}_{t}T_{\omega}\Theta(t,s)d\omega(s).$$

4 Applications

Theorem 10.*Let fractional differential equation*

$$\begin{cases} {}_{\kappa}T_{\omega}y(\kappa) = f(\kappa), \\ y(0) = y_0, \end{cases} \tag{9}$$

then problem (9) has one and only one solution x namely,

$$y(\kappa) = y_0 + \int_0^{\kappa} f(s)d\omega(s).$$

*Proof.*We have

$$\begin{aligned} {}_{\kappa}T_{\omega}y(\kappa) &= {}_{\kappa}T_{\omega}(y_0 + \int_0^{\kappa} f(s)d\omega(s)) \\ &= {}_{\kappa}T_{\omega}(y_0) + {}_{\kappa}T_{\omega}(\int_0^{\kappa} f(s)d\omega(s)), \end{aligned}$$

we have

$${}_{\kappa}T_{\omega}(\int_0^{\kappa} f(s)d\omega(s)) = f(\kappa).$$

Theorem 11. *Let fractional differential equation*

$$\begin{cases} {}_{\kappa}T_{\omega}y(\kappa) = Ay(\kappa) + Bu(\kappa), \\ y(0) = y_0, \end{cases} \tag{10}$$

where A and B matrices, then problem (10) has one and only one solution y namely,

$$y(\kappa) = e^{\frac{\kappa^{\omega}}{\omega}A}y_0 + \int_0^{\kappa} e^{(\frac{\kappa^{\omega}}{\omega} - \frac{s^{\omega}}{\omega})A}Bu(s)d\omega(s).$$

Proof. Let y be a solution of (10), we have then

$$\begin{aligned} {}_{\kappa}T_{\omega}y(\kappa) &= {}_{\kappa}T_{\omega}\left(e^{\frac{\kappa^{\omega}}{\omega}A}y_0 + \int_0^{\kappa} e^{(\frac{\kappa^{\omega}}{\omega} - \frac{s^{\omega}}{\omega})A}Bu(s)d\omega(s)\right) \\ &= {}_{\kappa}T_{\omega}(e^{\frac{\kappa^{\omega}}{\omega}A}y_0) + {}_{\kappa}T_{\omega}\left(\int_0^{\kappa} e^{(\frac{\kappa^{\omega}}{\omega} - \frac{s^{\omega}}{\omega})A}Bu(s)d\omega(s)\right) \\ &= {}_{\kappa}T_{\omega}(e^{\frac{\kappa^{\omega}}{\omega}A})y_0 + {}_{\kappa}T_{\omega}\left(\int_0^{\kappa} e^{(\frac{\kappa^{\omega}}{\omega} - \frac{s^{\omega}}{\omega})A}Bu(s)d\omega(s)\right), \end{aligned}$$

from Corollary 3, we have

$$\begin{aligned} {}_{\kappa}T_{\omega}\left(\int_0^{\kappa} e^{(\frac{\kappa^{\omega}}{\omega} - \frac{s^{\omega}}{\omega})A}Bu(s)d\omega(s)\right) &= e^{(\frac{\kappa^{\omega}}{\omega} - \frac{\kappa^{\omega}}{\omega})A}Bu(\kappa) + \int_0^{\kappa} {}_{\kappa}T_{\omega}(e^{(\frac{\kappa^{\omega}}{\omega} - \frac{s^{\omega}}{\omega})A}Bu(s))d\omega(s) \\ &= Bu(\kappa) + \int_0^{\kappa} Ae^{(\frac{\kappa^{\omega}}{\omega} - \frac{s^{\omega}}{\omega})A}Bu(s)d\omega(s) \\ &= Bu(\kappa) + A\int_0^{\kappa} e^{(\frac{\kappa^{\omega}}{\omega} - \frac{s^{\omega}}{\omega})A}Bu(s)d\omega(s), \end{aligned}$$

so

$$\begin{aligned} {}_{\kappa}T_{\omega}y(\kappa) &= Ae^{\frac{\kappa^{\omega}}{\omega}A}y_0 + Bu(\kappa) + A\int_0^{\kappa} e^{(\frac{\kappa^{\omega}}{\omega} - \frac{s^{\omega}}{\omega})A}Bu(s)d\omega(s) \\ &= Ay(\kappa) + Bu(\kappa). \end{aligned}$$

5 Conclusion

In this work, we introduced new properties and theorems related to a new fractional derivative, such as the CFD of the reciprocal function, power of function, and exponential-decay function. Two main types of functions were defined. These functions are often found in the analysis, mathematical physics, and ω-Leibniz integral rule. Finally, applications on solving the linear systems fractional differential equations were presented.

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